INSTRUCTOR'S MANUAL TO ACCOMPANY

INTRODUCTION TO REAL ANALYSIS

Fourth Edition

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PREFACE

This manual is offered as an aid in using the fourth edition of *Introduction to Real Analysis* as a text. Both of us have frequently taught courses from the earlier editions of the text and we share here our experience and thoughts as to how to use the book. We hope our comments will be useful.

We also provide partial solutions for almost all of the exercises in the book. Complete solutions are *almost never* presented here, but we hope that enough is given so that a complete solution is within reach. Of course, there is more than one correct way to attack a problem, and you may find better proofs for some of these exercises.

We also repeat the graphs that were given in the manual for the previous editions, which were prepared for us by Professor Horacio Porta, whom we wish to thank again.

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CONTENTS

CHAPTER 1
PRELIMINARIES

We suggest that this chapter be treated as review and covered quickly, without detailed classroom discussion. For one reason, many of these ideas will be already familiar to the students — at least informally. Further, we believe that, in practice, those notions of importance are best learned in the arena of real analysis, where their use and significance are more apparent. Dwelling on the formal aspect of sets and functions does not contribute very greatly to the students' understanding of real analysis.

If the students have already studied abstract algebra, number theory or combinatorics, they should be familiar with the use of mathematical induction. If not, then some time should be spent on mathematical induction.

The third section deals with finite, infinite and countable sets. These notions are important and should be briefly introduced. However, we believe that it is not necessary to go into the proofs of these results at this time.

Section 1.1

Students are usually familiar with the notations and operations of set algebra, so that a brief review is quite adequate. One item that should be mentioned is that two sets A and B are often proved to be equal by showing that: (i) if $x \in A$, then $x \in B$, and (ii) if $x \in B$, then $x \in A$. This type of element-wise argument is very common in real analysis, since manipulations with set identities is often not suitable when the sets are complicated.

Students are often not familiar with the notions of functions that are injective (= one-one) or surjective (= onto).

Sample Assignment: Exercises 1, 3, 9, 14, 15, 20.

Partial Solutions:

1. (a) $B \cap C = \{5, 11, 17, 23, \ldots\} = \{6k - 1 : k \in \mathbb{N}\}$, $A \cap (B \cap C) = \{5, 11, 17\}$
   (b) $(A \cap B) \setminus C = \{2, 8, 14, 20\}$
   (c) $(A \cap C) \setminus B = \{3, 7, 9, 13, 15, 19\}$
2. The sets are equal to (a) A, (b) $A \cap B$, (c) the empty set.
3. If $A \subseteq B$, then $x \in A$ implies $x \in B$, whence $x \in A \cap B$, so that $A \subseteq A \cap B \subseteq A$. Thus, if $A \subseteq B$, then $A = A \cap B$.

   Conversely, if $A = A \cap B$, then $x \in A$ implies $x \in A \cap B$, whence $x \in B$. Thus if $A = A \cap B$, then $A \subseteq B$.
4. If x is in $A \setminus (B \cap C)$, then x is in A but $x \notin B \cap C$, so that $x \in A$ and x is either not in B or not in C. Therefore either $x \in A \setminus B$ or $x \in A \setminus C$, which implies that $x \in (A \setminus B) \cup (A \setminus C)$. Thus $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.

Conversely, if x is in $(A \setminus B) \cup (A \setminus C)$, then $x \in A \setminus B$ or $x \in A \setminus C$. Thus $x \in A$ and either $x \notin B$ or $x \notin C$, which implies that $x \in A$ but $x \notin B \cap C$, so that $x \in A \setminus (B \cap C)$. Thus $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$.

Since the sets $A \setminus (B \cap C)$ and $(A \setminus B) \cup (A \setminus C)$ contain the same elements, they are equal.

5. (a) If $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B \cup C$. Hence we either have (i) $x \in A$ and $x \in B$, or we have (ii) $x \in A$ and $x \in C$. Therefore, either $x \in A \cap B$ or $x \in A \cap C$, so that $x \in (A \cap B) \cup (A \cap C)$. This shows that $A \cap (B \cup C)$ is a subset of $(A \cap B) \cup (A \cap C)$.

   Conversely, let y be an element of $(A \cap B) \cup (A \cap C)$. Then either (j) $y \in A \cap B$, or (jj) $y \in A \cap C$. It follows that $y \in A$ and either $y \in B$ or $y \in C$. Therefore, $y \in A$ and $y \in B \cup C$, so that $y \in A \cap (B \cup C)$. Hence $(A \cap B) \cup (A \cap C)$ is a subset of $A \cap (B \cup C)$.

   In view of Definition 1.1.1, we conclude that the sets $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are equal.
   (b) Similar to (a).
6. The set D is the union of $\{x : x \in A \text{ and } x \notin B\}$ and $\{x : x \notin A \text{ and } x \in B\}$.
7. Here $A_n = \{n+1, 2(n+1), \ldots\}$.
   (a) $A_1 = \{2,4,6,8,\ldots\}$, $A_2 = \{3,6,9,12,\ldots\}$, $A_1 \cap A_2 = \{6,12,18,24,\ldots\} = \{6k : k \in \mathbb{N}\} = A_5$.
   (b) $\bigcup A_n = \mathbb{N} \setminus \{1\}$, because if $n > 1$, then $n \in A_{n-1}$; moreover $1 \notin A_n$. Also $\bigcap A_n = \emptyset$, because $n \notin A_n$ for any $n \in \mathbb{N}$.
8. (a) The graph consists of four horizontal line segments.
   (b) The graph consists of three vertical line segments.
9. No. For example, both $(0, 1)$ and $(0, -1)$ belong to C.
10. (a) $f(E) = \{1/x^2 : 1 \le x \le 2\} = \{y : \frac{1}{4} \le y \le 1\} = [\frac{1}{4}, 1]$.
    (b) $f^{-1}(G) = \{x : 1 \le 1/x^2 \le 4\} = \{x : \frac{1}{4} \le x^2 \le 1\} = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$.
11. (a) $f(E) = \{x+2 : 0 \le x \le 1\} = [2,3]$, so $h(E) = g(f(E)) = g([2,3]) = \{y^2 : 2 \le y \le 3\} = [4,9]$.
    (b) $g^{-1}(G) = \{y : 0 \le y^2 \le 4\} = [-2,2]$, so $h^{-1}(G) = f^{-1}(g^{-1}(G)) = f^{-1}([-2,2]) = \{x : -2 \le x+2 \le 2\} = [-4,0]$.
12. If 0 is removed from E and F, then their intersection is empty, but the intersection of the images under f is $\{y : 0 < y \le 1\}$.
13. $E \setminus F = \{x : -1 \le x < 0\}$, $f(E) \setminus f(F)$ is empty, and $f(E \setminus F) = \{y : 0 < y \le 1\}$.
14. If $y \in f(E \cap F)$, then there exists $x \in E \cap F$ such that $y = f(x)$. Since $x \in E$ implies $y \in f(E)$, and $x \in F$ implies $y \in f(F)$, we have $y \in f(E) \cap f(F)$. This proves $f(E \cap F) \subseteq f(E) \cap f(F)$.
15. If $x \in f^{-1}(G) \cap f^{-1}(H)$, then $x \in f^{-1}(G)$ and $x \in f^{-1}(H)$, so that $f(x) \in G$ and $f(x) \in H$. Then $f(x) \in G \cap H$, and hence $x \in f^{-1}(G \cap H)$. This shows

that $f^{-1}(G) \cap f^{-1}(H) \subseteq f^{-1}(G \cap H)$. The opposite inclusion is shown in Example 1.1.8(b). The proof for unions is similar.

16. If $f(a) = f(b)$, then $a/\sqrt{a^2+1} = b/\sqrt{b^2+1}$, from which it follows that $a^2 = b^2$. Since a and b must have the same sign, we get $a = b$, and hence f is injective. If $-1 < y < 1$, then $x := y/\sqrt{1-y^2}$ satisfies $f(x) = y$ (why?), so that f takes $\mathbb{R}$ onto the set $\{y : -1 < y < 1\}$. If $x > 0$, then $x = \sqrt{x^2} < \sqrt{x^2+1}$, so it follows that $f(x) \in \{y : 0 < y < 1\}$.
17. One bijection is the familiar linear function that maps a to 0 and b to 1, namely, $f(x) := (x-a)/(b-a)$. Show that this function works.
18. (a) Let $f(x) = 2x$, $g(x) = 3x$.
    (b) Let $f(x) = x^2$, $g(x) = x$, $h(x) = 1$. (Many examples are possible.)
19. (a) If $x \in f^{-1}(f(E))$, then $f(x) \in f(E)$, so that there exists $x_1 \in E$ such that $f(x_1) = f(x)$. If f is injective, then $x_1 = x$, whence $x \in E$. Therefore, $f^{-1}(f(E)) \subseteq E$. Since $E \subseteq f^{-1}(f(E))$ holds for any f, we have set equality when f is injective. See Example 1.1.8(a) for an example.
    (b) If $y \in H$ and f is surjective, then there exists $x \in A$ such that $f(x) = y$. Then $x \in f^{-1}(H)$ so that $y \in f(f^{-1}(H))$. Therefore $H \subseteq f(f^{-1}(H))$. Since $f(f^{-1}(H)) \subseteq H$ for any f, we have set equality when f is surjective. See Example 1.1.8(a) for an example.
20. (a) Since $y = f(x)$ if and only if $x = f^{-1}(y)$, it follows that $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.
    (b) Since f is injective, then f^{-1} is injective on $R(f)$. And since f is surjective, then f^{-1} is defined on $R(f) = B$.
21. If $g(f(x_1)) = g(f(x_2))$, then $f(x_1) = f(x_2)$, so that $x_1 = x_2$, which implies that $g \circ f$ is injective. If $w \in C$, there exists $y \in B$ such that $g(y) = w$, and there exists $x \in A$ such that $f(x) = y$. Then $g(f(x)) = w$, so that $g \circ f$ is surjective. Thus $g \circ f$ is a bijection.
22. (a) If $f(x_1) = f(x_2)$, then $g(f(x_1)) = g(f(x_2))$, which implies $x_1 = x_2$, since $g \circ f$ is injective. Thus f is injective.
    (b) Given $w \in C$, since $g \circ f$ is surjective, there exists $x \in A$ such that $g(f(x)) = w$. If $y := f(x)$, then $y \in B$ and $g(y) = w$. Thus g is surjective.
23. We have $x \in f^{-1}(g^{-1}(H)) \iff f(x) \in g^{-1}(H) \iff g(f(x)) \in H \iff x \in (g \circ f)^{-1}(H)$.
24. If $g(f(x)) = x$ for all $x \in D(f)$, then $g \circ f$ is injective, and Exercise 22(a) implies that f is injective on $D(f)$. If $f(g(y)) = y$ for all $y \in D(g)$, then Exercise 22(b) implies that f maps $D(f)$ onto $D(g)$. Thus f is a bijection of $D(f)$ onto $D(g)$, and $g = f^{-1}$.

Section 1.2

The method of proof known as Mathematical Induction is used frequently in real analysis, but in many situations the details follow a routine patterns and are left to the reader by means of a phrase such as: "The proof is by Mathematical Induction". Since may students have only a hazy idea of what is involved, it may be a good idea to spend some time explaining and illustrating what constitutes a proof by induction.

Pains should be taken to emphasize that the induction hypothesis does *not* entail "assuming what is to be proved". The inductive step concerns the validity of going from the assertion for $k \in \mathbb{N}$ to that for $k+1$. The truth of falsity of the individual assertion is not an issue here.

Sample Assignment: Exercises 1, 2, 6, 11, 13, 14, 20.

Partial Solutions:

1. The assertion is true for $n=1$ because $1/(1\cdot 2)=1/(1+1)$. If it is true for $n=k$, then it follows for $k+1$ because $k/(k+1)+1/[(k+1)(k+2)]=(k+1)/(k+2)$.
2. The statement is true for $n=1$ because $[\frac{1}{2}\cdot 1\cdot 2]^2=1=1^3$. For the inductive step, use the fact that
$$\left[\tfrac{1}{2}k(k+1)\right]^2+(k+1)^3=\left[\tfrac{1}{2}(k+1)(k+2)\right]^2.$$
3. It is true for $n=1$ since $3=4-1$. If the equality holds for $n=k$, then add $8(k+1)-5=8k+3$ to both sides and show that $(4k^2-k)+(8k+3)=4(k+1)^2-(k+1)$ to deduce equality for the case $n=k+1$.
4. It is true for $n=1$ since $1=(4-1)/3$. If it is true for $n=k$, then add $(2k+1)^2$ to both sides and use some algebra to show that
$$\tfrac{1}{3}(4k^3-k)+(2k+1)^2=\tfrac{1}{3}[4k^3+12k^2+11k+3]=\tfrac{1}{3}[4(k+1)^3-(k+1)],$$
which establishes the case $n=k+1$.
5. Equality holds for $n=1$ since $1^2=(-1)^2(1\cdot 2)/2$. The proof is completed by showing $(-1)^{k+1}[k(k+1)]/2+(-1)^{k+2}(k+1)^2=(-1)^{k+2}[(k+1)(k+2)]/2$.
6. If $n=1$, then $1^3+5\cdot 1=6$ is divisible by 6. If k^3+5k is divisible by 6, then $(k+1)^3+5(k+1)=(k^3+5k)+3k(k+1)+6$ is also, because $k(k+1)$ is always even (why?) so that $3k(k+1)$ is divisible by 6, and hence the sum is divisible by 6.
7. If $5^{2k}-1$ is divisible by 8, then it follows that $5^{2(k+1)}-1=(5^{2k}-1)+24\cdot 5^{2k}$ is also divisible by 8.
8. $5^{k+1}-4(k+1)-1=5\cdot 5^k-4k-5=(5^k-4k-1)+4(5^k-1)$. Now show that 5^k-1 is always divisible by 4.
9. If $k^3+(k+1)^3+(k+2)^3$ is divisible by 9, then $(k+1)^3+(k+2)^3+(k+3)^3=k^3+(k+1)^3+(k+2)^3+9(k^2+3k+3)$ is also divisible by 9.
10. The sum is equal to $n/(2n+1)$.

11. The sum is $1+3+\cdots+(2n-1)=n^2$. Note that $k^2+(2k+1)=(k+1)^2$.
12. If $n_0>1$, let $S_1 := \{n\in\mathbb{N} : n-n_0+1\in S\}$ Apply 1.2.2 to the set S_1.
13. If $k<2^k$, then $k+1<2^k+1<2^k+2^k=2(2^k)=2^{k+1}$.
14. If $n=4$, then $2^4=16<24=4!$. If $2^k<k!$ and if $k\geq 4$, then $2^{k+1}=2\cdot 2^k<2\cdot k!<(k+1)\cdot k!=(k+1)!$. [Note that the inductive step is valid whenever $2<k+1$, including $k=2,3$, even though the statement is false for these values.]
15. For $n=5$ we have $7\leq 2^3$. If $k\geq 5$ and $2k-3\leq 2^{k-2}$, then $2(k+1)-3=(2k-3)+2\leq 2^{k-2}+2^{k-2}=2^{(k+1)-2}$.
16. It is true for $n=1$ and $n\geq 5$, but false for $n=2,3,4$. The inequality $2k+1<2^k$, wich holds for $k\geq 3$, is needed in the induction argument. [The inductive step is valid for $n=3,4$ even though the inequality $n^2<2^n$ is false for these values.]
17. $m=6$ trivially divides n^3-n for $n=1$, and it is the largest integer to divide $2^3-2=6$. If k^3-k is divisible by 6, then since k^2+k is even (why?), it follows that $(k+1)^3-(k+1)=(k^3-k)+3(k^2+k)$ is also divisible by 6.
18. $\sqrt{k}+1/\sqrt{k+1}=(\sqrt{k}\sqrt{k+1}+1)/\sqrt{k+1}>(k+1)/\sqrt{k+1}=\sqrt{k+1}$.
19. First note that since $2\in S$, then the number $1=2-1$ belongs to S. If $m\notin S$, then $m<2^m\in S$, so $2^m-1\in S$, etc.
20. If $1\leq x_{k-1}\leq 2$ and $1\leq x_k\leq 2$, then $2\leq x_{k-1}+x_k\leq 4$, so that $1\leq x_{k+1}=(x_{k-1}+x_k)/2\leq 2$.

Section 1.3

Every student of advanced mathematics needs to know the meaning of the words "finite", "infinite", "countable" and "uncountable". For most students at this level it is quite enough to learn the definitions and read the statements of the theorems in this section, but to skip the proofs. Probably every instructor will want to show that $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable (see Section 2.5).

Some students will not be able to comprehend that proofs are necessary for "obvious" statements about finite sets. Others will find the material absolutely fascinating and want to prolong the discussion forever. The teacher must avoid getting bogged down in a protracted discussion of cardinal numbers.

Sample Assignment: Exercises 1, 5, 7, 9, 11.

Partial Solutions:

1. If $T_1\neq\emptyset$ is finite, then the definition of a finite set applies to $T_2=\mathbb{N}_n$ for some n. If f is a bijection of T_1 onto T_2, and if g is a bijection of T_2 onto $\mathbb{N}_n$, then (by Exercise 1.1.21) the composite $g\circ f$ is a bijection of T_1 onto $\mathbb{N}_n$, so that T_1 is finite.

2. Part (b) Let f be a bijection of $\mathbb{N}_m$ onto A and let $C = \{f(k)\}$ for some $k \in \mathbb{N}_m$. Define g on $\mathbb{N}_{m-1}$ by $g(i) := f(i)$ for $i = 1, \ldots, k-1$, and $g(i) := f(i+1)$ for $i = k, \ldots, m-1$. Then g is a bijection of $\mathbb{N}_{m-1}$ onto $A \backslash C$. (Why?)
Part (c) First note that the union of two finite sets is a finite set. Now note that if C/B were finite, then $C = B \cup (C \setminus B)$ would also be finite.
3. (a) The element 1 can be mapped into any of the three elements of T, and 2 can then be mapped into any of the two remaining elements of T, after which the element 3 can be mapped into only one element of T. Hence there are $6 = 3 \cdot 2 \cdot 1$ different injections of S into T.
(b) Suppose a maps into 1. If b also maps into 1, then c must map into 2; if b maps into 2, then c can map into either 1 or 2. Thus there are 3 surjections that map a into 1, and there are 3 other surjections that map a into 2.
4. $f(n) := 2n + 13, n \in \mathbb{N}$.
5. $f(1) := 0$, $f(2n) := n, f(2n+1) := -n$ for $n \in \mathbb{N}$.
6. The bijection of Example 1.3.7(a) is one example. Another is the shift defined by $f(n) := n + 1$ that maps $\mathbb{N}$ onto $\mathbb{N} \setminus \{1\}$.
7. If T_1 is denumerable, take $T_2 = \mathbb{N}$. If f is a bijection of T_1 onto T_2, and if g is a bijection of T_2 onto $\mathbb{N}$, then (by Exercise 1.1.21) $g \circ f$ is a bijection of T_1 onto $\mathbb{N}$, so that T_1 is denumerable.
8. Let $A_n := \{n\}$ for $n \in \mathbb{N}$, so $\bigcup A_n = \mathbb{N}$.
9. If $S \cap T = \emptyset$ and $f : \mathbb{N} \to S, g \colon \mathbb{N} \to T$ are bijections onto S and T, respectively, let $h(n) := f((n+1)/2)$ if n is odd and $h(n) := g(n/2)$ if n is even. It is readily seen that h is a bijection of $\mathbb{N}$ onto $S \cup T$; hence $S \cup T$ is denumerable. What if $S \cap T \neq \emptyset$?
10. (a) $m + n - 1 = 9$ and $m = 6$ imply $n = 4$. Then $h(6,4) = \frac{1}{2} \cdot 8 \cdot 9 + 6 = 42$.
(b) $h(m,3) = \frac{1}{2}(m+1)(m+2) + m = 19$, so that $m^2 + 5m - 36 = 0$. Thus $m = 4$.
11. (a) $\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ has $2^2 = 4$ elements.
(b) $\mathcal{P}(\{1,2,3\})$ has $2^3 = 8$ elements.
(c) $\mathcal{P}(\{1,2,3,4\})$ has $2^4 = 16$ elements.
12. Let $S_{n+1} := \{x_1, \ldots, x_n, x_{n+1}\} = S_n \cup \{x_{n+1}\}$ have $n+1$ elements. Then a subset of S_{n+1} either (i) contains x_{n+1}, or (ii) does not contain x_{n+1}. The induction hypothesis implies that there are 2^n subsets of type (i), since each such subset is the union of $\{x_{n+1}\}$ and a subset of S_n. There are also 2^n subsets of type (ii). Thus there is a total of $2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$ subsets of S_{n+1}.
13. For each $m \in \mathbb{N}$, the collection of all subsets of $\mathbb{N}_m$ is finite. (See Exercise 12.) Every finite subset of $\mathbb{N}$ is a subset of $\mathbb{N}_m$ for a sufficiently large m. Therefore Theorem 1.3.12 implies that $\mathcal{F}(\mathbb{N}) = \bigcup_{m=1}^{\infty} \mathcal{P}(\mathbb{N}_m)$ is countable.

CHAPTER 2

THE REAL NUMBERS

Students will be familiar with much of the factual content of the first few sections, but the process of deducing these facts from a basic list of axioms will be new to most of them. The ability to construct proofs usually improves gradually during the course, and there are much more significant topics forthcoming. A few selected theorems should be proved in detail, since some experience in writing formal proofs is important to students at this stage. However, one should not spend too much time on this material.

Sections 2.3 and 2.4 on the Completeness Property form the heart of this chapter. *These sections should be covered thoroughly.* Also the Nested Intervals Property in Section 2.5 should be treated carefully.

Section 2.1

One goal of Section 2.1 is to acquaint students with the idea of deducing consequences from a list of basic axioms. Students who have not encountered this type of formal reasoning may be somewhat uncomfortable at first, since they often regard these results as "obvious". Since there is much more to come, a sampling of results will suffice at this stage, making it clear that it is only a sampling. The classic proof of the irrationality of $\sqrt{2}$ should certainly be included in the discussion, and students should be asked to modify this argument for $\sqrt{3}$, etc.

Sample Assignment: Exercises 1(a,b), 2(a,b), 3(a,b), 6, 13, 16(a,b), 20, 23.

Partial Solutions:

1. (a) Apply appropriate algebraic properties to get $b = 0 + b = (-a + a) + b = -a + (a + b) = -a + 0 = -a$.
(b) Apply (a) to $(-a) + a = 0$ with $b = a$ to conclude that $a = -(-a)$.
(c) Apply (a) to the equation $a + (-1)a = a(1 + (-1)) = a \cdot 0 = 0$ to conclude that $(-1)a = -a$.
(d) Apply (c) with $a = -1$ to get $(-1)(-1) = -(-1)$. Then apply (b) with $a = 1$ to get $(-1)(-1) = 1$.
2. (a) $-(a + b) = (-1)(a + b) = (-1)a + (-1)b = (-a) + (-b)$.
(b) $(-a) \cdot (-b) = ((-1)a) \cdot ((-1)b) = (-1)(-1)(ab) = ab$.
(c) Note that $(-a)(-(1/a)) = a(1/a) = 1$.
(d) $-(a/b) = (-1)(a(1/b)) = ((-1)a)(1/b) = (-a)/b$.
3. (a) Add -5 to both sides of $2x + 5 = 8$ and use (A2),(A4),(A3) to get $2x = 3$. Then multiply both sides by $1/2$ to get $x = 3/2$.
(b) Write $x^2 - 2x = x(x - 2) = 0$ and apply Theorem 2.1.3(b). Alternatively, note that $x = 0$ satisfies the equation, and if $x \neq 0$, then multiplication by $1/x$ gives $x = 2$.