Solution Manual for Nonlinear Control

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Contents

Chapter 1

Introduction

1.1 Take $x_1 = y$, $x_2 = \dot{y}$, $\ldots$, $x_n = y^{(n-1)}$. Then

$$f(t,x,u) = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ g(t,x,u) \end{bmatrix}, \quad h = x_1$$

1.2 **(a)** $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, $x_4 = \dot{q}_2$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{k}{J}(x_1 - x_3) + \frac{1}{J}u \end{aligned}$$

(b)

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (-(MgL/I)\cos x_1 - k/I) & 0 & k/I & 0 \\ 0 & 0 & 0 & 1 \\ k/J & 0 & -k/J & 0 \end{bmatrix}$$

$[\partial f/\partial x]$ is globally bounded. Hence, f is globally Lipschitz.

(c) $x_2 = x_4 = 0$, $x_1 - x_3 = 0 \Rightarrow \sin x_1 = 0$. The equilibrium points are $(n\pi, 0, n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$.

1.3 **(a)** $x_1 = \delta$, $x_2 = \dot{\delta}$, $x_3 = E_q$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (P - Dx_2 - \eta_1 x_3 \sin x_1)/M \\ \dot{x}_3 &= (-\eta_2 x_3 + \eta_3 \cos x_1 + E_F)/\tau \end{aligned}$$

(b) f is continuously differentiable $\forall x$; hence it is locally Lipschitz $\forall x$. $[\partial f_2][\partial x_1] = -\eta_1 x_3 \cos x_1/M$ is not globally bounded; hence, f is not globally Lipschitz.

(c) Equilibrium points:

$$0 = x_2, \qquad 0 = P - \eta_1 x_3 \sin x_1, \qquad 0 = -\eta_2 x_3 + \eta_3 \cos x_1 + E_F$$

Substituting x_3 from the third equation into the second one, we obtain

$$P = \left(a + b\sqrt{1-y^2}\right) y \stackrel{\text{def}}{=} g(y)$$

where

$$y = \sin x_1, \qquad a = \frac{\eta_1 E_F}{\eta_2} > P, \qquad b = \frac{\eta_1 \eta_3}{\eta_2}$$

$$0 \le x_1 \le \frac{\pi}{2} \Longleftrightarrow 0 \le y \le 1$$

By calculating $g'(y)$ and $g''(y)$ it can be seen that $g(y)$ starts from zero at $y = 0$, increases until it reaches a maximum and then decreases to $g(1) = a$. Because $P < a$, the equation $P = g(y)$ has a unique solution y^* with $0 < y^* < 1$. For $0 \le x \le \pi/2$, the equation $y^* = \sin x_1$ has a unique solution x_1^*. Thus, the unique equilibrium point is $(x_1^*, 0, (\eta_3/\eta_2)\cos x_1^* + E_F/\eta_2)$.

1.4 **(a)** From Kirchoff's Current Law, $i_s = v_C/R + i_c + i_L$. Let $x_1 = \phi_L$, $x_2 = v_C$, and $u = i_s$.

$$\dot{x}_1 = \frac{d\phi_L}{dt} = v_L = v_C = x_2$$

$$\dot{x}_2 = \frac{dv_C}{dt} = \frac{i_C}{C} = \frac{1}{C}\left(i_s - \frac{v_C}{R} - i_L\right) = -\frac{1}{CR}x_2 - \frac{I_0}{C}\sin kx_1 + \frac{1}{C}u$$

$$f(x,u) = \begin{bmatrix} x_2 \\ -\frac{I_0}{C}\sin kx_1 - \frac{1}{CR}x_2 + \frac{1}{C}u \end{bmatrix}$$

(b) f is continuously differentiable; hence it is locally Lipschitz.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -(I_0 k/C)\cos kx_1 & -1/(CR) \end{bmatrix}$$

$[\partial f/\partial x]$ is globally bounded. Hence, f is globally Lipschitz.

(c) Equilibrium points:

$$0 = x_2, \qquad 0 = I_0 \sin kx_1 + I_s \quad \Rightarrow \quad \sin kx_1 = \frac{I_s}{I_0} < 1$$

Let a and b be the solutions of $\sin y = I_s/I_0$ in $0 < y < \pi$. Then the equilibrium points are

$$\left(\frac{a + 2n\pi}{k}, 0\right), \quad \left(\frac{b + 2n\pi}{k}, 0\right), \quad n = 0, \pm 1, \pm 2, \cdots$$

1.5 The Problem statement should say "in Part (c), $I_s > 0$."

(a) From Kirchoff's Current Law, $i_s = v_C/R + i_c + i_L$. Let $x_1 = \phi_L$, $x_2 = v_C$, and $u = i_s$.

$$\dot{x}_1 = \frac{d\phi_L}{dt} = v_L = v_C = x_2$$

$$\dot{x}_2 = \frac{dv_C}{dt} = \frac{i_C}{C} = \frac{1}{C}\left(i_s - \frac{v_C}{R} - i_L\right) = -\frac{1}{CR}x_2 - \frac{1}{C}(k_1x_1 + k_2x_1^3) + \frac{1}{C}u$$

$$f(x,u) = \begin{bmatrix} x_2 \\ -\frac{1}{C}(k_1x_1 + k_2x_1^3) - \frac{1}{CR}x_2 + \frac{1}{C}u \end{bmatrix}$$

(b) f is continuously differentiable; hence it is locally Lipschitz.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -(1/C)(k_1 + 3k_2x_1^2) & -1/(CR) \end{bmatrix}$$

$[\partial f/\partial x]$ is not globally bounded. Hence, f is not globally Lipschitz.

(c) Equilibrium points:

$$0 = x_2, \qquad 0 = -k_1x_1 - k_2x_1^3 + I_s$$

There is a unique equilibrium point $(x_1^*, 0)$ where x_1^* is the unique solution of $k_1x_1^* + k_2{x_1^*}^3 = I_s$.

1.6 Projecting the force Mg in the direction of F, Newton't law yields the equation of motion

$$M\dot{v} = F - Mg\sin\theta - k_1\text{sgn}(v) - k_2v - k_3v^2$$

where k_1, k_2, and k_3 are positive constants. let $x = v$, $u = F$, and $w = g\sin\theta$. The state equation is

$$\dot{x} = -\frac{k_1}{M}\text{sgn}(x) - \frac{k_2}{M}x - \frac{k_3}{M}x^2 + \frac{1}{M}u - w$$

1.7 **(a)** The state model of $G(s)$ is

$$\dot{z} = Az + Bu, \qquad y = Cz$$

Moreover,

$$u = \sin e, \qquad e = \theta_i - \theta_o, \qquad \dot{\theta}_o = y = Cz$$

The state model of the closed-loop system is

$$\dot{z} = Az + B\sin e, \qquad \dot{e} = -Cz$$

(b) Equilibrium points:

$$0 = Az + B\sin e, \qquad Cz = 0$$

$$z = -A^{-1}B\sin e \quad \Longrightarrow \quad -CA^{-1}B\sin e = 0$$

Since $G(s) = C(sI - A)^{-1}B$, $G(0) = -CA^{-1}B$. Therefore

$$G(0)\sin e = 0 \quad \Longleftrightarrow \quad \sin e = 0 \quad \Longleftrightarrow \quad e = n\pi,\ n = 0, \pm1, \pm2, \ldots$$

At equilibrium, $z = -A^{-1}B\sin e = 0$. Hence, the equilibrium points are $(z, e) = (0, n\pi)$.

1.8 By Newton's law,

$$m\ddot{y} = mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

where k is the spring constant. Let $x_1 = y$ and $x_2 = \dot{y}$.

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = g - \frac{k}{m}x_1 - \frac{c_1}{m}c_2 - \frac{c_2}{m}x_2|x_2|$$

1.9 **(a)** Substitution of $\dot{v} = A(h)\dot{h}$ and $w_o = k\sqrt{p - p_a} = k\sqrt{\rho g h}$ in $\dot{v} = w_i - w_o$, results in

$$A(h)\dot{h} = w_i - k\sqrt{\rho g h}$$

With $x = h$, $u = w_i$, and $y = h$, the state model is

$$\dot{x} = \frac{1}{A(x)}\left(u - k\sqrt{\rho g x}\right), \qquad y = x$$

(b) With $x = p - p_a$, $u = w_i$, and $y = h$, using $\dot{p} = \rho g\dot{h}$, the state model is

$$\dot{x} = \frac{\rho g}{A(\frac{x}{\rho g})}(u - k\sqrt{x}), \qquad y = \frac{x}{\rho g}$$

(c) From part (a), the equilibrium points satisfy

$$0 = u - k\sqrt{\rho g x}$$

For $x = r$, $u = k\sqrt{\rho g r}$.

1.10 **(a)**

$$\begin{aligned}\dot{x} &= \dot{p} = \frac{\rho g\dot{v}}{A} = \frac{\rho g}{A}(w_i - w_0) \\ &= \frac{\rho g}{A}\left\{\alpha\left[1 - \left(\frac{p - p_a}{\beta}\right)^2\right] - k\sqrt{p - p_a}\right\} \\ &= \frac{\rho g}{A}\left[\alpha\left(1 - \frac{x^2}{\beta^2}\right) - k\sqrt{x}\right]\end{aligned}$$

(b) At equilibrium,

$$0 = \alpha\left(1 - \frac{x^2}{\beta^2}\right) - k\sqrt{x}$$

$$1 - \frac{x^2}{\beta^2} = \frac{k}{\alpha}\sqrt{x}$$

The left-hand side is monotonically decreasing over $[0, \beta]$ and reaches zero at $x = \beta$. The right-hand side is monotonically increasing. Therefore, the forgoing equation has a unique solution $x^* \in (0, \beta)$.

1.11 (a) Let $x_1 = p_1 - p_a$ and $x_2 = p_2 - p_a$.

$$
\begin{aligned}
\dot{x}_1 &= \dot{p}_1 = \frac{\rho g}{A}\dot{v}_1 = \frac{\rho g}{A}(w_p - w_1) \\
&= \frac{\rho g}{A}\left\{\alpha\left[1 - \left(\frac{p_1 - p_a}{\beta}\right)^2\right] - k_1\sqrt{p_1 - p_2}\right\} \\
&= \frac{\rho g}{A}\left[\alpha\left(1 - \frac{x_1^2}{\beta^2}\right) - k_1\sqrt{x_1 - x_2}\right] \\
\dot{x}_2 &= \dot{p}_2 = \frac{\rho g}{A}\dot{v}_2 = \frac{\rho g}{A}(w_1 - w_2) \\
&= \frac{\rho g}{A}\left(k_1\sqrt{x_1 - x_2} - k_2\sqrt{x_2}\right)
\end{aligned}
$$

(b) At equilibrium,

$$
0 = \alpha\left(1 - \frac{x_1^2}{\beta^2}\right) - k_1\sqrt{x_1 - x_2}, \qquad 0 = k_1\sqrt{x_1 - x_2} - k_2\sqrt{x_2}
$$

From the second equation,

$$
x_2 = \frac{k_1^2}{k_1^2 + k_2^2}x_1 \quad \Longrightarrow \quad \sqrt{x_1 - x_2} = \frac{k_2\sqrt{x_1}}{\sqrt{k_1^2 + k_2^2}}
$$

Substitution of $\sqrt{x_1 - x_2}$ in the first equation results in

$$
1 - \frac{x_1^2}{\beta^2} = \frac{k_1 k_2\sqrt{x_1}}{\sqrt{k_1^2 + k_2^2}}
$$

The left-hand side is monotonically decreasing over $[0, \beta]$ and reaches zero at $x_1 = \beta$. The right-hand side is monotonically increasing. Therefore, the forgoing equation has a unique solution $x_1^* \in (0, \beta)$. Hence, there is a unique equilibrium point at (x_1^*, x_2^*), where $x_2^* = x_1^* k_1^2/(k_1^2 + k_2^2)$.

1.12 (a)

$$
f(x, u) = \begin{bmatrix} x_2 \\ -\sin x_1 - bx_2 + cu \end{bmatrix}
$$

Partial derivatives of f are continuous and globally bounded; hence, f is globally Lipschitz.

(b) $\eta(x_1, x_2)$ is discontinuous; hence the right-hand-side function is not locally Lipschitz.

(c) The right-hand-side function is locally Lipschitz if $h(x_1)$ is continuously differentiable. For typical h, as in Figure A.4(b), $\partial h/\partial x_1$ is not globally bounded; in this case, it is not globally Lipschitz

(d) The right-hand-side function f is continuously differentiable; hence it is locally Lipschitz. $\partial f_2/\partial z_2 = -\varepsilon z_2^2$ is not globally bounded; hence, f is not globally Lipschitz.

(e) The right-hand-side function f is continuously differentiable; hence it is locally Lipschitz. $\partial f_1/\partial x_2 = u$; $\partial f_2/\partial x_1 = -u$. Since $0 < u < 1$, the partial derivatives are bounded; hence, f is globally Lipschitz.

(f) The right-hand-side function f is continuously differentiable; hence it is locally Lipschitz. $\partial f_1/\partial x_2 = x_1\nu'(x_2)$ is not globally bounded; hence, f is not globally Lipschitz.

(g) The right-hand-side function f is continuously differentiable; hence it is locally Lipschitz. $\partial f_1/\partial x_2 = -d_1 x_3$ is not globally bounded; hence, f is not globally Lipschitz.

(h) The right-hand-side function f is continuously differentiable; hence it is locally Lipschitz. $\partial f_2/\partial x_3 = -8cx_3/(1+x_1)^2$ is not globally bounded; hence, f is not globally Lipschitz.

(i) The right-hand-side function f is continuously differentiable; hence it is locally Lipschitz. $\partial f_3/\partial x_1 = x_3/T$ is not globally bounded; hence, f is not globally Lipschitz.

(j) The right-hand-side function f is continuously differentiable; hence it is locally Lipschitz. The partial derivatives of $C(x_1,x_2)x_2$ are not globally bounded; hence, f is not globally Lipschitz.

(k) The right-hand-side function f is continuously differentiable; hence it is locally Lipschitz. $\partial f_2/\partial x_2 = -2(mL)^2 x_2 \sin x_1 \cos x_1/\Delta(x_1)$ is not globally bounded; hence, f is not globally Lipschitz.

(l) The right-hand-side function f is continuously differentiable; hence it is locally Lipschitz. $\partial f_2/\partial x_2 = -2(mL)^2 x_2 \sin x_1 \cos x_1/\Delta(x_1)$ is not globally bounded; hence, f is not globally Lipschitz.

1.13

$$y = z_1 = x_1 \quad \Longrightarrow \quad T_1(x) = x_1$$

$$\dot{z}_1 = \dot{x}_1 \quad \Longrightarrow \quad z_2 = x_2 + g_1(x_1) \quad \Longrightarrow \quad T_2(x) = x_2 + g_1(x_1)$$

$$\dot{z}_2 = \dot{x}_2 + \frac{\partial g_1}{\partial x_1}\dot{x}_1 = x_3 + g_2(x_1+x_2) + \frac{\partial g_1}{\partial x_1}[x_2 + g_1(x_1)]$$

$$\Longrightarrow \quad T_3(x) = x_3 + g_2(x_1,x_2) + \frac{\partial g_1}{\partial x_1}[x_2 + g_1(x_1)]$$

$$T(x) = \begin{bmatrix} x_1 \\ x_2 + g_1(x_1) \\ x_3 + g_2(x_1,x_2) + \frac{\partial g_1}{\partial x_1}[x_2 + g_1(x_1)] \end{bmatrix}$$

$$\frac{\partial T}{\partial x} = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}$$

$[\partial T/\partial x]$ is nonsingular for all x and $\|T(x)\| \to \infty$ as $\|x\| \to \infty$. Hence, T is a global diffeomorphism. To show that $\|T(x)\| \to \infty$ as $\|x\| \to \infty$, note that if $\|x\| \to \infty$ then $|x_i| \to \infty$ for at least one of the components of x. If $|x_1| \to \infty$, then $|T_1(x)| \to \infty$. If $|x_1|$ does not tend to ∞, but $|x_2|$ does, then, $|T_2(x)| \to \infty$. If both $|x_1|$ and $|x_2|$ do not go to ∞, but $|x_3|$ does, then, $|T_3(x)| \to \infty$.

1.14

$$y = z_1 = x_1 \quad \Longrightarrow \quad T_1(x) = x_1$$

$$\dot{z}_1 = \dot{x}_1 \quad \Longrightarrow \quad z_2 = \sin x_2 \quad \Longrightarrow \quad T_2(x) = \sin x_2$$

$$T(x) = \begin{bmatrix} x_1 \\ \sin x_2 \end{bmatrix}, \qquad \frac{\partial T}{\partial x} = \begin{bmatrix} 1 & 0 \\ 0 & \cos x_2 \end{bmatrix}$$

$[\partial T/\partial x]$ is nonsingular for $-\pi/2 < x_2 < \pi/2$. The inverse transformation is given by

$$x_1 = z_1, \qquad x_2 = \sin^{-1}(z_2)$$

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = (-x_1^2 + u)\cos x_2 = -z_1^2 \cos(\sin^{-1}(z_2)) + \cos(\sin^{-1}(z_2))u$$

$$a(z) = -z_1^2 \cos(\sin^{-1}(z_2)) = -z_1^2 \sqrt{1 - z_2^2}, \qquad b(z) = \cos(\sin^{-1}(z_2)) = \sqrt{1 - z_2^2}$$