Chapter 1

Introduction

1.1 Mathematical Models and Solutions

1.(a) Let $u(t)$ be the temperature of the coffee at time t (measured in minutes), and $T_0 = 70$ be the ambient temperature. The initial value problem is $u' = -k(u - T_0)$, $u(0) = 200$.

(b) According to the text, the solution of the differential equation is $u(t) = T_0 + ce^{-kt}$. $200 = u(0) = T_0 + c$, thus $c = 130$. Also, $190 = u(1) = 70 + 130e^{-k}$, thus $e^{-k} = 12/13$. The coffee reaches temperature 170 when $170 = 70 + 130e^{-kt} = 70 + 130(12/13)^t$, thus $t = \ln(10/13)/\ln(12/13) \approx 3.28$ minutes.

2. Let us measure time in hours. When the ambient temperature is 120°F, the solution of the differential equation is $u = 120 - 80e^{-kt}$. We know that $90 = u(3/4) = 120 - 80e^{-3k/4}$, thus $e^{-k} = (3/8)^{4/3}$. When the ambient temperature is 100°F, the solution is $u = 100 - 60e^{-kt}$. Then $90 = 100 - 60e^{-kt}$, i.e. $e^{-kt} = 1/6$ when $t = \ln(1/6)/\ln(3/8)^{4/3} \approx 1.37$ hours.

3. Let $t = 0$ be 11:09pm, and let us measure time in hours. The temperature of the body is then given by $u(t) = 68 + 12e^{-kt}$. Also, $78.5 = u(1) = 68 + 12e^{-k}$. This gives that $e^{-k} = 7/8$. The time of death is given by the equation $98.6 = 68 + 12e^{-kt}$; we obtain that $t = \ln(51/20)/\ln(7/8) \approx -7.01$ hours, i.e. at around 4:08pm.

4.(a) The solution of the differential equation $p' = rp$, when $p(0) = p_0$ is $p(t) = p_0 e^{rt}$. If the population doubles in 30 days, then $p(30) = p_0 e^{30r} = 2p_0$, so $r = \ln 2/30$ (day^{-1}).

(b) The same computation shows that $r = \ln 2/N$ (day^{-1}).

5.(a) The general solution is $p(t) = 900 + ce^{t/2}$. Plugging in for the initial condition, we have $p(t) = 900 + (p_0 - 900)e^{t/2}$. With $p_0 = 850$, the solution is $p(t) = 900 - 50e^{t/2}$. To find the time when the population becomes extinct, we need to find the time T when $p(T) = 0$. Therefore, $900 = 50e^{T/2}$, which implies $e^{T/2} = 18$, and, therefore, $T = 2\ln 18 \approx 5.78$ months.

(b) Using the general solution, $p(t) = 900 + (p_0 - 900)e^{t/2}$, we see that the population will become extinct at the time T when $900 = (900 - p_0)e^{T/2}$. That is, $T = 2\ln[900/(900 - p_0)]$ months.

(c) Using the general solution, $p(t) = 900 + (p_0 - 900)e^{t/2}$, we see that the population after 1 year (12 months) will be $p(6) = 900 + (p_0 - 900)e^6$. If we want to know the initial population which will lead to extinction after 1 year, we set $p(6) = 0$ and solve for p_0. Doing so, we have $(900 - p_0)e^6 = 900$ which implies $p_0 = 900(1 - e^{-6}) \approx 897.8$.

6.(a) The equation is $a' = -ra$, $a(0) = a_0$. The solution is $a(t) = a_0 e^{-rt}$.

(b) We have to solve the equation $a_0/2 = a(T_{1/2}) = a_0 e^{-rT_{1/2}}$; we obtain $T_{1/2} = \ln 2/r$.

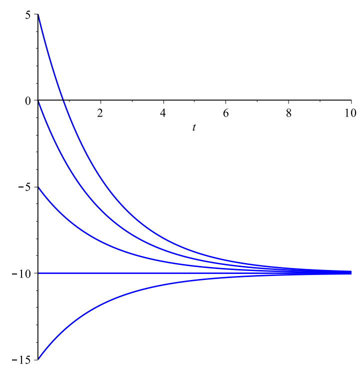
7.(a) The general solution of the equation is $Q(t) = ce^{-rt}$. Given that $Q(0) = 100$, we have $c = 100$. Assuming that $Q(1) = 82.04$, we have $82.04 = 100e^{-r}$. Solving this equation for r, we have $r = -\ln(82.04/100) = 0.19796$ per week or $r = 0.02828$ per day.

(b) Using the form of the general solution and r found above, we have $Q(t) = 100e^{-0.02828t}$.

(c) Let T be the time it takes the isotope to decay to half of its original amount. From part (b), we conclude that $0.5 = e^{-0.2828T}$ which implies that $T = -\ln(0.5)/0.2828 \approx 24.5$ days.

8.(a) First, $mv' = m(v_0 + mg/\gamma)(-\gamma/m)e^{-\gamma t/m} = -\gamma(v_0 + mg/\gamma)e^{-\gamma t/m}$. Also, $-mg - \gamma v = -mg - \gamma((v_0 + mg\gamma)e^{-\gamma t/m} - mg/\gamma) = -\gamma(v_0 + mg/\gamma)e^{-\gamma t/m}$. So the function satisfies the given differential equation. We can also see that $v(0) = (v_0 + mg/\gamma) - mg/\gamma = v_0$.

(b)



(c) The ball reaches its maximum height when $v = 0$. This happens when $(v_0+mg/\gamma)e^{-\gamma t/m} = mg/\gamma$. Dividing both sides by $e^{-\gamma t/m}mg/\gamma$, we obtain $v_0\gamma/(mg) + 1 = e^{\gamma t/m}$. Taking the logarithm of both sides and dividing by γ/m we get that $t = t_{\max} = (m/\gamma)\ln(1+\gamma v_0/(mg))$.

(d) Using the previous parts, $\gamma = -mg/v_{\text{term}} = -0.145(9.8)/(-33)(\text{kg/sec}) \approx 0.0431(\text{kg/sec})$.

(e) Using the expression for the velocity, we can get the function describing the height of the thrown ball. Because $v = h'$, we get that $h(t) = (-m/\gamma)(v_0 + mg/\gamma)e^{-\gamma t/m} - mgt/\gamma + h_0 + (m/\gamma)(v_0 + mg/\gamma)$, where the constant was chosen to satisfy the initial condition $h(0) = h_0$. Using part (c), the time needed to reach maximum height is $(m/\gamma)\ln(1 + \gamma v_0/(mg))$, by plugging this into the height function we obtain that $h_{\max} \approx 31.16$ (m).

9.(a) Following the discussion in the text, the equation is given by $mv' = mg - kv^2$.

(b) After a long time, $v' \to 0$. Therefore, $mg - kv^2 \to 0$, or $v \to \sqrt{mg/k}$.

(c) We need to solve the equation $\sqrt{0.025(9.8)/k} = 35$. Solving this equation, we see that $k = 0.0002$ kg/m.

10. Using the model from the text, we obtain the equation $Q'(t) = 4 - 5(Q(t)/V(t))$, $Q(0) = 0$. The amount of brine is given by the equation $V'(t) = -1$, $V(0) = 200$.

11. Using the model from the text, we obtain the equation $Q' = (1 + (\sin t)/2)/2 - 2Q/100$, $Q(0) = 50$.

12.(a) Using the model from the text, we get the equation $Q'(t) = 300(0.01) - 300Q/1000000$, $Q(0) = Q_0$, where Q_0 is the unknown original amount of chemical in the pond.

(b) The limiting amount can be found by setting the right hand side of the differential equation to zero. We obtain $Q = 10000\text{g} = 10\text{kg}$. The limiting amount does not depend on Q_0.

13.(a) We obtain that $C(t) = C_0 e^{-kt}$; this function satisfies $dC/dt = -kC$ and $C(0) = C_0$.

(b) $C_2 = C_0 + C(T) = C_0 + C_0 e^{-kT} = C_0(1 + e^{-kT})$

(c) Using induction, $C_n = C_0 + C_{n-1}e^{-kT} = C_0(1 + e^{-kT} + e^{-2kT} + \ldots + e^{-(n-1)kT}) = C_0(1 - e^{-nkT})/(1 - e^{-kT})$. Thus $\lim_{n\to\infty} C_n = C_0/(1 - e^{-kT})$.

14.(a) Let $q(t)$ be the total amount of the drug (in milligrams) in the body at a given time t (measured in hours). The drug enters the body at the rate of 5 mg/cm^3 $\cdot$100 cm^3/hr = 500 mg/hr, and the drug leaves the body at the rate of $0.4q$ mg/hr. Therefore, the governing differential equation is given by $dq/dt = 500 - 0.4q$.

(b) If $q > 1250$, then $q' < 0$. If $q < 1250$, then $q' > 0$. Therefore, $q \to 1250$.

15. We compute: $P'(t) = (P_0 + k/r)e^{rt}r = r[(P_0 + k/r)e^{rt} - (k/r)] + k = rP + k$, and $P(0) = P_0 + k/r - k/r = P_0$.

16. Using the model from problem 15, $k = 0$, $P_0 = 1050$ and $r = 0.04$. Then $P(394) \approx 7.34 \cdot 10^9$. If $r = 0.06$, we obtain $P(394) = 1.94 \cdot 10^{13}$.

17. Using the model from problem 15, $k = -200(12)$, $P_0 = 20,000$ and $r = 0.05$. Then solving the equation $P(t) = (P_0 + k/r)e^{rt} - k/r = 0$ we obtain $t \approx 10.78$ years; the total amount paid is $(10.78)(12)(200) = \$25,872$.

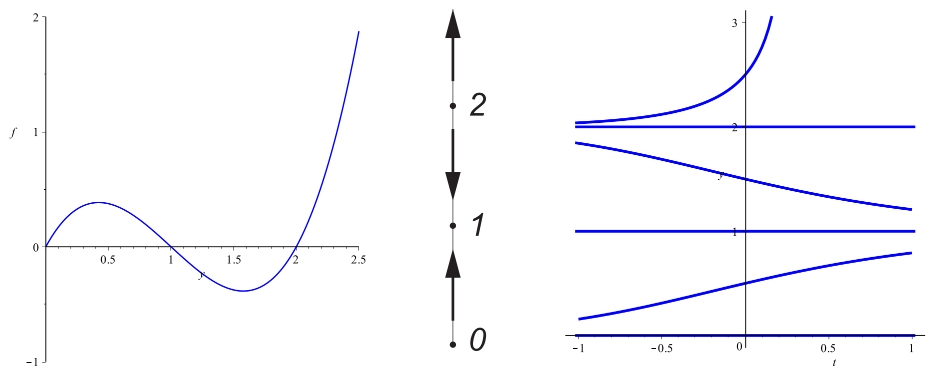
18. The discrete approximation is $P(t + \Delta t) \approx P(t) + (r\Delta t)P(t) + k\Delta t$; following the hint, we can write $P(t + \Delta t) = P(t) + P'(t)\Delta t + (1/2)P''(\tilde{t})(\Delta t)^2$ according to Taylor's theorem. Substituting this into the discrete approximation and subtracting $P(t)$ from both sides we obtain $P'(t)\Delta t + (1/2)P''(\tilde{t})(\Delta t)^2 \approx (r\Delta t)P(t) + k\Delta t$; then after dividing by Δt we get $P'(t) + (1/2)P''(\tilde{t})\Delta t \approx rP(t) + k$. If we let $\Delta t \to 0$, we obtain the differential equation.

19. The surface area of a spherical raindrop of radius r is given by $S = 4\pi r^2$. The volume of a spherical raindrop is given by $V = 4\pi r^3/3$. Therefore, we see that the surface area $S = cV^{2/3}$ for some constant c. If the raindrop evaporates at a rate proportional to its surface area, then $dV/dt = -kV^{2/3}$ for some $k > 0$.

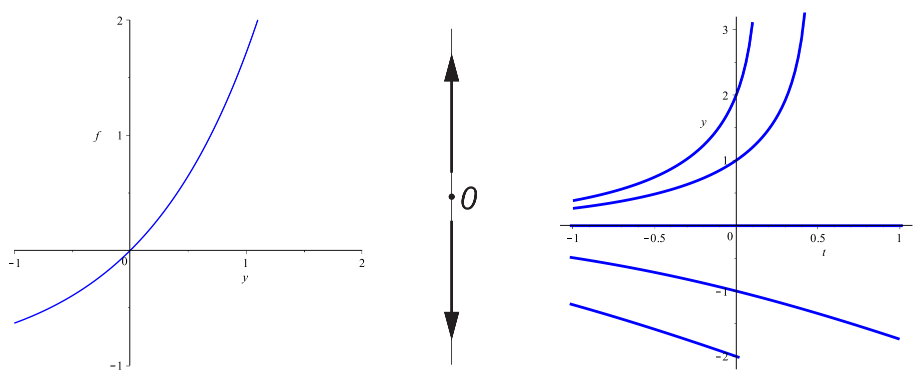
20. The equation of motion is given by $ma = mdv/dt = mg - kv - g\rho_0 4r^3\pi/3$, where ρ_0 is density of the seawater, r is the radius of the buoy, and k is the proportionality constant of the drag force. Also, $v(0) = 10$ m/s. The buoy will sink when $mg \geq g\rho_0 4r^3\pi/3$, i.e. when $m \geq \rho_0 4r^3\pi/3 \approx 536.69$ kg. The terminal sink velocity for m values bigger than this are given by the equation $0 = mg - kv - g\rho_0 4r^3\pi/3$, i.e. $v = (mg - g\rho_0 4r^3\pi/3)/k$.

1.2 Qualitative Methods: Phase Lines and Direction Fields

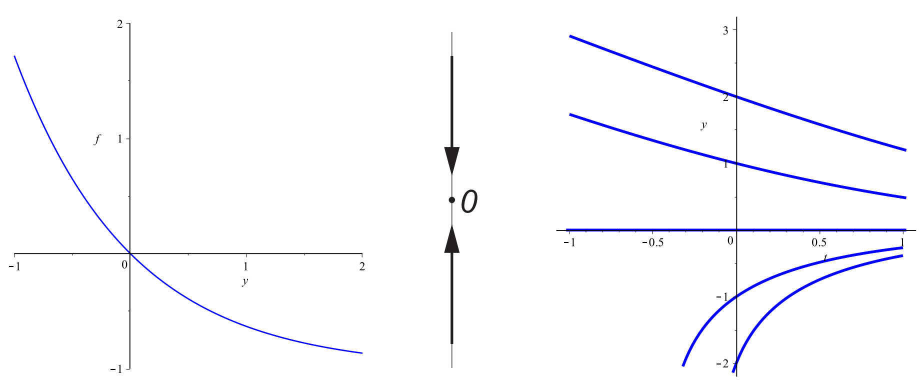
1. $y = 0$, $y = 2$ unstable, $y = 1$ asymptotically stable.



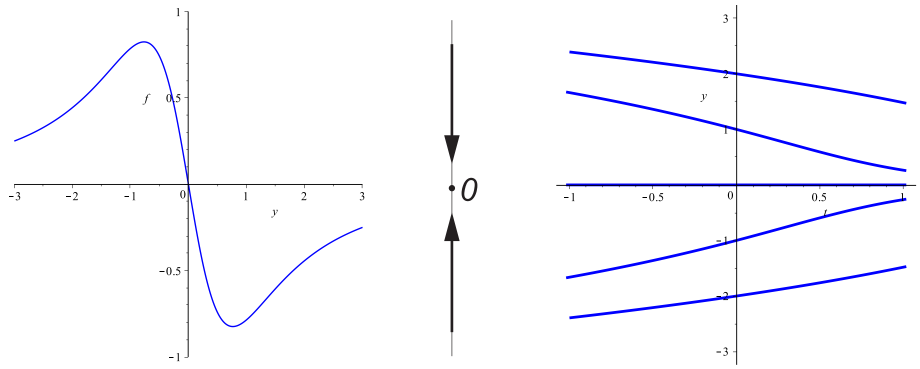
2. $y = 0$ unstable.



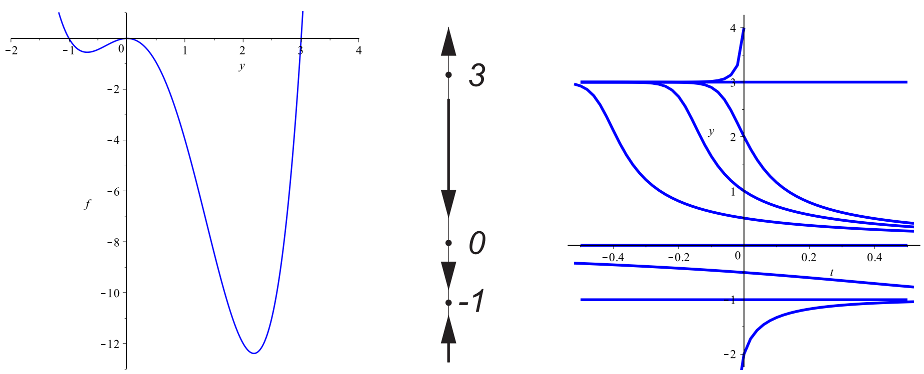
3. $y = 0$ asymptotically stable.



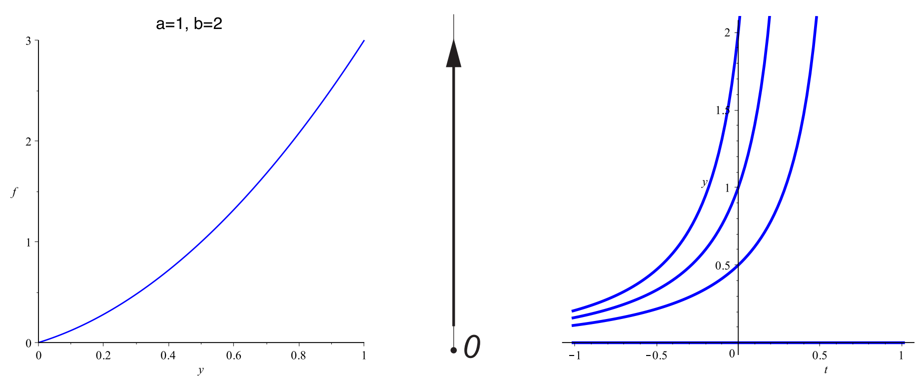
4. $y = 0$ asymptotically stable.



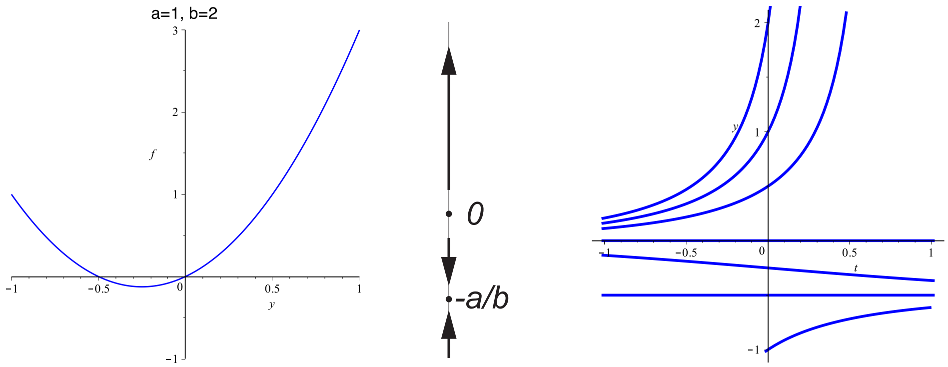
5. $y = -1$ asymptotically stable, $y = 0$ semistable, $y = 3$ unstable.



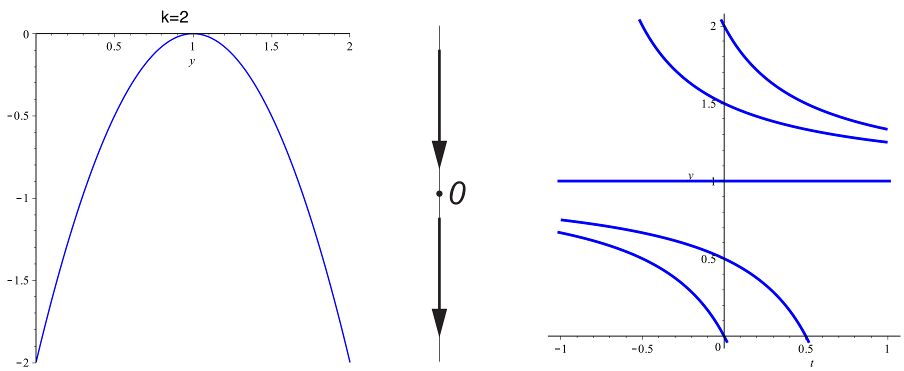
6. $y = 0$ unstable.



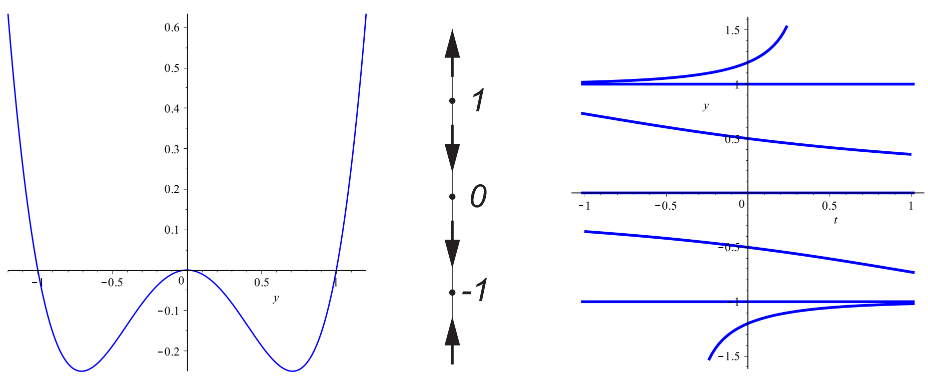
7. $y = 0$ unstable, $y = -a/b$ asymptotically stable.



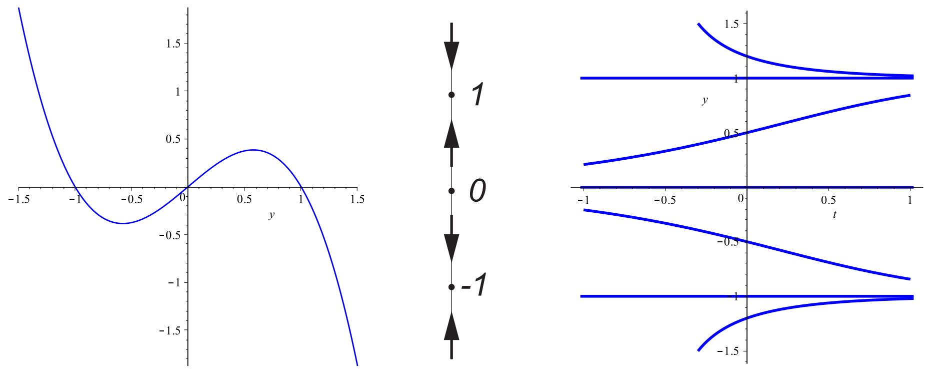
8. $y = 1$ semistable.



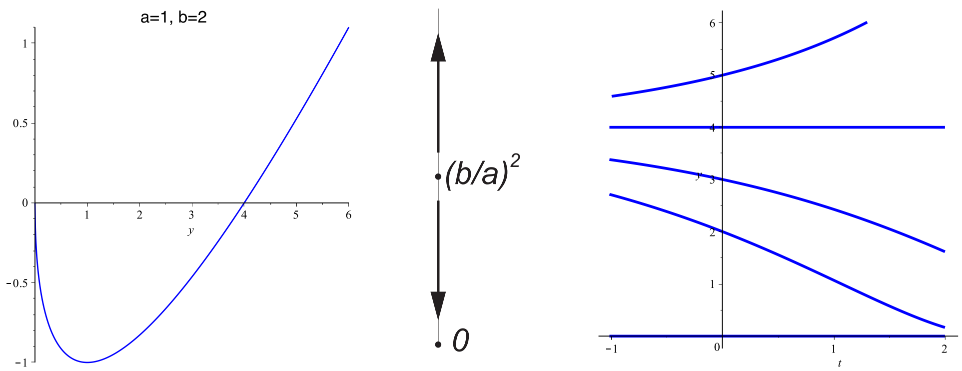
9. $y = -1$ asymptotically stable, $y = 0$ semistable, $y = 1$ unstable.



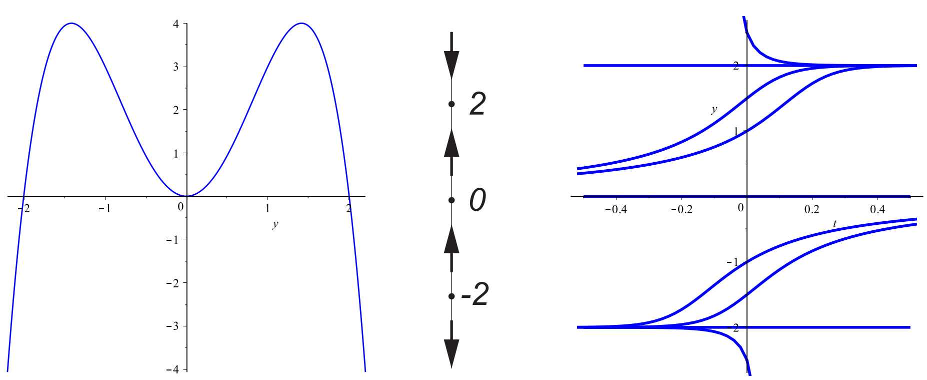
10. $y = -1$ asymptotically stable, $y = 0$ unstable, $y = 1$ asymptotically stable.



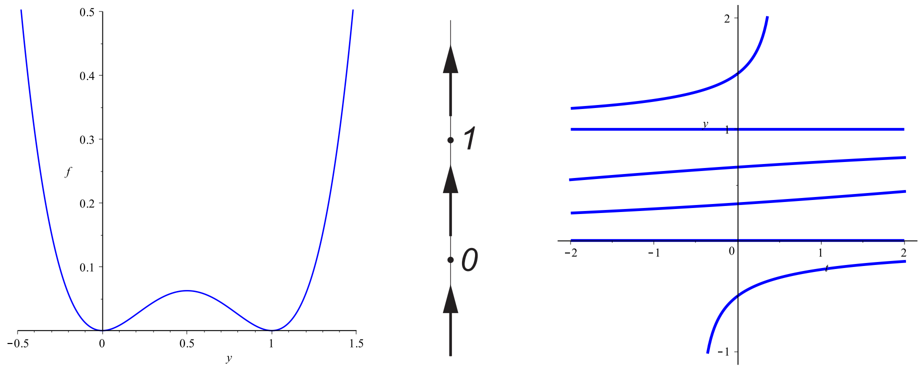
11. $y = 0$ asymptotically stable, $y = (b/a)^2$ unstable.



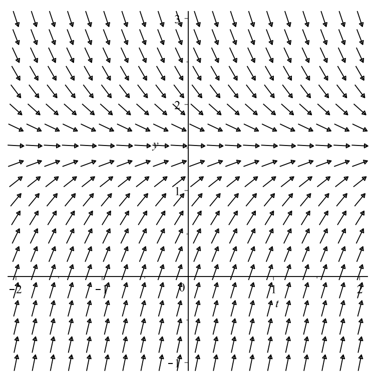
12. $y = -2$ unstable, $y = 0$ semistable, $y = 2$ asymptotically stable.



13. $y = 0$ semistable, $y = 1$ semistable.

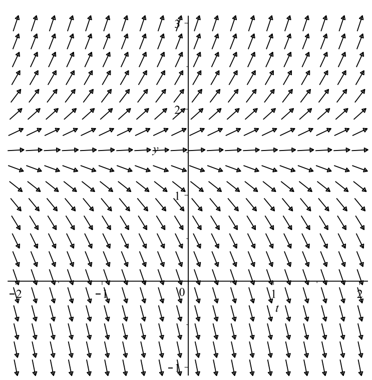


14.



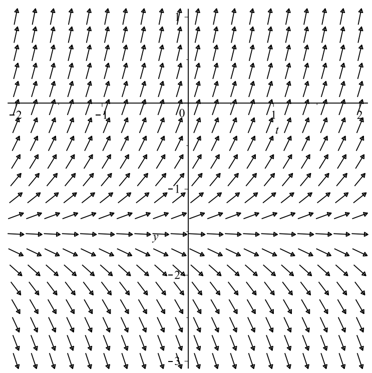
For $y > 3/2$, the slopes are negative, thus the solutions decrease. For $y < 3/2$, the slopes are positive, thus the solutions increase. As a result, $y \to 3/2$ as $t \to \infty$ for all initial conditions y_0.

15.



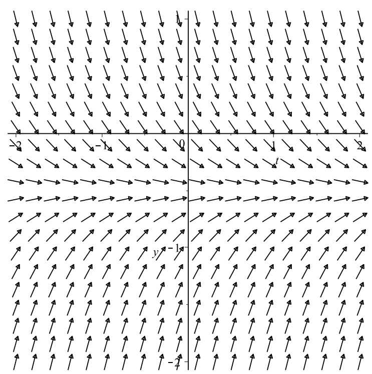
For $y > 3/2$, the slopes are positive, therefore the solutions increase. For $y < 3/2$, the slopes are negative, therefore the solutions decrease. As a result, y diverges from $3/2$ as $t \to \infty$ if $y(0) \neq 3/2$.

16.



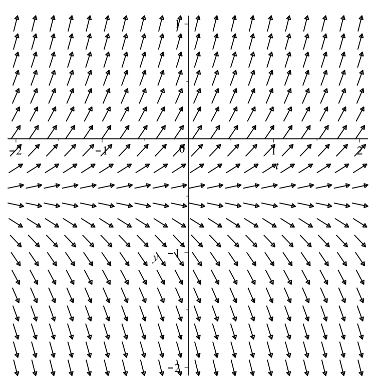
For $y > -3/2$, the slopes are positive, thus the solutions increase. For $y < -3/2$, the slopes are negative, thus the solutions decrease. As a result, y diverges from the equilibrium $-3/2$ as $t \to \infty$ if $y(0) \neq -3/2$.

17.



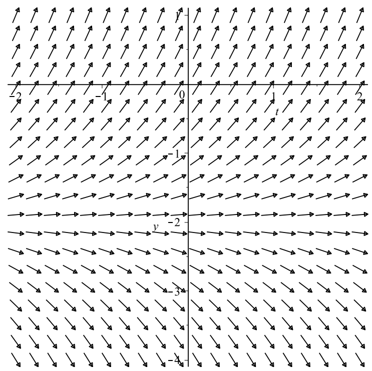
For $y > -1/2$, the slopes are negative, therefore the solutions decrease. For $y < -1/2$, the slopes are positive, therefore the solutions increase. As a result, $y \to -1/2$ as $t \to \infty$ for all initial conditions y_0.

18.



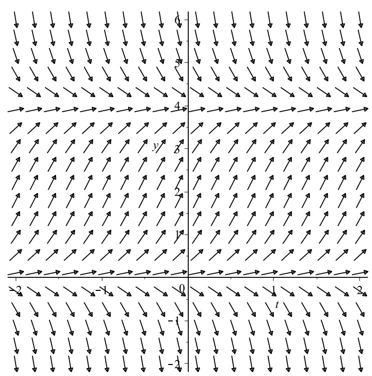
For $y > -1/2$, the slopes are positive, thus the solutions increase. For $y < -1/2$, the slopes are negative, thus the solutions decrease. As a result, y diverges from the equilibrium $-1/2$ as $t \to \infty$ if $y(0) \neq -1/2$.

19.



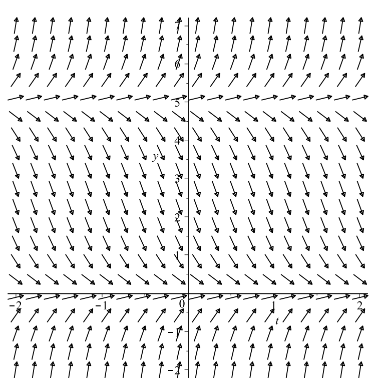
For $y > -2$, the slopes are positive, therefore the solutions increase. For $y < -2$, the slopes are negative, therefore the solutions decrease. As a result, y diverges from -2 as $t \to \infty$ if $y(0) \neq -2$.

20.



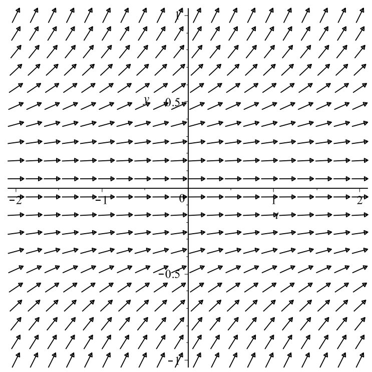
$y = 0$ and $y = 4$ are equilibrium solutions; $y \to 4$ if initial value is positive; y diverges from 0 if initial value is negative.

21.



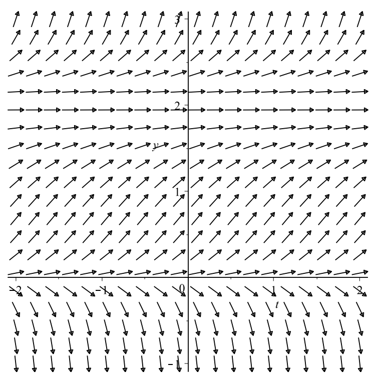
$y = 0$ and $y = 5$ are equilibrium solutions; y diverges from 5 if the initial value is greater than 5; $y \to 0$ if the initial value is less than 5.

22.



$y = 0$ is equilibrium solution; $y \to 0$ if initial value is negative; y diverges from 0 if initial value is positive.

23.



$y = 0$ and $y = 2$ are equilibrium solutions; y diverges from 0 if the initial value is negative; $y \to 2$ if the initial value is between 0 and 2; y diverges from 2 if the initial value is greater than 2.

24. (j) - only equilibrium is $y = 2$; $y' > 0$ when $y < 2$.

25. (c) - only equilibrium is $y = 2$; $y' < 0$ when $y < 2$.

26. (g) - only equilibrium is $y = -2$; $y' > 0$ when $y < -2$.

27. (b) - only equilibrium is $y = -2$; $y' < 0$ when $y < -2$.

28. (h) - equilibria at $y = 0$, $y = 3$; $y' > 0$ when $0 < y < 3$.

29. (e) - equilibria at $y = 0$, $y = 3$; $y' < 0$ when $0 < y < 3$.

30. With

$$\phi(t) = T_0 + \frac{kA}{k^2 + \omega^2}(k \sin \omega t - \omega \cos \omega t) + ce^{-kt},$$

differentiation gives that

$$\phi'(t) = \frac{kA}{k^2 + \omega^2}(k\omega \cos \omega t + \omega^2 \sin \omega t) - kce^{-kt},$$

thus

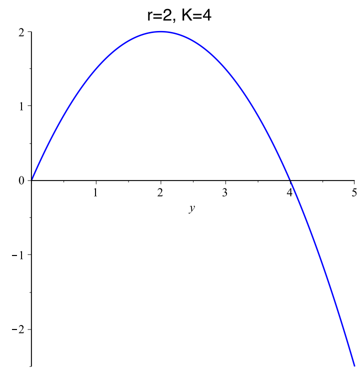
$$\phi'(t) + k\phi(t) = kT_0 + kA \sin \omega t.$$

31. Using the fact that

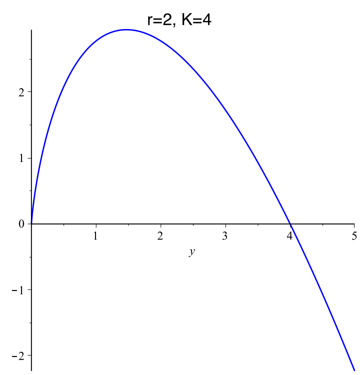
$$R \sin(\omega t - \delta) = R \cos \delta \sin \omega t - R \sin \delta \cos \omega t,$$

where $R^2 \cos^2 \delta + R^2 \sin^2 \delta = R^2 = A^2 + B^2$, the desired result follows.

32. $y = 0$ unstable, $y = K$ asymptotically stable. The figure shows the case $r = 2$, $K = 4$.

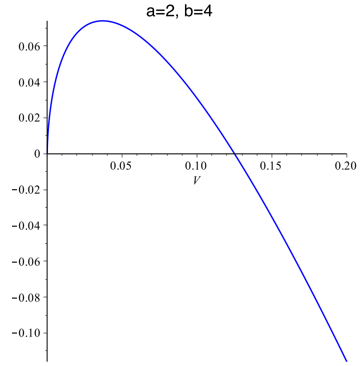


33.(a) $y = 0$ unstable, $y = K$ asymptotically stable. The figure shows the case $r = 2$, $K = 4$.



(b) We have to show that for $0 < y \leq K$, $ry(1 - y/K) \leq ry \ln(K/y)$. This is the same as $1 - y/K \leq \ln(K/y)$; thus we have to show that $e^{1-y/K} \leq K/y$, which is equivalent to $e^{y/K-1} \geq y/K$. The function e^x is concave up, thus the tangent line at $x = 0$ is below the function, i.e. $e^x \geq 1 + x$ for every x value. Plugging in $x = y/K - 1$ we obtain the inequality we need.

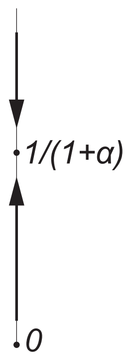
34. $y = 0$ unstable, $y = (a/b)^3$ asymptotically stable. The figure shows the case $a = 2$, $b = 4$.

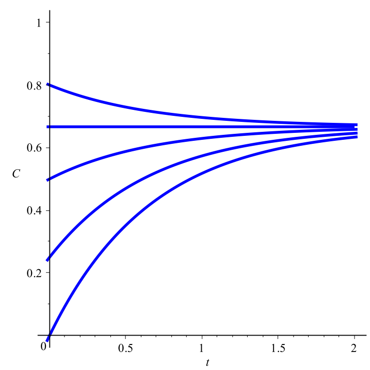


35.(a) We compute:

$$\frac{dC}{d\tau} = \frac{dC}{dt}\frac{dt}{d\tau} = \frac{1}{c_i}\frac{dc}{dt}\frac{dt}{d\tau} = \frac{1}{c_i}(\frac{r_i}{V}c_i - r_i\frac{c}{V} - kc)\frac{V}{r_i} = 1 - \frac{c}{c_i} - \frac{kV}{r_i}\frac{c}{c_i} = 1 - C - \alpha C.$$

(b) The equilibrium is at $C = 1/(1+\alpha)$; it is asymptotically stable.





36.(a) The volume of the cone is $V = \pi a^2 h/3$, thus $3aV/\pi h = a^3$, and then $a^2 = (3a/\pi h)^{2/3}V^{2/3}$. If the rate of evaporation is proportional to the surface area, then rate out $= \alpha\pi a^2 = \alpha\pi(3a/\pi h)^{2/3}V^{2/3}$. We obtain

$$\frac{dV}{dt} = \text{rate in} - \text{rate out} = k - \alpha\pi\left(\frac{3a}{\pi h}\right)^{2/3} V^{2/3}.$$

(b) The equilibrium volume can be found by setting $dV/dt = 0$. We see that the equilibrium volume is

$$V = \left(\frac{k}{\alpha\pi}\right)^{3/2}\left(\frac{\pi h}{3a}\right).$$

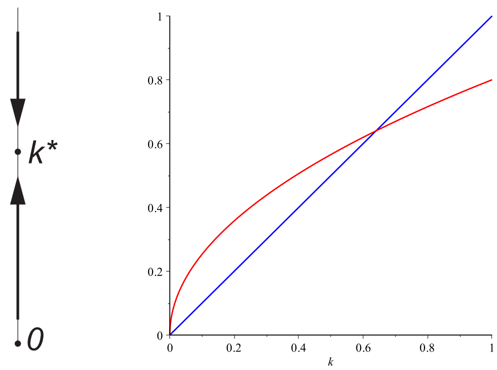
To find the equilibrium height, we use the fact that the height and radius of the conical pond maintain a constant ratio. Therefore, if h_e, a_e represent the equilibrium values for the h and a, we must have $h_e/a_e = h/a$. Further, we notice that the equilibrium volume can be written as

$$V = \pi a_e^2 \frac{h_e}{3} = \left(\frac{k}{\alpha\pi}\right)^{3/2}\left(\frac{\pi h_e}{3a_e}\right),$$

thus $a_e = (k\alpha\pi)^{1/2}$, and then $h_e = (k/\alpha\pi)^{1/2}(h/a)$. Since $f'(V) = -\frac{2}{3}\alpha\pi(3a/\pi h)^{2/3}V^{-1/3} < 0$, the equilibrium is asymptotically stable.

(c) In order to guarantee that the pond does not overflow, we need the rate of water in to be less than or equal to the rate of water out. Therefore, we need $k - \alpha\pi a^2 \leq 0$.

37.(a)



(b) We compute, using the product rule:

$$\frac{dY}{dt} = \frac{dA}{dt}Lf(k) + A\frac{dL}{dt}f(k) + ALf'(k)\frac{dk}{dt} = gALf(k) + nALf(k) = (n+g)Y,$$

because $dk/dt = 0$ at $k = k^*$.

1.3 Definitions, Classification, and Terminology

1. The differential equation is second order, since the highest derivative in the equation is of order two. The equation is linear since the left hand side is a linear function of y and its derivatives and the right hand side is just a function of t.

2. The differential equation is second order since the highest derivative in the equation is of order two. The equation is nonlinear because of the term $y^2 d^2y/dt^2$.

3. The differential equation is fourth order since the highest derivative in the equation is of order four. The equation is linear since the left hand side is a linear function of y and its derivatives and the right hand side does not depend on y.

4. The differential equation is first order since the only derivative in the equation is of order one. The equation is nonlinear because of the y^2 term.

5. The differential equation is second order since the highest derivative in the equation is of order two. The equation is nonlinear because of the term $\sin(t+y)$ which is not a linear function of y.

6. The differential equation is third order since the highest derivative in the equation is of order three. The equation is linear because the left hand side is a linear function of y and its derivatives, and the right hand side is only a function of t.

7. $a_0 = 1$, $a_1 = 1/(1+t)$, $g = 2\sin t$; nonhomogeneous.

8. $a_0 = 1$, $a_1 = 0$, $a_2 = -t$, $g = 0$; homogeneous.

9. $a_0 = x^2$, $a_1 = -3x$, $a_2 = 4$, $g = \ln x$; nonhomogeneous.

10. $a_0 = 1 - x^2$, $a_1 = -2x$, $a_2 = n(n+1)$, $g = 0$; homogeneous.

11. $a_0 = 1$, $a_1 = 0$, $a_2 = \cos t$, $a_3 = 0$, $a_4 = 1$, $g = e^{-t}\sin t$; nonhomogenous.

12. $a_0 = p(x)$, $a_1 = p'(x)$, $a_2 = -q(x) + \lambda r(x)$, $g = 0$; homogeneous.

13. If $y_1 = e^t$, then $y_1' = e^t$ and $y_1'' = e^t$. Therefore, $y_1'' - y_1 = 0$. Also, $y_2 = \cosh t$ implies $y_2' = \sinh t$ and $y_2' = \cosh t$. Therefore, $y_2'' - y_2 = 0$.

14. If $y_1 = e^{-3t}$, then $y_1' = -3e^{-3t}$ and $y_1'' = 9e^{-3t}$. Therefore, $y_1''+2y_1'-3y_1 = (9-6-3)y_1 = 0$. Also, $y_2 = e^t$ implies $y_2' = y_2'' = e^t$. Therefore, $y_1'' + 2y_1' - 3y_1 = (1+2-3)e^t = 0$.

15. If $y = 3t + t^2$, then $y' = 3 + 2t$. Therefore, $ty' - y = t(3+2t) - (3t + t^2) = t^2$.

16. If $y_1 = t/3$, then $y_1' = 1/3$ and $y_1'' = y_1''' = y_1'''' = 0$. Therefore, $y_1'''' + 4y_1''' + 3y = t$. Also, $y_2 = e^{-t} + t/3$ implies $y_2' = -e^{-t} + 1/3$, $y_2'' = e^{-t}$, $y_2''' = -e^{-t}$, and $y_2'''' = e^{-t}$. Therefore, $y_2'''' + 4y_2''' + 3y = e^{-t} - 4e^{-t} + 3(e^{-t} + t/3) = t$.

17. If $y_1 = t^{1/2}$, then $y_1' = t^{-1/2}/2$ and $y_1'' = -t^{-3/2}/4$. Therefore, $2t^2y_1'' + 3ty_1' - y_1 = 2t^2(-t^{-3/2}/4) + 3t(t^{-1/2}/2) - t^{1/2} = (-1/2+3/2-1)t^{1/2} = 0$. Also, $y_2 = t^{-1}$ implies $y_2' = -t^{-2}$ and $y_2'' = 2t^{-3}$. Therefore, $2t^2y_2'' + 3ty_2' - y_2 = 2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1} = (4-3-1)t^{-1} = 0$.

18. If $y_1 = t^{-2}$, then $y_1' = -2t^{-3}$ and $y_1'' = 6t^{-4}$. Therefore, $t^2y_1'' + 5ty_1' + 4y_1 = t^2(6t^{-4}) + 5t(-2t^{-3}) + 4t^{-2} = (6-10+4)t^{-2} = 0$. Also, $y_2 = t^{-2}\ln t$ implies $y_2' = t^{-3} - 2t^{-3}\ln t$ and $y_2'' = -5t^{-4} + 6t^{-4}\ln t$. Therefore, $t^2y_2'' + 5ty_2' + 4y_2 = t^2(-5t^{-4} + 6t^{-4}\ln t) + 5t(t^{-3} - 2t^{-3}\ln t) + 4(t^{-2}\ln t) = (-5+5)t^{-2} + (6-10+4)t^{-2}\ln t = 0$.

19. If $y = (\cos t)\ln\cos t + t\sin t$, then $y' = -(\sin t)\ln\cos t + t\cos t$ and $y'' = -(\cos t)\ln\cos t - t\sin t + \sec t$. Therefore, $y'' + y = -(\cos t)\ln\cos t - t\sin t + \sec t + (\cos t)\ln\cos t + t\sin t = \sec t$.

20. If $y = e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}$, then $y' = 2te^{t^2}\int_0^t e^{-s^2}\,ds + 1 + 2te^{t^2}$. Therefore, $y' - 2ty = 2te^{t^2}\int_0^t e^{-s^2}\,ds + 1 + 2te^{t^2} - 2t(e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}) = 1$.

21. Let $y = e^{rt}$. Then $y' = re^{rt}$. Substituting these terms into the differential equation, we have $y' + 2y = re^{rt} + 2e^{rt} = (r+2)e^{rt} = 0$. This equation implies $r = -2$.

22. Let $y = e^{rt}$. Then $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Substituting these terms into the differential equation, we have $y'' - y = (r^2 - 1)e^{rt} = 0$. This equation implies $r = \pm 1$.

23. Let $y = e^{rt}$. Then $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Substituting these terms into the differential equation, we have $y'' + y' - 6y = (r^2 + r - 6)e^{rt} = 0$. In order for r to satisfy this equation, we need $r^2 + r - 6 = 0$. That is, we need $r = 2, -3$.

24. Let $y = e^{rt}$. Then $y' = re^{rt}$, $y'' = r^2e^{rt}$ and $y''' = r^3e^{rt}$. Substituting these terms into the differential equation, we have $y''' - 3y'' + 2y' = (r^3 - 3r^2 + 2r)e^{rt} = 0$. In order for r to satisfy this equation, we need $r^3 - 3r^2 + 2r = 0$. That is, we need $r = 0, 1, 2$.

25. Let $y = t^r$. Then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these terms into the differential equation, we have $t^2y'' + 4ty' + 2y = t^2(r(r-1)t^{r-2}) + 4t(rt^{r-1}) + 2t^r = (r(r-1) + 4r + 2)t^r = 0$. In order for r to satisfy this equation, we need $r(r-1) + 4r + 2 = 0$. Simplifying this expression, we need $r^2 + 3r + 2 = 0$. The solutions of this equation are $r = -1, -2$.

26. Let $y = t^r$. Then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these terms into the differential equation, we have $t^2y'' - 4ty' + 4y = t^2(r(r-1)t^{r-2}) - 4t(rt^{r-1}) + 4t^r = (r(r-1)-4r+4)t^r = 0$. In order for r to satisfy this equation, we need $r(r-1)-4r+4 = 0$. Simplifying this expression, we need $r^2 - 5r + 4 = 0$. The solutions of this equation are $r = 1, 4$.

27. If $y = Ce^{-2t}$, then $y' = -2Ce^{-2t}$. Thus $y' + 2y = 0$. Also, $1 = y(0) = Ce^0 = C$.

28. If $y = Ce^{\cos t}$, then $y' = (-\sin t)Ce^{\cos t}$. Thus $y' + (\sin t)y = 0$. Also, $1 = y(\pi) = Ce^{\cos \pi} = Ce^{-1}$ and then $C = e$.

29. If $y = \sin t/t^2 + C/t^2$, then $y' = \cos t/t^2 - 2\sin t/t^3 - 2C/t^3$. Thus $y' + (2/t)y = \cos t/t^2$. Also, $1/2 = y(1) = \sin 1 + C$ and then $C = 1/2 - \sin 1$.

30. If $y = 1 - 1/t + Ce^{-t}/t$, then $y' = 1/t^2 - Ce^{-t}/t - Ce^{-t}/t^2$. Thus $ty' + (t+1)y = t$. Also, $1 = y(\ln 2) = 1 - 1/\ln 2 + Ce^{-\ln 2}/\ln 2$ and then $C = 2$.

31. If $y = e^{-t^2/4}\int_0^t e^{s^2/4}\,ds + Ce^{-t^2/4}$, then $y' = -(t/2)e^{-t^2/4}\int_0^t e^{s^2/4}\,ds + e^{-t^2/4}e^{t^2/4} - (t/2)Ce^{-t^2/4}$. Thus $2y' + ty = 2$. Also, $1 = y(0) = Ce^0 = C$.

32. If $\phi(t) = c_1e^{-t} + c_2e^{-2t}$, then $\phi'(t) = -c_1e^{-t} - 2c_2e^{-2t}$ and $\phi''(t) = c_1e^{-t} + 4c_2e^{-2t}$. Thus $\phi'' + 3\phi' + 2\phi = 0$.

(a) $-1 = y(0) = c_1 + c_2$ and $4 = y'(0) = -c_1 - 2c_2$, thus $c_1 = 2$ and $c_2 = -3$.

(b) $2 = y(0) = c_1 + c_2$ and $0 = y'(0) = -c_1 - 2c_2$, thus $c_1 = 4$ and $c_2 = -2$.

33. If $\phi(t) = c_1e^t + c_2te^t$, then $\phi'(t) = c_1e^t + c_2e^t + c_2te^t$ and $\phi''(t) = c_1e^t + 2c_2e^t + c_2te^t$. Thus $\phi'' - 2\phi' + \phi = 0$.

(a) $3 = y(0) = c_1$ and $1 = y'(0) = c_1 + c_2$, thus $c_2 = -2$.

(b) $1 = y(0) = c_1$ and $-4 = y'(0) = c_1 + c_2$, thus $c_2 = -5$.

34. If $\phi(t) = c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t$, then $\phi'(t) = -c_1e^{-t}\cos 2t - 2c_1e^{-t}\sin 2t - c_2e^{-t}\sin 2t + 2c_2e^{-t}\cos 2t$ and $\phi''(t) = c_1e^{-t}\cos 2t + 2c_1e^{-t}\sin 2t + 2c_1e^{-t}\sin 2t - 4c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t - 2c_2e^{-t}\cos 2t - 2c_2e^{-t}\cos 2t - 4c_2e^{-t}\sin 2t$. Thus $\phi'' + 2\phi' + 5\phi = 0$.

(a) $1 = y(0) = c_1$ and $1 = y'(0) = -c_1 + 2c_2$, thus $c_2 = 1$.

(b) $2 = y(0) = c_1$ and $5 = y'(0) = -c_1 + 2c_2$, thus $c_2 = 7/2$.