PREFACE

This *Complete Solutions Manual* contains solutions to all of the exercises in my textbook *Applied Calculus for the Managerial, Life, and Social Sciences: A Brief Approach, Tenth Edition*. The corresponding *Student Solutions Manual* contains solutions to the odd-numbered exercises and the even-numbered exercises in the "Before Moving On" quizzes. It also offers problem-solving tips for many sections.

I would like to thank Andy Bulman-Fleming for checking the accuracy of the answers to the new exercises in this edition of the text, rendering the art, and typesetting this manual. I also wish to thank my development editor Laura Wheel and my editor Rita Lombard of Cengage Learning for their help and support in bringing this supplement to market.

Please submit any errors in the solutions manual or suggestions for improvements to me in care of the publisher: Math Editorial, Cengage Learning, 20 Channel Center Street, Boston, MA, 02210.

Soo T. Tan

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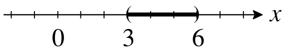
1 PRELIMINARIES

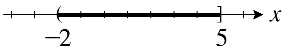
1.1 Precalculus Review I

Exercises page 13

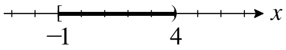
1. The interval (3, 6) is shown on the number line below. Note that this is an open interval indicated by “(” and “)”.



2. The interval $(-2, 5]$ is shown on the number line below.

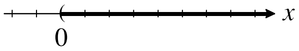


3. The interval $[-1, 4)$ is shown on the number line below. Note that this is a half-open interval indicated by “[’’ (closed) and “)”(open).

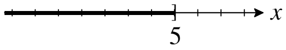


4. The closed interval $\left[-\frac{6}{5}, -\frac{1}{2}\right]$ is shown on the number line below.

5. The infinite interval $(0, \infty)$ is shown on the number line below.



6. The infinite interval $(-\infty, 5]$ is shown on the number line below.



7. $27^{2/3} = \left(3^3\right)^{2/3} = 3^2 = 9.$

8. $8^{-4/3} = \left(\frac{1}{8^{4/3}}\right) = \frac{1}{2^4} = \frac{1}{16}.$

9. $\left(\frac{1}{\sqrt{3}}\right)^0 = 1$. Recall that any number raised to the zeroth power is 1.

10. $\left(7^{1/2}\right)^4 = 7^{4/2} = 7^2 = 49.$

11. $\left[\left(\frac{1}{8}\right)^{1/3}\right]^{-2} = \left(\frac{1}{2}\right)^{-2} = \left(2^2\right) = 4.$

12. $\left[\left(-\frac{1}{3}\right)^2\right]^{-3} = \left(\frac{1}{9}\right)^{-3} = (9)^3 = 729.$

13. $\left(\frac{7^{-5} \cdot 7^2}{7^{-2}}\right)^{-1} = \left(7^{-5+2+2}\right)^{-1} = \left(7^{-1}\right)^{-1} = 7^1 = 7.$

14. $\left(\frac{9}{16}\right)^{-1/2} = \left(\frac{16}{9}\right)^{1/2} = \frac{4}{3}.$

15. $\left(125^{2/3}\right)^{-1/2} = 125^{(2/3)(-1/2)} = 125^{-1/3} = \frac{1}{125^{1/3}}$

$= \frac{1}{5}.$

16. $\sqrt[3]{2^6} = \left(2^6\right)^{1/3} = 2^{6(1/3)} = 2^2 = 4.$

17. $\frac{\sqrt{32}}{\sqrt{8}} = \sqrt{\frac{32}{8}} = \sqrt{4} = 2.$

18. $\sqrt[3]{\frac{-8}{27}} = \frac{\sqrt[3]{-8}}{\sqrt[3]{27}} = -\frac{2}{3}.$

19. $\dfrac{16^{5/8}16^{1/2}}{16^{7/8}} = 16^{(5/8)+(1/2)-(7/8)} = 16^{1/4} = 2.$

20. $\left(\dfrac{9^{-3}\cdot 9^{5}}{9^{-2}}\right)^{-1/2} = 9^{(-3+5+2)(-1/2)} = 9^{4(-1/2)} = \dfrac{1}{81}.$

21. $16^{1/4}\cdot 8^{-1/3} = 2\cdot\left(\frac{1}{8}\right)^{1/3} = 2\cdot\frac{1}{2} = 1.$

22.
$$\begin{aligned}\frac{6^{2.5}\cdot 6^{-1.9}}{6^{-1.4}} &= 6^{2.5-1.9-(-1.4)} = 6^{2.5-1.9+1.4} = 6^{2}\\ &= 36.\end{aligned}$$

23. True.

24. True. $3^2\times 2^2 = (3\times 2)^2 = 6^2 = 36.$

25. False. $x^3\times 2x^2 = 2x^{3+2} = 2x^5 \neq 2x^6.$

26. False. $3^3+3 = 27+3 = 30 \neq 3^4.$

27. False. $\dfrac{2^{4x}}{1^{3x}} = \dfrac{2^{4x}}{1} = 2^{4x}.$

28. True. $\left(2^2\times 3^2\right)^2 = (4\times 9)^2 = 36^2 = \left(6^2\right)^2 = 6^4.$

29. False. $\dfrac{1}{4^{-3}} = 4^3 = 64.$

30. True. $\dfrac{4^{3/2}}{2^4} = \dfrac{8}{16} = \dfrac{1}{2}.$

31. False. $\left(1.2^{1/2}\right)^{-1/2} = (1.2)^{-1/4} \neq 1.$

32. True.
$$5^{2/3}\times 25^{2/3} = 5^{2/3}\left(5^2\right)^{2/3} = 5^{2/3}\times 5^{4/3} = 5^2 = 25.$$

33. $(xy)^{-2} = \dfrac{1}{(xy)^2}.$

34. $3s^{1/3}\cdot s^{-7/3} = 3s^{(1/3)-(7/3)} = 3s^{-6/3} = 3s^{-2} = \dfrac{3}{s^2}.$

35. $\dfrac{x^{-1/3}}{x^{1/2}} = x^{(-1/3)-(1/2)} = x^{-5/6} = \dfrac{1}{x^{5/6}}.$

36.
$$\begin{aligned}\sqrt{x^{-1}}\cdot\sqrt{9x^{-3}} &= x^{-1/2}\cdot 3x^{-3/2} = 3x^{(-1/2)+(-3/2)}\\ &= 3x^{-2} = \frac{3}{x^2}.\end{aligned}$$

37. $12^0\,(s+t)^{-3} = 1\cdot\dfrac{1}{(s+t)^3} = \dfrac{1}{(s+t)^3}.$

38.
$$\begin{aligned}(x-y)\left(x^{-1}+y^{-1}\right) &= (x-y)\left(\frac{1}{x}+\frac{1}{y}\right)\\ &= (x-y)\left(\frac{y+x}{xy}\right) = \frac{(x-y)(x+y)}{xy} = \frac{x^2-y^2}{xy}.\end{aligned}$$

39. $\dfrac{x^{7/3}}{x^{-2}} = x^{(7/3)+2} = x^{(7/3)+(6/3)} = x^{13/3}.$

40. $\left(49x^{-2}\right)^{-1/2} = (49)^{-1/2}\,x^{(-2)(-1/2)} = \frac{1}{7}x.$

41. $\left(x^2y^{-3}\right)\left(x^{-5}y^3\right) = x^{2-5}y^{-3+3} = x^{-3}y^0 = x^{-3} = \dfrac{1}{x^3}.$

42. $\dfrac{5x^6y^3}{2x^2y^7} = \frac{5}{2}x^{6-2}y^{3-7} = \frac{5}{2}x^4y^{-4} = \dfrac{5x^4}{2y^4}.$

43. $\dfrac{x^{3/4}}{x^{-1/4}} = x^{(3/4)-(-1/4)} = x^{4/4} = x.$

44. $\left(\dfrac{x^3y^2}{z^2}\right)^2 = \dfrac{x^{3\cdot 2}y^{2\cdot 2}}{z^{2(2)}} = \dfrac{x^6y^4}{z^4}.$

45.
$$\begin{aligned}\left(\frac{x^3}{-27y^{-6}}\right)^{-2/3} &= x^{3(-2/3)}\left(-\frac{1}{27}\right)^{-2/3}y^{6(-2/3)}\\ &= x^{-2}\left(-\frac{1}{3}\right)^{-2}y^{-4} = \frac{9}{x^2y^4}.\end{aligned}$$

46. $\left(\dfrac{e^x}{e^{x-2}}\right)^{-1/2} = e^{[x-(x-2)](-1/2)} = e^{-1} = \dfrac{1}{e}.$

47. $\left(\dfrac{x^{-3}}{y^{-2}}\right)^2\left(\dfrac{y}{x}\right)^4 = \dfrac{x^{-3\cdot 2}y^4}{y^{-2\cdot 2}x^4} = \dfrac{y^{4+4}}{x^{4+6}} = \dfrac{y^8}{x^{10}}.$

48. $\dfrac{(r^n)^4}{r^{5-2n}} = r^{4n-(5-2n)} = r^{4n+2n-5} = r^{6n-5}.$

49. $\sqrt[3]{x^{-2}} \cdot \sqrt{4x^5} = x^{-2/3} \cdot 4^{1/2} \cdot x^{5/2} = x^{(-2/3)+(5/2)} \cdot 2$
$= 2x^{11/6}$.

50. $\sqrt{81x^6y^{-4}} = (81)^{1/2} \cdot x^{6/2} \cdot y^{-4/2} = \dfrac{9x^3}{y^2}$.

51. $-\sqrt[4]{16x^4y^8} = -\left(16^{1/4} \cdot x^{4/4} \cdot y^{8/4}\right) = -2xy^2$.

52. $\sqrt[3]{x^{3a+b}} = x^{(3a+b)(1/3)} = x^{a+(b/3)}$.

53. $\sqrt[6]{64x^8y^3} = 64^{1/6} \cdot x^{8/6}y^{3/6} = 2x^{4/3}y^{1/2}$.

54. $\sqrt[3]{27r^6} \cdot \sqrt{s^2t^4} = 27^{1/3}\left(r^6\right)^{1/3}\left(s^2\right)^{1/2}\left(t^4\right)^{1/2}$
$= 3r^2st^2$.

55. $2^{3/2} = 2\left(2^{1/2}\right) \approx 2\,(1.414) = 2.828$.

56. $8^{1/2} = \left(2^3\right)^{1/2} = 2^{3/2} = 2\left(2^{1/2}\right) \approx 2.828$.

57. $9^{3/4} = \left(3^2\right)^{3/4} = 3^{6/4} = 3^{3/2} = 3 \cdot 3^{1/2}$
$\approx 3\,(1.732) = 5.196$.

58. $6^{1/2} = (2 \cdot 3)^{1/2} = 2^{1/2} \cdot 3^{1/2}$
$\approx (1.414)\,(1.732) \approx 2.449$.

59. $10^{3/2} = 10^{1/2} \cdot 10 \approx (3.162)\,(10) = 31.62$.

60. $1000^{3/2} = \left(10^3\right)^{3/2} = 10^{9/2} = 10^4 \times 10^{1/2}$
$\approx (10000)\,(3.162) = 31{,}620$.

61. $10^{2.5} = 10^2 \cdot 10^{1/2} \approx 100\,(3.162) = 316.2$.

62. $(0.0001)^{-1/3} = \left(10^{-4}\right)^{-1/3} = 10^{4/3} = 10 \cdot 10^{1/3}$
$\approx 10\,(2.154) = 21.54$.

63. $\dfrac{3}{2\sqrt{x}} \cdot \dfrac{\sqrt{x}}{\sqrt{x}} = \dfrac{3\sqrt{x}}{2x}$.

64. $\dfrac{3}{\sqrt{xy}} \cdot \dfrac{\sqrt{xy}}{\sqrt{xy}} = \dfrac{3\sqrt{xy}}{xy}$.

65. $\dfrac{2y}{\sqrt{3y}} \cdot \dfrac{\sqrt{3y}}{\sqrt{3y}} = \dfrac{2y\sqrt{3y}}{3y} = \dfrac{2\sqrt{3y}}{3}$.

66. $\dfrac{5x^2}{\sqrt{3x}} \cdot \dfrac{\sqrt{3x}}{\sqrt{3x}} = \dfrac{5x^2\sqrt{3x}}{3x} = \dfrac{5x\sqrt{3x}}{3}$.

67. $\dfrac{1}{\sqrt[3]{x}} \cdot \dfrac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \dfrac{\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \dfrac{\sqrt[3]{x^2}}{x}$.

68. $\sqrt{\dfrac{2x}{y}} = \dfrac{\sqrt{2x}}{\sqrt{y}} \cdot \dfrac{\sqrt{y}}{\sqrt{y}} = \dfrac{\sqrt{2xy}}{y}$.

69. $\dfrac{2\sqrt{x}}{3} \cdot \dfrac{\sqrt{x}}{\sqrt{x}} = \dfrac{2x}{3\sqrt{x}}$.

70. $\dfrac{\sqrt[3]{x}}{24} \cdot \dfrac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \dfrac{x}{24\sqrt[3]{x^2}}$.

71. $\sqrt{\dfrac{2y}{x}} = \dfrac{\sqrt{2y}}{\sqrt{x}} \cdot \dfrac{\sqrt{2y}}{\sqrt{2y}} = \dfrac{2y}{\sqrt{2xy}}$.

72. $\sqrt[3]{\dfrac{2x}{3y}} = \dfrac{\sqrt[3]{2x}}{\sqrt[3]{3y}} \cdot \dfrac{\sqrt[3]{(2x)^2}}{\sqrt[3]{(2x)^2}} = \dfrac{2x}{\sqrt[3]{12x^2y}}$.

73. $\dfrac{\sqrt[3]{x^2z}}{y} \cdot \dfrac{\sqrt[3]{xz^2}}{\sqrt[3]{xz^2}} = \dfrac{\sqrt[3]{x^3z^3}}{y\sqrt[3]{xz^2}} = \dfrac{xz}{y\sqrt[3]{xz^2}}$.

74. $\dfrac{\sqrt[3]{x^2y}}{2x} \cdot \dfrac{\sqrt[3]{xy^2}}{\sqrt[3]{xy^2}} = \dfrac{xy}{2x\sqrt[3]{xy^2}} = \dfrac{y}{2\sqrt[3]{xy^2}}$.

75. $\left(7x^2 - 2x + 5\right) + \left(2x^2 + 5x - 4\right) = 7x^2 - 2x + 5 + 2x^2 + 5x - 4 = 9x^2 + 3x + 1$.

76. $\left(3x^2 + 5xy + 2y\right) + \left(4 - 3xy - 2x^2\right) = 3x^2 + 5xy + 2y + 4 - 3xy - 2x^2 = x^2 + 2xy + 2y + 4$.

77. $\left(5y^2 - 2y + 1\right) - \left(y^2 - 3y - 7\right) = 5y^2 - 2y + 1 - y^2 + 3y + 7 = 4y^2 + y + 8$.

78. $3\,(2a - b) - 4\,(b - 2a) = 6a - 3b - 4b + 8a = 14a - 7b = 7\,(2a - b)$.

79. $x - \{2x - [-x - (1 - x)]\} = x - \{2x - [-x - 1 + x]\} = x - (2x + 1) = x - 2x - 1 = -x - 1.$

80. $$\begin{aligned} 3x^2 - \left\{x^2 + 1 - x\left[x - (2x - 1)\right]\right\} + 2 &= 3x^2 - \left[x^2 + 1 - x\,(x - 2x + 1)\right] + 2 \\ &= 3x^2 - \left[x^2 + 1 - x\,(-x + 1)\right] + 2 = 3x^2 - \left(x^2 + 1 + x^2 - x\right) + 2 \\ &= 3x^2 - \left(2x^2 - x + 1\right) + 2 = x^2 - 1 + x + 2 = x^2 + x + 1. \end{aligned}$$

81. $\left(\frac{1}{3} - 1 + e\right) - \left(-\frac{1}{3} - 1 + e^{-1}\right) = \frac{1}{3} - 1 + e + \frac{1}{3} + 1 - \frac{1}{e} = \frac{2}{3} + e - \frac{1}{e} = \frac{3e^2 + 2e - 3}{3e}.$

82. $-\frac{3}{4}y - \frac{1}{4}x + 100 + \frac{1}{2}x + \frac{1}{4}y - 120 = -\frac{3}{4}y + \frac{1}{4}y - \frac{1}{4}x + \frac{1}{2}x + 100 - 120 = -\frac{1}{2}y + \frac{1}{4}x - 20.$

83. $3\sqrt{8} + 8 - 2\sqrt{y} + \frac{1}{2}\sqrt{x} - \frac{3}{4}\sqrt{y} = 3\sqrt{8} + 8 + \frac{1}{2}\sqrt{x} - \frac{11}{4}\sqrt{y} = 6\sqrt{2} + 8 + \frac{1}{2}\sqrt{x} - \frac{11}{4}\sqrt{y}.$

84. $$\begin{aligned} \tfrac{8}{9}x^2 + \tfrac{2}{3}x + \tfrac{16}{3}x^2 - \tfrac{16}{3}x - 2x + 2 &= \frac{8x^2 + 6x + 48x^2 - 48x - 18x + 18}{9} = \frac{56x^2 - 60x + 18}{9} \\ &= \tfrac{2}{9}\left(28x^2 - 30x + 9\right). \end{aligned}$$

85. $(x + 8)(x - 2) = x(x - 2) + 8(x - 2) = x^2 - 2x + 8x - 16 = x^2 + 6x - 16.$

86. $(5x + 2)(3x - 4) = 5x(3x - 4) + 2(3x - 4) = 15x^2 - 20x + 6x - 8 = 15x^2 - 14x - 8.$

87. $(a + 5)^2 = (a + 5)(a + 5) = a(a + 5) + 5(a + 5) = a^2 + 5a + 5a + 25 = a^2 + 10a + 25.$

88. $$\begin{aligned} (3a - 4b)^2 &= (3a - 4b)(3a - 4b) = 3a(3a - 4b) - 4b(3a - 4b) = 9a^2 - 12ab - 12ab + 16b^2 \\ &= 9a^2 - 24ab + 16b^2. \end{aligned}$$

89. $(x + 2y)^2 = (x + 2y)(x + 2y) = x(x + 2y) + 2y(x + 2y) = x^2 + 2xy + 2yx + 4y^2 = x^2 + 4xy + 4y^2.$

90. $(6 - 3x)^2 = (6 - 3x)(6 - 3x) = 6(6 - 3x) - 3x(6 - 3x) = 36 - 18x - 18x + 9x^2 = 36 - 36x + 9x^2.$

91. $(2x + y)(2x - y) = 2x(2x - y) + y(2x - y) = 4x^2 - 2xy + 2xy - y^2 = 4x^2 - y^2.$

92. $(3x + 2)(2 - 3x) = 3x(2 - 3x) + 2(2 - 3x) = 6x - 9x^2 + 4 - 6x = -9x^2 + 4.$

93. $\left(2x^2 - 1\right)\left(3x^2\right) + \left(x^2 + 3\right)(4x) = 6x^4 - 3x^2 + 4x^3 + 12x = 6x^4 + 4x^3 - 3x^2 + 12x = x\left(6x^3 + 4x^2 - 3x + 12\right).$

94. $\left(x^2 - 1\right)(2x) - x^2(2x) = 2x^3 - 2x - 2x^3 = -2x.$

95. $6x\left(\frac{1}{2}\right)\left(2x^2 + 3\right)^{-1/2}(4x) + 6\left(2x^2 + 3\right)^{1/2} = 3\left(2x^2 + 3\right)^{-1/2}\left[x(4x) + 2\left(2x^2 + 3\right)\right] = \frac{6\left(4x^2 + 3\right)}{\left(2x^2 + 3\right)^{1/2}}.$

96. $\left(x^{1/2} + 1\right)\left(\frac{1}{2}x^{-1/2}\right) - \left(x^{1/2} - 1\right)\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2}x^{-1/2}\left[\left(x^{1/2} + 1\right) - \left(x^{1/2} - 1\right)\right] = \frac{1}{2}x^{-1/2}(2) = \frac{1}{x^{1/2}} = \frac{1}{\sqrt{x}}.$

97. $100\left(-10te^{-0.1t} - 100e^{-0.1t}\right) = -1000(10 + t)e^{-0.1t}.$

98. $$\begin{aligned} 2\left(t + \sqrt{t}\right)^2 - 2t^2 &= 2\left(t + \sqrt{t}\right)\left(t + \sqrt{t}\right) - 2t^2 = 2\left(t^2 + 2t\sqrt{t} + t\right) - 2t^2 = 2t^2 + 4t\sqrt{t} + 2t - 2t^2 \\ &= 4t\sqrt{t} + 2t = 2t\left(2\sqrt{t} + 1\right). \end{aligned}$$

99. $4x^5 - 12x^4 - 6x^3 = 2x^3\left(2x^2 - 6x - 3\right)$.

100. $4x^2y^2z - 2x^5y^2 + 6x^3y^2z^2 = 2x^2y^2\left(2z - x^3 + 3xz^2\right)$.

101. $7a^4 - 42a^2b^2 + 49a^3b = 7a^2\left(a^2 + 7ab - 6b^2\right)$.

102. $3x^{2/3} - 2x^{1/3} = x^{1/3}\left(3x^{1/3} - 2\right)$.

103. $e^{-x} - xe^{-x} = e^{-x}\left(1 - x\right)$.

104. $2ye^{xy^2} + 2xy^3e^{xy^2} = 2ye^{xy^2}\left(1 + xy^2\right)$.

105. $2x^{-5/2} - \frac{3}{2}x^{-3/2} = \frac{1}{2}x^{-5/2}\left(4 - 3x\right)$.

106. $\frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) = \frac{1}{2}\cdot\frac{2}{3}u^{1/2}\left(u - 3\right) = \frac{1}{3}u^{1/2}\left(u - 3\right)$.

107. $6ac + 3bc - 4ad - 2bd = 3c\left(2a + b\right) - 2d\left(2a + b\right) = \left(2a + b\right)\left(3c - 2d\right)$.

108. $3x^3 - x^2 + 3x - 1 = x^2\left(3x - 1\right) + 1\left(3x - 1\right) = \left(x^2 + 1\right)\left(3x - 1\right)$.

109. $4a^2 - b^2 = \left(2a + b\right)\left(2a - b\right)$, a difference of two squares.

110. $12x^2 - 3y^2 = 3\left(4x^2 - y^2\right) = 3\left(2x + y\right)\left(2x - y\right)$.

111. $10 - 14x - 12x^2 = -2\left(6x^2 + 7x - 5\right) = -2\left(3x + 5\right)\left(2x - 1\right)$.

112. $x^2 - 2x - 15 = \left(x - 5\right)\left(x + 3\right)$.

113. $3x^2 - 6x - 24 = 3\left(x^2 - 2x - 8\right) = 3\left(x - 4\right)\left(x + 2\right)$.

114. $3x^2 - 4x - 4 = \left(3x + 2\right)\left(x - 2\right)$.

115. $12x^2 - 2x - 30 = 2\left(6x^2 - x - 15\right) = 2\left(3x - 5\right)\left(2x + 3\right)$.

116. $\left(x + y\right)^2 - 1 = \left(x + y - 1\right)\left(x + y + 1\right)$.

117. $9x^2 - 16y^2 = \left(3x\right)^2 - \left(4y\right)^2 = \left(3x - 4y\right)\left(3x + 4y\right)$.

118. $8a^2 - 2ab - 6b^2 = 2\left(4a^2 - ab - 3b^2\right) = 2\left(a - b\right)\left(4a + 3b\right)$.

119. $x^6 + 125 = \left(x^2\right)^3 + \left(5\right)^3 = \left(x^2 + 5\right)\left(x^4 - 5x^2 + 25\right)$.

120. $x^3 - 27 = x^3 - 3^3 = \left(x - 3\right)\left(x^2 + 3x + 9\right)$.

121. $\left(x^2 + y^2\right)x - xy\left(2y\right) = x^3 + xy^2 - 2xy^2 = x^3 - xy^2$.

122. $2kr\left(R - r\right) - kr^2 = 2kRr - 2kr^2 - kr^2 = 2kRr - 3kr^2 = kr\left(2R - 3r\right)$.

123. $$\begin{aligned}2(x-1)(2x+2)^3[4(x-1)+(2x+2)] &= 2(x-1)(2x+2)^3(4x-4+2x+2)\\ &= 2(x-1)(2x+2)^3(6x-2) = 4(x-1)(3x-1)(2x+2)^3\\ &= 32(x-1)(3x-1)(x+1)^3.\end{aligned}$$

124. $5x^2\left(3x^2+1\right)^4(6x)+\left(3x^2+1\right)^5(2x)=(2x)\left(3x^2+1\right)^4\left[15x^2+\left(3x^2+1\right)\right]=2x\left(3x^2+1\right)^4\left(18x^2+1\right).$

125. $$\begin{aligned}4(x-1)^2(2x+2)^3(2)+(2x+2)^4(2)(x-1) &= 2(x-1)(2x+2)^3[4(x-1)+(2x+2)]\\ &= 2(x-1)(2x+2)^3(6x-2) = 4(x-1)(3x-1)(2x+2)^3\\ &= 32(x-1)(3x-1)(x+1)^3.\end{aligned}$$

126. $$\begin{aligned}\left(x^2+1\right)\left(4x^3-3x^2+2x\right)-\left(x^4-x^3+x^2\right)(2x) &= 4x^5-3x^4+2x^3+4x^3-3x^2+2x-2x^5+2x^4-2x^3\\ &= 2x^5-x^4+4x^3-3x^2+2x.\end{aligned}$$

127. $\left(x^2+2\right)^2\left[5\left(x^2+2\right)^2-3\right](2x)=\left(x^2+2\right)^2\left[5\left(x^4+4x^2+4\right)-3\right](2x)=(2x)\left(x^2+2\right)^2\left(5x^4+20x^2+17\right).$

128. $$\begin{aligned}\left(x^2-4\right)\left(x^2+4\right)(2x+8)-\left(x^2+8x-4\right)\left(4x^3\right) &= \left(x^4-16\right)(2x+8)-4x^5-32x^4+16x^3\\ &= 2x^5+8x^4-32x-128-4x^5-32x^4+16x^3 = -2x^5-24x^4+16x^3-32x-128\\ &= -2\left(x^5+12x^4-8x^3+16x+64\right).\end{aligned}$$

129. We factor the left-hand side of $x^2+x-12=0$ to obtain $(x+4)(x-3)=0$, so $x=-4$ or $x=3$. We conclude that the roots are $x=-4$ and $x=3$.

130. We factor the left-hand side of $3x^2-x-4=0$ to obtain $(3x-4)(x+1)=0$. Thus, $3x=4$ or $x=-1$, and we conclude that the roots are $x=\frac{4}{3}$ and $x=-1$.

131. $4t^2+2t-2=(2t-1)(2t+2)=0$. Thus, the roots are $t=\frac{1}{2}$ and $t=-1$.

132. $-6x^2+x+12=(3x+4)(-2x+3)=0$. Thus, $x=-\frac{4}{3}$ and $x=\frac{3}{2}$ are the roots of the equation.

133. $\frac{1}{4}x^2-x+1=\left(\frac{1}{2}x-1\right)\left(\frac{1}{2}x-1\right)=0$. Thus $\frac{1}{2}x=1$, and so $x=2$ is a double root of the equation.

134. $\frac{1}{2}a^2+a-12=a^2+2a-24=(a+6)(a-4)=0$. Thus, $a=-6$ and $a=4$ are the roots of the equation.

135. We use the quadratic formula to solve the equation $4x^2+5x-6=0$. In this case, $a=4$, $b=5$, and $c=-6$. Therefore, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=\dfrac{-5\pm\sqrt{5^2-4(4)(-6)}}{2(4)}=\dfrac{-5\pm\sqrt{121}}{8}=\dfrac{-5\pm 11}{8}$. Thus, $x=-\frac{16}{8}=-2$ and $x=\frac{6}{8}=\frac{3}{4}$ are the roots of the equation.

136. We use the quadratic formula to solve the equation $3x^2-4x+1=0$. Here $a=3$, $b=-4$, and $c=1$, so $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=\dfrac{-(-4)\pm\sqrt{(-4)^2-4(3)(1)}}{2(3)}=\dfrac{4\pm\sqrt{4}}{6}$. Thus, $x=\frac{6}{6}=1$ and $x=\frac{2}{6}=\frac{1}{3}$ are the roots of the equation.

137. We use the quadratic formula to solve the equation $8x^2 - 8x - 3 = 0$. Here $a = 8$, $b = -8$, and $c = -3$, so $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-(-8) \pm \sqrt{(-8)^2 - 4(8)(-3)}}{2(8)} = \dfrac{8 \pm \sqrt{160}}{16} = \dfrac{8 \pm 4\sqrt{10}}{16} = \dfrac{2 \pm \sqrt{10}}{4}$. Thus, $x = \frac{1}{2} + \frac{1}{4}\sqrt{10}$ and $x = \frac{1}{2} - \frac{1}{4}\sqrt{10}$ are the roots of the equation.

138. We use the quadratic formula to solve the equation $x^2 - 6x + 6 = 0$. Here $a = 1$, $b = -6$, and $c = 6$. Therefore, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(6)}}{2(1)} = \dfrac{6 \pm 2\sqrt{3}}{2} = 3 \pm \sqrt{3}$. Thus, the roots are $3 + \sqrt{3}$ and $3 - \sqrt{3}$.

139. We use the quadratic formula to solve $2x^2 + 4x - 3 = 0$. Here $a = 2$, $b = 4$, and $c = -3$, so $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-4 \pm \sqrt{4^2 - 4(2)(-3)}}{2(2)} = \dfrac{-4 \pm \sqrt{40}}{4} = \dfrac{-4 \pm 2\sqrt{10}}{4} = \dfrac{-2 \pm \sqrt{10}}{2}$. Thus, $x = -1 + \frac{1}{2}\sqrt{10}$ and $x = -1 - \frac{1}{2}\sqrt{10}$ are the roots of the equation.

140. We use the quadratic formula to solve the equation $2x^2 + 7x - 15 = 0$. Then $a = 2$, $b = 7$, and $c = -15$. Therefore, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-7 \pm \sqrt{7^2 - 4(2)(-15)}}{2(2)} = \dfrac{-7 \pm \sqrt{169}}{4} = \dfrac{-7 \pm 13}{4}$. We conclude that $x = \frac{3}{2}$ and $x = -5$ are the roots of the equation.

141. The total revenue is given by $\left(0.2t^2 + 150t\right) + \left(0.5t^2 + 200t\right) = 0.7t^2 + 350t$ thousand dollars t months from now, where $0 \leq t \leq 12$.

142. In month t, the revenue of the second gas station will exceed that of the first gas station by $\left(0.5t^2 + 200t\right) - \left(0.2t^2 + 150t\right) = 0.3t^2 + 50t$ thousand dollars, where $0 < t \leq 12$.

143. **a.** $f(30{,}000) = \left(5.6 \times 10^{11}\right)(30{,}000)^{-1.5} \approx 107{,}772$, or 107,772 families.

b. $f(60{,}000) = \left(5.6 \times 10^{11}\right)(60{,}000)^{-1.5} \approx 38{,}103$, or 38,103 families.

c. $f(150{,}000) = \left(5.6 \times 10^{11}\right)(150{,}000)^{-1.5} \approx 9639$, or 9639 families.

144. $-t^3 + 6t^2 + 15t = -t\left(t^2 - 6t - 15\right)$.

145. $8000x - 100x^2 = 100x(80 - x)$.

146. True. The two real roots are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

147. True. If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is not a real number.

148. True, because $(a + b)(b - a) = b^2 - a^2$.

1.2 Precalculus Review II

Exercises page 23

1. $\dfrac{x^2+x-2}{x^2-4}=\dfrac{(x+2)(x-1)}{(x+2)(x-2)}=\dfrac{x-1}{x-2}.$

2. $\dfrac{2a^2-3ab-9b^2}{2ab^2+3b^3}=\dfrac{(2a+3b)(a-3b)}{b^2(2a+3b)}=\dfrac{a-3b}{b^2}.$

3. $\dfrac{12t^2+12t+3}{4t^2-1}=\dfrac{3\left(4t^2+4t+1\right)}{4t^2-1}=\dfrac{3(2t+1)(2t+1)}{(2t+1)(2t-1)}=\dfrac{3(2t+1)}{2t-1}.$

4. $\dfrac{x^3+2x^2-3x}{-2x^2-x+3}=\dfrac{x\left(x^2+2x-3\right)}{-\left(2x^2+x-3\right)}=\dfrac{x(x+3)(x-1)}{-(2x+3)(x-1)}=-\dfrac{x(x+3)}{2x+3}.$

5. $\dfrac{(4x-1)(3)-(3x+1)(4)}{(4x-1)^2}=\dfrac{12x-3-12x-4}{(4x-1)^2}=-\dfrac{7}{(4x-1)^2}.$

6. $\dfrac{\left(1+x^2\right)^2(2)-2x(2)\left(1+x^2\right)(2x)}{\left(1+x^2\right)^4}=\dfrac{\left(1+x^2\right)(2)\left(1+x^2-4x^2\right)}{\left(1+x^2\right)^4}=\dfrac{\left(1+x^2\right)(2)\left(-3x^2+1\right)}{\left(1+x^2\right)^4}=\dfrac{2\left(1-3x^2\right)}{\left(1+x^2\right)^3}.$

7. $\dfrac{2a^2-2b^2}{b-a}\cdot\dfrac{4a+4b}{a^2+2ab+b^2}=\dfrac{2(a+b)(a-b)4(a+b)}{-(a-b)(a+b)(a+b)}=-8.$

8. $\dfrac{x^2-6x+9}{x^2-x-6}\cdot\dfrac{3x+6}{2x^2-7x+3}=\dfrac{3(x-3)^2(x+2)}{(x-3)(x+2)(2x-1)(x-3)}=\dfrac{3}{2x-1}.$

9. $\dfrac{3x^2+2x-1}{2x+6}\div\dfrac{x^2-1}{x^2+2x-3}=\dfrac{(3x-1)(x+1)}{2(x+3)}\cdot\dfrac{(x+3)(x-1)}{(x+1)(x-1)}=\dfrac{3x-1}{2}.$

10. $\dfrac{3x^2-4xy-4y^2}{x^2y}\div\dfrac{(2y-x)^2}{x^3y}=\dfrac{(3x+2y)(x-2y)}{x^2y}\cdot\dfrac{x^3y}{(2y-x)(2y-x)}=\dfrac{x(3x+2y)}{x-2y}.$

11. $\dfrac{58}{3(3t+2)}+\dfrac{1}{3}=\dfrac{58+3t+2}{3(3t+2)}=\dfrac{3t+60}{3(3t+2)}=\dfrac{t+20}{3t+2}.$

12. $\dfrac{a+1}{3a}+\dfrac{b-2}{5b}=\dfrac{5b(a+1)+3a(b-2)}{15ab}=\dfrac{5ab+5b+3ab-6a}{15ab}=\dfrac{-6a+8ab+5b}{15ab}.$

13. $$\dfrac{2x}{2x-1}-\dfrac{3x}{2x+5}=\dfrac{2x(2x+5)-3x(2x-1)}{(2x-1)(2x+5)}=\dfrac{4x^2+10x-6x^2+3x}{(2x-1)(2x+5)}=\dfrac{-2x^2+13x}{(2x-1)(2x+5)}$$
$$=-\dfrac{x(2x-13)}{(2x-1)(2x+5)}.$$

14. $\dfrac{-xe^x}{x+1}+e^x=\dfrac{-xe^x+(x+1)e^x}{x+1}=\dfrac{-xe^x+xe^x+e^x}{x+1}=\dfrac{e^x}{x+1}.$

15. $\dfrac{4}{x^2-9}-\dfrac{5}{x^2-6x+9}=\dfrac{4}{(x+3)(x-3)}-\dfrac{5}{(x-3)^2}=\dfrac{4(x-3)-5(x+3)}{(x-3)^2(x+3)}=-\dfrac{x+27}{(x-3)^2(x+3)}.$

16. $$\frac{x}{1-x}+\frac{2x+3}{x^2-1}=\frac{-x\,(x+1)+2x+3}{(x+1)\,(x-1)}=\frac{-x^2-x+2x+3}{x^2-1}=-\frac{x^2-x-3}{x^2-1}.$$

17. $$\frac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}=\frac{\dfrac{x+1}{x}}{\dfrac{x-1}{x}}=\frac{x+1}{x}\cdot\frac{x}{x-1}=\frac{x+1}{x-1}.$$

18. $$\frac{\dfrac{1}{x}+\dfrac{1}{y}}{1-\dfrac{1}{xy}}=\frac{\dfrac{x+y}{xy}}{\dfrac{xy-1}{xy}}=\frac{x+y}{xy}\cdot\frac{xy}{xy-1}=\frac{x+y}{xy-1}.$$

19. $$\frac{4x^2}{2\sqrt{2x^2+7}}+\sqrt{2x^2+7}=\frac{4x^2+2\sqrt{2x^2+7}\sqrt{2x^2+7}}{2\sqrt{2x^2+7}}=\frac{4x^2+4x^2+14}{2\sqrt{2x^2+7}}=\frac{4x^2+7}{\sqrt{2x^2+7}}.$$

20. $$\begin{aligned}6\,(2x+1)^2\sqrt{x^2+x}+\frac{(2x+1)^4}{2\sqrt{x^2+x}}&=\frac{6\,(2x+1)^2\sqrt{x^2+x}\,(2)\sqrt{x^2+x}+(2x+1)^4}{2\sqrt{x^2+x}}\\&=\frac{(2x+1)^2\left[12\left(x^2+x\right)+4x^2+4x+1\right]}{2\sqrt{x^2+x}}=\frac{(2x+1)^2\left(16x^2+16x+1\right)}{2\sqrt{x^2+x}}.\end{aligned}$$

21. $$5\left[\frac{\left(t^2+1\right)(1)-t\,(2t)}{\left(t^2+1\right)^2}\right]=\frac{5\left(t^2+1-2t^2\right)}{\left(t^2+1\right)^2}=\frac{5\left(1-t^2\right)}{\left(t^2+1\right)^2}=-\frac{5\left(t^2-1\right)}{\left(t^2+1\right)^2}.$$

22. $$\frac{2x\,(x+1)^{-1/2}-(x+1)^{1/2}}{x^2}=\frac{(x+1)^{-1/2}\,(2x-x-1)}{x^2}=\frac{(x+1)^{-1/2}\,(x-1)}{x^2}=\frac{x-1}{x^2\sqrt{x+1}}.$$

23. $$\frac{\left(x^2+1\right)^2(-2)+(2x)\,2\left(x^2+1\right)(2x)}{\left(x^2+1\right)^4}=\frac{2\left(x^2+1\right)\left[-\left(x^2+1\right)+4x^2\right]}{\left(x^2+1\right)^4}=\frac{2\left(3x^2-1\right)}{\left(x^2+1\right)^3}.$$

24. $$\frac{\left(x^2+1\right)^{1/2}-2x^2\left(x^2+1\right)^{-1/2}}{1-x^2}=\frac{\left(x^2+1\right)^{-1/2}\left(x^2+1-2x^2\right)}{1-x^2}=\frac{\left(x^2+1\right)^{-1/2}\left(-x^2+1\right)}{1-x^2}=\frac{1}{\sqrt{x^2+1}}.$$

25. $$3\left(\frac{2x+1}{3x+2}\right)^2\left[\frac{(3x+2)\,(2)-(2x+1)\,(3)}{(3x+2)^2}\right]=\frac{3\,(2x+1)^2\,(6x+4-6x-3)}{(3x+2)^4}=\frac{3\,(2x+1)^2}{(3x+2)^4}.$$

26. $$\frac{(2x+1)^{1/2}-(x+2)\,(2x+1)^{-1/2}}{2x+1}=\frac{(2x+1)^{-1/2}\,(2x+1-x-2)}{2x+1}=\frac{(2x+1)^{-1/2}\,(x-1)}{2x+1}=\frac{x-1}{(2x+1)^{3/2}}.$$

27. $$\begin{aligned}&100\left[\frac{\left(t^2+20t+100\right)(2t+10)-\left(t^2+10t+100\right)(2t+20)}{\left(t^2+20t+100\right)^2}\right]\\&\qquad=100\left[\frac{2t^3+40t^2+200t+10t^2+200t+1000-2t^3-20t^2-200t-20t^2-200t-2000}{\left(t^2+20t+100\right)^2}\right]\\&\qquad=\frac{100\left(10t^2-1000\right)}{\left(t^2+20t+100\right)^2}=\frac{1000\,(t-10)}{(t+10)^3}.\end{aligned}$$

28. $$\frac{2(2x-3)^{1/3}-(x-1)(2x-3)^{-2/3}}{(2x-3)^{2/3}}=\frac{(2x-3)^{-2/3}[2(2x-3)-(x-1)]}{(2x-3)^{2/3}}=\frac{(2x-3)^{-2/3}(4x-6-x+1)}{(2x-3)^{2/3}}$$
$$=\frac{3x-5}{(2x-3)^{4/3}}.$$

29. $\frac{1}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1}=\frac{\sqrt{3}+1}{3-1}=\frac{\sqrt{3}+1}{2}.$

30. $\frac{1}{\sqrt{x}+5}\cdot\frac{\sqrt{x}-5}{\sqrt{x}-5}=\frac{\sqrt{x}-5}{x-25}.$

31. $\frac{1}{\sqrt{x}-\sqrt{y}}\cdot\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}=\frac{\sqrt{x}+\sqrt{y}}{x-y}.$

32. $\frac{a}{1-\sqrt{a}}\cdot\frac{1+\sqrt{a}}{1+\sqrt{a}}=\frac{a\left(1+\sqrt{a}\right)}{1-a}.$

33. $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\cdot\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{a-b}.$

34. $\frac{2\sqrt{a}+\sqrt{b}}{2\sqrt{a}-\sqrt{b}}\cdot\frac{2\sqrt{a}+\sqrt{b}}{2\sqrt{a}+\sqrt{b}}=\frac{\left(2\sqrt{a}+\sqrt{b}\right)^2}{4a-b}.$

35. $\frac{\sqrt{x}}{3}\cdot\frac{\sqrt{x}}{\sqrt{x}}=\frac{x}{3\sqrt{x}}.$

36. $\frac{\sqrt[3]{y}}{x}\cdot\frac{\sqrt[3]{y^2}}{\sqrt[3]{y^2}}=\frac{y}{x\sqrt[3]{y^2}}.$

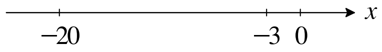
37. $\frac{1-\sqrt{3}}{3}\cdot\frac{1+\sqrt{3}}{1+\sqrt{3}}=\frac{1^2-\left(\sqrt{3}\right)^2}{3\left(1+\sqrt{3}\right)}=-\frac{2}{3\left(1+\sqrt{3}\right)}.$

38. $\frac{\sqrt{x}-1}{x}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\frac{x-1}{x\left(\sqrt{x}+1\right)}.$

39. $\frac{1+\sqrt{x+2}}{\sqrt{x+2}}\cdot\frac{1-\sqrt{x+2}}{1-\sqrt{x+2}}=\frac{1-(x+2)}{\sqrt{x+2}\left(1-\sqrt{x+2}\right)}=-\frac{x+1}{\sqrt{x+2}\left(1-\sqrt{x+2}\right)}.$

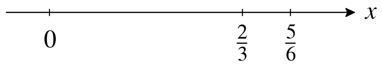
40. $\frac{\sqrt{x+3}-\sqrt{x}}{3}\cdot\frac{\sqrt{x+3}+\sqrt{x}}{\sqrt{x+3}+\sqrt{x}}=\frac{x+3-x}{3\left(\sqrt{x+3}+\sqrt{x}\right)}=\frac{1}{\sqrt{x+3}+\sqrt{x}}.$

41. The statement is false because -3 is greater than -20. See the number line below.



42. The statement is true because -5 is equal to -5.

43. The statement is false because $\frac{2}{3}=\frac{4}{6}$ is less than $\frac{5}{6}$.



44. The statement is false because $-\frac{5}{6}=-\frac{10}{12}$ is greater than $-\frac{11}{12}$.

45. We are given $2x+4<8$. Add -4 to each side of the inequality to obtain $2x<4$, then multiply each side of the inequality by $\frac{1}{2}$ to obtain $x<2$. We write this in interval notation as $(-\infty,2)$.

46. We are given $-6>4+5x$. Add -4 to each side of the inequality to obtain $-6-4>5x$, so $-10>5x$. Dividing by 2, we obtain $-2>x$, so $x<-2$. We write this in interval notation as $(-\infty,-2)$.

47. We are given the inequality $-4x\geq 20$. Multiply both sides of the inequality by $-\frac{1}{4}$ and reverse the sign of the inequality to obtain $x\leq -5$. We write this in interval notation as $(-\infty,-5]$.

48. $-12 \le -3x \Rightarrow 4 \ge x$, or $x \le 4$. We write this in interval notation as $(-\infty, 4]$.

49. We are given the inequality $-6 < x - 2 < 4$. First add 2 to each member of the inequality to obtain $-6 + 2 < x < 4 + 2$ and $-4 < x < 6$, so the solution set is the open interval $(-4, 6)$.

50. We add -1 to each member of the given double inequality $0 \le x + 1 \le 4$ to obtain $-1 \le x \le 3$, and the solution set is $[-1, 3]$.

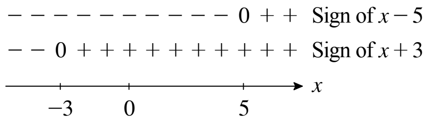
51. We want to find the values of x that satisfy at least one of the inequalities $x + 1 > 4$ and $x + 2 < -1$. Adding -1 to both sides of the first inequality, we obtain $x + 1 - 1 > 4 - 1$, so $x > 3$. Similarly, adding -2 to both sides of the second inequality, we obtain $x + 2 - 2 < -1 - 2$, so $x < -3$. Therefore, the solution set is $(-\infty, -3) \cup (3, \infty)$.

52. We want to find the values of x that satisfy at least one of the inequalities $x + 1 > 2$ and $x - 1 < -2$. Solving these inequalities, we find that $x > 1$ or $x < -1$, and the solution set is $(-\infty, -1) \cup (1, \infty)$.

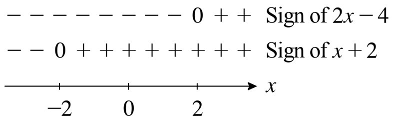
53. We want to find the values of x that satisfy the inequalities $x + 3 > 1$ and $x - 2 < 1$. Adding -3 to both sides of the first inequality, we obtain $x + 3 - 3 > 1 - 3$, or $x > -2$. Similarly, adding 2 to each side of the second inequality, we obtain $x - 2 + 2 < 1 + 2$, so $x < 3$. Because both inequalities must be satisfied, the solution set is $(-2, 3)$.

54. We want to find the values of x that satisfy the inequalities $x - 4 \le 1$ and $x + 3 > 2$. Solving these inequalities, we find that $x \le 5$ and $x > -1$, and the solution set is $(-1, 5]$.

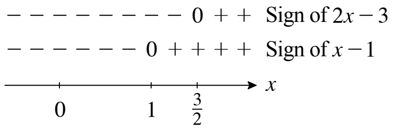
55. We want to find the values of x that satisfy the inequality $(x + 3)(x - 5) \le 0$. From the sign diagram, we see that the given inequality is satisfied when $-3 \le x \le 5$, that is, when the signs of the two factors are different or when one of the factors is equal to zero.



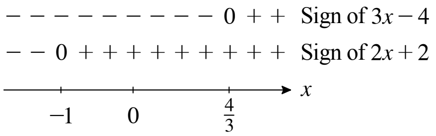
56. We want to find the values of x that satisfy the inequality $(2x - 4)(x + 2) \ge 0$. From the sign diagram, we see that the given inequality is satisfied when $x \le -2$ or $x \ge 2$; that is, when the signs of both factors are the same or one of the factors is equal to zero.



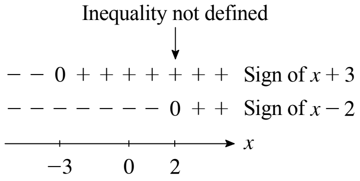
57. We want to find the values of x that satisfy the inequality $(2x - 3)(x - 1) \ge 0$. From the sign diagram, we see that the given inequality is satisfied when $x \le 1$ or $x \ge \frac{3}{2}$; that is,when the signs of both factors are the same, or one of the factors is equal to zero.



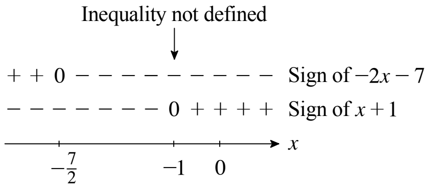
58. We want to find the values of x that satisfy the inequalities $(3x - 4)(2x + 2) \le 0$.
From the sign diagram, we see that the given inequality is satisfied when $-1 \le x \le \frac{4}{3}$, that is, when the signs of the two factors are different or when one of the factors is equal to zero.



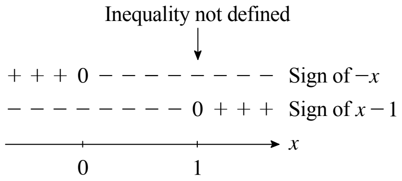
59. We want to find the values of x that satisfy the inequalities $\dfrac{x+3}{x-2} \geq 0$. From the sign diagram, we see that the given inequality is satisfied when $x \leq -3$ or $x > 2$, that is, when the signs of the two factors are the same. Notice that $x = 2$ is not included because the inequality is not defined at that value of x.



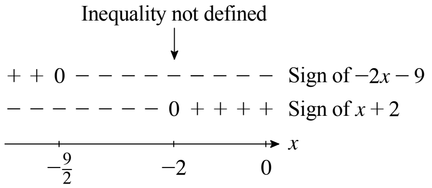
60. We want to find the values of x that satisfy the inequality $\dfrac{2x-3}{x+1} \geq 4$. We rewrite the inequality as $\dfrac{2x-3}{x+1} - 4 \geq 0$, $\dfrac{2x-3-4x-4}{x+1} \geq 0$, and $\dfrac{-2x-7}{x+1} \geq 0$. From the sign diagram, we see that the given inequality is satisfied when $-\frac{7}{2} \leq x < -1$; that is, when the signs of the two factors are the same. Notice that $x = -1$ is not included because the inequality is not defined at that value of x.



61. We want to find the values of x that satisfy the inequality $\dfrac{x-2}{x-1} \leq 2$. Subtracting 2 from each side of the given inequality and simplifying gives $\dfrac{x-2}{x-1} - 2 \leq 0$, $\dfrac{x-2-2(x-1)}{x-1} \leq 0$, and $-\dfrac{x}{x-1} \leq 0$. From the sign diagram, we see that the given inequality is satisfied when $x \leq 0$ or $x > 1$; that is, when the signs of the two factors differ. Notice that $x = 1$ is not included because the inequality is undefined at that value of x.



62. We want to find the values of x that satisfy the inequality$\dfrac{2x-1}{x+2} \leq 4$. Subtracting 4 from each side of the given inequality and simplifying gives $\dfrac{2x-1}{x+2} - 4 \leq 0$, $\dfrac{2x-1-4(x+2)}{x+2} \leq 0$, $\dfrac{2x-1-4x-8}{x+2} \leq 0$, and finally $\dfrac{-2x-9}{x+2} \leq 0$. From the sign diagram, we see that the given inequality is satisfied when $x \leq -\frac{9}{2}$ or $x > -2$.



63. $|-6+2| = 4$.

64. $4 + |-4| = 4 + 4 = 8$.

65. $\dfrac{|-12+4|}{|16-12|} = \dfrac{|-8|}{|4|} = 2$.

66. $\left|\dfrac{0.2-1.4}{1.6-2.4}\right| = \left|\dfrac{-1.2}{-0.8}\right| = 1.5$.

67. $\sqrt{3}\,|-2| + 3\left|-\sqrt{3}\right| = \sqrt{3}\,(2) + 3\sqrt{3} = 5\sqrt{3}$.

68. $|-1| + \sqrt{2}\,|-2| = 1 + 2\sqrt{2}$.

69. $|\pi - 1| + 2 = \pi - 1 + 2 = \pi + 1$.

70. $|\pi - 6| - 3 = 6 - \pi - 3 = 3 - \pi$.

71. $\left|\sqrt{2}-1\right| + \left|3-\sqrt{2}\right| = \sqrt{2} - 1 + 3 - \sqrt{2} = 2$.

72. $\left|2\sqrt{3}-3\right|-\left|\sqrt{3}-4\right|=2\sqrt{3}-3-\left(4-\sqrt{3}\right)=3\sqrt{3}-7.$

73. False. If $a > b$, then $-a < -b$, $-a+b < -b+b$, and $b-a < 0$.

74. False. Let $a=-2$ and $b=-3$. Then $a/b=\frac{-2}{-3}=\frac{2}{3}<1$.

75. False. Let $a=-2$ and $b=-3$. Then $a^2=4$ and $b^2=9$, and $4<9$. Note that we need only to provide a counterexample to show that the statement is not always true.

76. False. Let $a=-2$ and $b=-3$. Then $\frac{1}{a}=-\frac{1}{2}$ and $\frac{1}{b}=-\frac{1}{3}$, and $-\frac{1}{2}<-\frac{1}{3}$.

77. True. There are three possible cases.
Case 1: If $a>0$ and $b>0$, then $a^3>b^3$, since $a^3-b^3=(a-b)\left(a^2+ab+b^2\right)>0$.
Case 2: If $a>0$ and $b<0$, then $a^3>0$ and $b^3<0$, and it follows that $a^3>b^3$.
Case 3: If $a<0$ and $b<0$, then $a^3-b^3=(a-b)\left(a^2+ab+b^2\right)>0$, and we see that $a^3>b^3$. (Note that $a-b>0$ and $ab>0$.)

78. True. If $a>b$, then it follows that $-a<-b$ because an inequality symbol is reversed when both sides of the inequality are multiplied by a negative number.

79. False. If we take $a=-2$, then $|-a|=|-(-2)|=|2|=2\neq a$.

80. True. If $b<0$, then $b^2>0$, and $\left|b^2\right|=b^2$.

81. True. If $a-4<0$, then $|a-4|=4-a=|4-a|$. If $a-4>0$, then $|4-a|=a-4=|a-4|$.

82. False. If we let $a=-2$, then $|a+1|=|-2+1|=|-1|=1\neq|-2|+1=3$.

83. False. If we take $a=3$ and $b=-1$, then $|a+b|=|3-1|=2\neq|a|+|b|=3+1=4$.

84. False. If we take $a=3$ and $b=-1$, then $|a-b|=4\neq|a|-|b|=3-1=2$.

85. If the car is driven in the city, then it can be expected to cover $(18.1)(20)=362\ \frac{\text{miles}}{\text{gal}}\cdot\text{gal}$, or 362 miles, on a full tank. If the car is driven on the highway, then it can be expected to cover $(18.1)(27)=488.7\ \frac{\text{miles}}{\text{gal}}\cdot\text{gal}$, or 488.7 miles, on a full tank. Thus, the driving range of the car may be described by the interval $[362, 488.7]$.

86. Simplifying $5(C-25)\geq1.75+2.5C$, we obtain $5C-125\geq1.75+2.5C$, $5C-2.5C\geq1.75+125$, $2.5C\geq126.75$, and finally $C\geq50.7$. Therefore, the minimum cost is \$50.70.

87. $6(P-2500)\leq4(P+2400)$ can be rewritten as $6P-15{,}000\leq4P+9600$, $2P\leq24{,}600$, or $P\leq12{,}300$. Therefore, the maximum profit is \$12,300.

88. **a.** We want to find a formula for converting Centigrade temperatures to Fahrenheit temperatures. Thus, $C=\frac{5}{9}(F-32)=\frac{5}{9}F-\frac{160}{9}$. Therefore, $\frac{5}{9}F=C+\frac{160}{9}$, $5F=9C+160$, and $F=\frac{9}{5}C+32$. Calculating the lower temperature range, we have $F=\frac{9}{5}(-15)+32=5$, or 5 degrees. Calculating the upper temperature range, $F=\frac{9}{5}(-5)+32=23$, or 23 degrees. Therefore, the temperature range is $5^\circ<F<23^\circ$.

b. For the lower temperature range, $C = \frac{5}{9}(63 - 32) = \frac{155}{9} \approx 17.2$, or 17.2 degrees. For the upper temperature range, $C = \frac{5}{9}(80 - 32) = \frac{5}{9}(48) \approx 26.7$, or 26.7 degrees. Therefore, the temperature range is $17.2^\circ < C < 26.7^\circ$.

89. Let x represent the salesman's monthly sales in dollars. Then $0.15(x - 12{,}000) \geq 6000$, $15(x - 12{,}000) \geq 600{,}000$, $15x - 180{,}000 \geq 600{,}000$, $15x \geq 780{,}000$, and $x \geq 52{,}000$. We conclude that the salesman must have sales of at least \$52,000 to reach his goal.

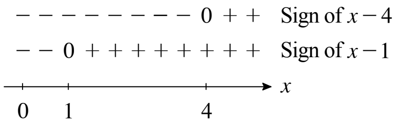
90. Let x represent the wholesale price of the car. Then $\dfrac{\text{Selling price}}{\text{Wholesale price}} - 1 \geq \text{Markup}$; that is, $\dfrac{11{,}200}{x} - 1 \geq 0.30$, whence $\dfrac{11{,}200}{x} \geq 1.30$, $1.3x \leq 11{,}200$, and $x \leq 8615.38$. We conclude that the maximum wholesale price is \$8615.38.

91. The rod is acceptable if $0.49 \leq x \leq 0.51$ or $-0.01 \leq x - 0.5 \leq 0.01$. This gives the required inequality, $|x - 0.5| \leq 0.01$.

92. $|x - 0.1| \leq 0.01$ is equivalent to $-0.01 \leq x - 0.1 \leq 0.01$ or $0.09 \leq x \leq 0.11$. Therefore, the smallest diameter a ball bearing in the batch can have is 0.09 inch, and the largest diameter is 0.11 inch.

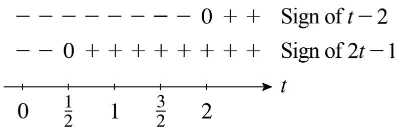
93. We want to solve the inequality $-6x^2 + 30x - 10 \geq 14$. (Remember that x is expressed in thousands.) Adding -14 to both sides of this inequality, we have $-6x^2 + 30x - 10 - 14 \geq 14 - 14$, or $-6x^2 + 30x - 24 \geq 0$. Dividing both sides of the inequality by -6 (which reverses the sign of the inequality), we have $x^2 - 5x + 4 \leq 0$. Factoring this last expression, we have $(x - 4)(x - 1) \leq 0$.

From the sign diagram, we see that x must lie between 1 and 4. (The inequality is satisfied only when the two factors have opposite signs.) Because x is expressed in thousands of units, we see that the manufacturer must produce between 1000 and 4000 units of the commodity.



94. We solve the inequality $\dfrac{0.2t}{t^2 + 1} \geq 0.08$, obtaining $0.08t^2 + 0.08 \leq 0.2t$, $0.08t^2 - 0.2t + 0.08 \leq 0$, $2t^2 - 5t + 2 \leq 0$, and $(2t - 1)(t - 2) \leq 0$.

From the sign diagram, we see that the required solution is $\left[\frac{1}{2}, 2\right]$, so the concentration of the drug is greater than or equal to 0.08 mg/cc between $\frac{1}{2}$ hr and 2 hr after injection.



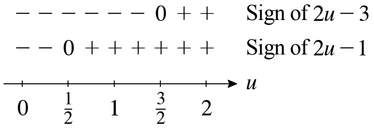
95. We solve the inequalities $25 \leq \dfrac{0.5x}{100 - x} \leq 30$, obtaining $2500 - 25x \leq 0.5x \leq 3000 - 30x$, which is equivalent to $2500 - 25x \leq 0.5x$ and $0.5x \leq 3000 - 30x$. Simplifying further, $25.5x \geq 2500$ and $30.5x \leq 3000$, so $x \geq \dfrac{2500}{25.5} \approx 98.04$ and $x \leq \dfrac{3000}{30.5} \approx 98.36$. Thus, the city could expect to remove between 98.04% and 98.36% of the toxic pollutant.

96. We simplify the inequality $20t - 40\sqrt{t} + 50 \le 35$ to $20t - 40\sqrt{t} + 15 \le 0$ (1). Let $u = \sqrt{t}$. Then $u^2 = t$, so we have $20u^2 - 40u + 15 \le 0$, $4u^2 - 8u + 3 \le 0$, and $(2u - 3)(2u - 1) \le 0$.

From the sign diagram, we see that we must have u in $\left[\frac{1}{2}, \frac{3}{2}\right]$.

Because $t = u^2$, we see that the solution to Equation (1) is $\left[\frac{1}{4}, \frac{9}{4}\right]$.

Thus, the average speed of a vehicle is less than or equal to 35 miles per hour between 6:15 a.m. and 8:15 a.m.



97. We solve the inequality $\dfrac{136}{1 + 0.25(t - 4.5)^2} + 28 \ge 128$ or $\dfrac{136}{1 + 0.25(t - 4.5)^2} \ge 100$. Next, $136 \ge 100\left[1 + 0.25(t - 4.5)^2\right]$, so $136 \ge 100 + 25(t - 4.5)^2$, $36 \ge 25(t - 4.5)^2$, $(t - 4.5)^2 \le \frac{36}{25}$, and $t - 4.5 \le \pm\frac{6}{5}$. Solving this last inequality, we have $t \le 5.7$ and $t \ge 3.3$. Thus, the amount of nitrogen dioxide is greater than or equal to 128 PSI between 10:18 a.m. and 12:42 p.m.

98. False. Take $a = 2$, $b = 3$, and $c = 4$. Then $\dfrac{a}{b + c} = \dfrac{2}{3 + 4} = \dfrac{2}{7}$, but $\dfrac{a}{b} + \dfrac{a}{c} = \dfrac{2}{3} + \dfrac{2}{4} = \dfrac{8 + 6}{12} = \dfrac{14}{12} = \dfrac{7}{6}$.

99. False. Take $a = 1$, $b = 2$, and $c = 3$. Then $a < b$, but $a - c = 1 - 3 = -2 \not> 2 - 3 = -1 = b - c$.

100. True. $|b - a| = |(-1)(a - b)| = |-1|\,|a - b| = |a - b|$.

101. True. $|a - b| = |a + (-b)| \le |a| + |-b| = |a| + |b|$.

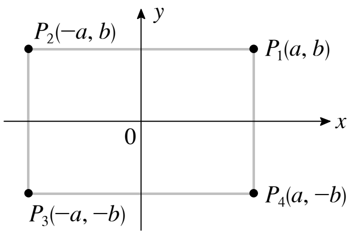
102. False. Take $a = 3$ and $b = 1$. Then $\sqrt{a^2 - b^2} = \sqrt{9 - 1} = \sqrt{8} = 2\sqrt{2}$, but $|a| - |b| = 3 - 1 = 2$.

1.3 The Cartesian Coordinate System

Concept Questions page 29

1. a. $a < 0$ and $b > 0$ **b.** $a < 0$ and $b < 0$ **c.** $a > 0$ and $b < 0$

2. a.



b. $d(P_1(a, b), (0, 0)) = \sqrt{(0 - a)^2 + (0 - b)^2} = \sqrt{a^2 + b^2}$,
$d(P_2(-a, b), (0, 0)) = \sqrt{[0 - (-a)]^2 + (0 - b)^2} = \sqrt{a^2 + b^2}$,
$d(P_3(-a, -b), (0, 0)) = \sqrt{[0 - (-a)]^2 + [0 - (-b)]^2} = \sqrt{a^2 + b^2}$,
and $d(P_4(a, -b), (0, 0)) = \sqrt{(0 - a)^2 + [0 - (-b)]^2} = \sqrt{a^2 + b^2}$,
so the points $P_1(a, b)$, $P_2(-a, b)$, $P_3(-a, -b)$, and $P_4(a, -b)$ are all the same distance from the origin.

Exercises page 30

1. The coordinates of A are $(3, 3)$ and it is located in Quadrant I.

2. The coordinates of B are $(-5, 2)$ and it is located in Quadrant II.

3. The coordinates of C are $(2, -2)$ and it is located in Quadrant IV.

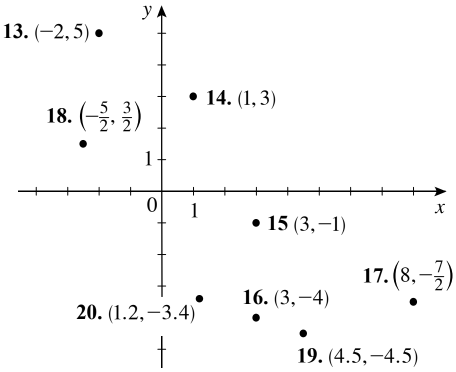
4. The coordinates of D are $(-2, 5)$ and it is located in Quadrant II.

5. The coordinates of E are $(-4, -6)$ and it is located in Quadrant III.

6. The coordinates of F are $(8, -2)$ and it is located in Quadrant IV.

7. A **8.** $(-5, 4)$ **9.** E, F, and G **10.** E **11.** F **12.** D

For Exercises 13–20, refer to the following figure.



21. Using the distance formula, we find that $\sqrt{(4-1)^2+(7-3)^2}=\sqrt{3^2+4^2}=\sqrt{25}=5$.

22. Using the distance formula, we find that $\sqrt{(4-1)^2+(4-0)^2}=\sqrt{3^2+4^2}=\sqrt{25}=5$.

23. Using the distance formula, we find that $\sqrt{[4-(-1)]^2+(9-3)^2}=\sqrt{5^2+6^2}=\sqrt{25+36}=\sqrt{61}$.

24. Using the distance formula, we find that $\sqrt{[10-(-2)]^2+(6-1)^2}=\sqrt{12^2+5^2}=\sqrt{144+25}=\sqrt{169}=13$.

25. The coordinates of the points have the form $(x, -6)$. Because the points are 10 units away from the origin, we have $(x-0)^2+(-6-0)^2=10^2$, $x^2=64$, or $x=\pm 8$. Therefore, the required points are $(-8, -6)$ and $(8, -6)$.

26. The coordinates of the points have the form $(3, y)$. Because the points are 5 units away from the origin, we have $(3-0)^2+(y-0)^2=5^2$, $y^2=16$, or $y=\pm 4$. Therefore, the required points are $(3, 4)$ and $(3, -4)$.

27. The points are shown in the diagram. To show that the four sides are equal, we compute

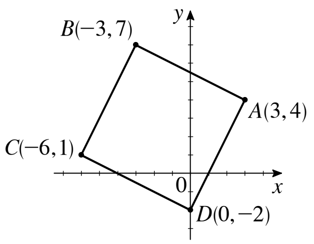
$d(A, B)=\sqrt{(-3-3)^2+(7-4)^2}=\sqrt{(-6)^2+3^2}=\sqrt{45}$,

$d(B, C)=\sqrt{[-6-(-3)]^2+(1-7)^2}=\sqrt{(-3)^2+(-6)^2}=\sqrt{45}$,

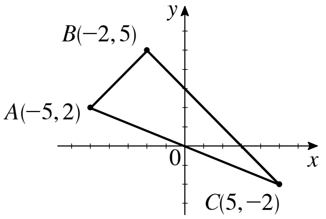
$d(C, D)=\sqrt{[0-(-6)]^2+[(-2)-1]^2}=\sqrt{(6)^2+(-3)^2}=\sqrt{45}$,

and $d(A, D)=\sqrt{(0-3)^2+(-2-4)^2}=\sqrt{(3)^2+(-6)^2}=\sqrt{45}$.

Next, to show that ΔABC is a right triangle, we show that it satisfies the Pythagorean Theorem. Thus, $d(A, C)=\sqrt{(-6-3)^2+(1-4)^2}=\sqrt{(-9)^2+(-3)^2}=\sqrt{90}=3\sqrt{10}$ and $[d(A, B)]^2+[d(B, C)]^2=90=[d(A, C)]^2$. Similarly, $d(B, D)=\sqrt{90}=3\sqrt{10}$, so ΔBAD is a right triangle as well. It follows that $\angle B$ and $\angle D$ are right angles, and we conclude that $ADCB$ is a square.



28. The triangle is shown in the figure. To prove that ΔABC is a right triangle, we show that $[d(A,C)]^2 = [d(A,B)]^2 + [d(B,C)]^2$ and the result will then follow from the Pythagorean Theorem. Now $[d(A,C)]^2 = (-5-5)^2 + [2-(-2)]^2 = 100 + 16 = 116$. Next, we find $[d(A,B)]^2 + [d(B,C)]^2 = [-2-(-5)]^2 + (5-2)^2 + [5-(-2)]^2 + (-2-5)^2 = 9 + 9 + 49 + 49 = 116$, and the result follows.



29. The equation of the circle with radius 5 and center (2, −3) is given by $(x-2)^2 + [y-(-3)]^2 = 5^2$, or $(x-2)^2 + (y+3)^2 = 25$.

30. The equation of the circle with radius 3 and center (−2, −4) is given by $[x-(-2)]^2 + [y-(-4)]^2 = 9$, or $(x+2)^2 + (y+4)^2 = 9$.

31. The equation of the circle with radius 5 and center (0, 0) is given by $(x-0)^2 + (y-0)^2 = 5^2$, or $x^2 + y^2 = 25$.

32. The distance between the center of the circle and the point (2, 3) on the circumference of the circle is given by $d = \sqrt{(3-0)^2 + (2-0)^2} = \sqrt{13}$. Therefore $r = \sqrt{13}$ and the equation of the circle centered at the origin that passes through (2, 3) is $x^2 + y^2 = 13$.

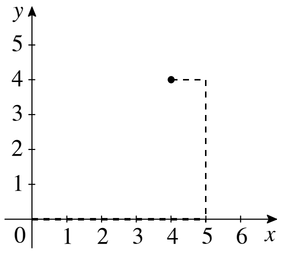
33. The distance between the points (5, 2) and (2, −3) is given by $d = \sqrt{(5-2)^2 + [2-(-3)]^2} = \sqrt{3^2 + 5^2} = \sqrt{34}$. Therefore $r = \sqrt{34}$ and the equation of the circle passing through (5, 2) and (2, −3) is $(x-2)^2 + [y-(-3)]^2 = 34$, or $(x-2)^2 + (y+3)^2 = 34$.

34. The equation of the circle with center $(-a, a)$ and radius $2a$ is given by $[x-(-a)]^2 + (y-a)^2 = (2a)^2$, or $(x+a)^2 + (y-a)^2 = 4a^2$.

35. **a.** The coordinates of the suspect's car at its final destination are $x = 4$ and $y = 4$.

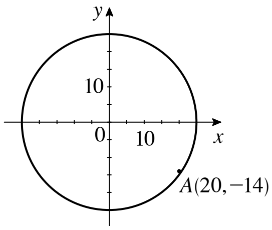
b. The distance traveled by the suspect was $5 + 4 + 1$, or 10 miles.

c. The distance between the original and final positions of the suspect's car was $d = \sqrt{(4-0)^2 + (4-0)^2} = \sqrt{32} = 4\sqrt{2}$, or approximately 5.66 miles.

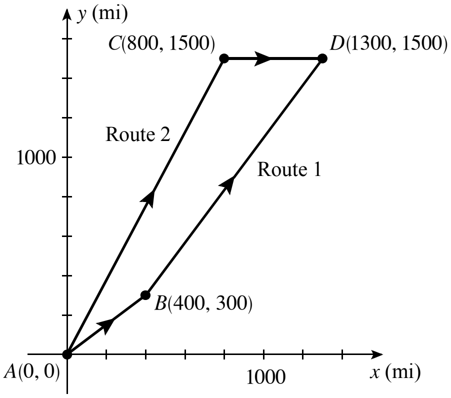


36. Referring to the diagram on page 31 of the text, we see that the distance from A to B is given by $d(A,B) = \sqrt{400^2 + 300^2} = \sqrt{250{,}000} = 500$. The distance from B to C is given by $d(B,C) = \sqrt{(-800-400)^2 + (800-300)^2} = \sqrt{(-1200)^2 + (500)^2} = \sqrt{1{,}690{,}000} = 1300$. The distance from C to D is given by $d(C,D) = \sqrt{[-800-(-800)]^2 + (800-0)^2} = \sqrt{0 + 800^2} = 800$. The distance from D to A is given by $d(D,A) = \sqrt{[(-800)-0]^2 + (0-0)} = \sqrt{640{,}000} = 800$. Therefore, the total distance covered on the tour is $d(A,B) + d(B,C) + d(C,D) + d(D,A) = 500 + 1300 + 800 + 800 = 3400$, or 3400 miles.

37. Suppose that the furniture store is located at the origin O so that your house is located at $A(20, -14)$. Because $d(O, A) = \sqrt{20^2 + (-14)^2} = \sqrt{596} \approx 24.4$, your house is located within a 25-mile radius of the store and you will not incur a delivery charge.



38.



Referring to the diagram, we see that the distance the salesman would cover if he took Route 1 is given by

$$d(A, B) + d(B, D) = \sqrt{400^2 + 300^2} + \sqrt{(1300 - 400)^2 + (1500 - 300)^2}$$
$$= \sqrt{250{,}000} + \sqrt{2{,}250{,}000} = 500 + 1500 = 2000$$

or 2000 miles. On the other hand, the distance he would cover if he took Route 2 is given by

$$d(A, C) + d(C, D) = \sqrt{800^2 + 1500^2} + \sqrt{(1300 - 800)^2} = \sqrt{2{,}890{,}000} + \sqrt{250{,}000}$$
$$= 1700 + 500 = 2200$$

or 2200 miles. Comparing these results, we see that he should take Route 1.

39. The cost of shipping by freight train is $(0.66)(2000)(100) = 132{,}000$, or \$132,000.
The cost of shipping by truck is $(0.62)(2200)(100) = 136{,}400$, or \$136,400.
Comparing these results, we see that the automobiles should be shipped by freight train. The net savings are $136{,}400 - 132{,}000 = 4400$, or \$4400.

40. The length of cable required on land is $d(S, Q) = 10{,}000 - x$ and the length of cable required under water is $d(Q, M) = \sqrt{\left(x^2 - 0\right) + (0 - 3000)^2} = \sqrt{x^2 + 3000^2}$. The cost of laying cable is thus $3(10{,}000 - x) + 5\sqrt{x^2 + 3000^2}$.
If $x = 2500$, then the total cost is given by $3(10{,}000 - 2500) + 5\sqrt{2500^2 + 3000^2} \approx 42{,}025.62$, or \$42,025.62.
If $x = 3000$, then the total cost is given by $3(10{,}000 - 3000) + 5\sqrt{3000^2 + 3000^2} \approx 42{,}213.20$, or \$42,213.20.

41. To determine the VHF requirements, we calculate $d = \sqrt{25^2 + 35^2} = \sqrt{625 + 1225} = \sqrt{1850} \approx 43.01$.
Models B, C, and D satisfy this requirement.
To determine the UHF requirements, we calculate $d = \sqrt{20^2 + 32^2} = \sqrt{400 + 1024} = \sqrt{1424} \approx 37.74$. Models C and D satisfy this requirement.
Therefore, Model C allows him to receive both channels at the least cost.

42. **a.** Let the positions of ships A and B after t hours be $A\,(0, y)$ and $B\,(x, 0)$, respectively. Then $x = 30t$ and $y = 20t$. Therefore, the distance in miles between the two ships is $D = \sqrt{(30t)^2 + (20t)^2} = \sqrt{900t^2 + 400t^2} = 10\sqrt{13}t$.

b. The required distance is obtained by letting $t = 2$, giving $D = 10\sqrt{13}\,(2)$, or approximately 72.11 miles.

43. **a.** Let the positions of ships A and B be $(0, y)$ and $(x, 0)$, respectively. Then $y = 25\left(t + \frac{1}{2}\right)$ and $x = 20t$. The distance D in miles between the two ships is

$$D = \sqrt{(x-0)^2 + (0-y)^2} = \sqrt{x^2 + y^2} = \sqrt{400t^2 + 625\left(t + \tfrac{1}{2}\right)^2} \quad (1).$$

b. The distance between the ships 2 hours after ship A has left port is obtained by letting $t = \frac{3}{2}$ in Equation (1), yielding $D = \sqrt{400\left(\frac{3}{2}\right)^2 + 625\left(\frac{3}{2} + \frac{1}{2}\right)^2} = \sqrt{3400}$, or approximately 58.31 miles.

44. **a.** The distance in feet is given by $\sqrt{(4000)^2 + x^2} = \sqrt{16{,}000{,}000 + x^2}$.

b. Substituting the value $x = 20{,}000$ into the above expression gives $\sqrt{16{,}000{,}000 + (20{,}000)^2} \approx 20{,}396$, or 20,396 ft.

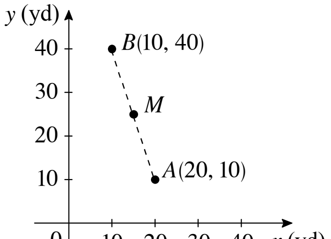
45. **a.** Suppose that $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are endpoints of the line segment and that the point $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$ is the midpoint of the line segment PQ. The distance between P and Q is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. The distance between P and M is

$$\sqrt{\left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_1 + y_2}{2} - y_1\right)^2} = \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2} = \tfrac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

which is one-half the distance from P to Q. Similarly, we obtain the same expression for the distance from M to P.

b. The midpoint is given by $\left(\dfrac{4-3}{2}, \dfrac{-5+2}{2}\right)$, or $\left(\dfrac{1}{2}, -\dfrac{3}{2}\right)$.

46. **a.**



b. The coordinates of the position of the prize are $x = \dfrac{20 + 10}{2}$ and $y = \dfrac{10 + 40}{2}$, or $x = 15$ yards and $y = 25$ yards.

c. The distance from the prize to the house is

$$\begin{aligned} d\,(M\,(15, 25), (0, 0)) &= \sqrt{(15-0)^2 + (25-0)^2} = \sqrt{850} \\ &\approx 29.15 \text{ (yards).} \end{aligned}$$

47. True. Plot the points.

48. True. Plot the points.

49. False. The distance between $P_1\,(a, b)$ and $P_3\,(kc, kd)$ is

$$\begin{aligned} d &= \sqrt{(kc - a)^2 + (kd - b)^2} \\ &\neq |k|\,D = |k|\sqrt{(c-a)^2 + (d-b)^2} = \sqrt{k^2\,(c-a)^2 + k^2\,(d-b)^2} = \sqrt{[k\,(c-a)]^2 + [k\,(d-b)]^2}. \end{aligned}$$

50. True. $kx^2 + ky^2 = a^2$ gives $x^2 + y^2 = \dfrac{a^2}{k} < a^2$ if $k > 1$. So the radius of the circle with equation $kx^2 + ky^2 = a^2$ is a circle of radius smaller than a centered at the origin if $k > 1$. Therefore, it lies inside the circle of radius a with equation $x^2 + y^2 = a^2$.

51. Referring to the figure in the text, we see that the distance between the two points is given by the length of the hypotenuse of the right triangle. That is, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

52. $(x - h)^2 + (y - k)^2 = r^2$; $x^2 - 2xh + h^2 + y^2 - 2ky + k^2 = r^2$. This has the form $x^2 + y^2 + Cx + Dy + E = 0$, where $C = -2h$, $D = -2k$, and $E = h^2 + k^2 - r^2$.

1.4 Straight Lines

Concept Questions page 42

1. The slope is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, where $P(x_1, y_1)$ and $P(x_2, y_2)$ are any two distinct points on the nonvertical line. The slope of a vertical line is undefined.

2. a. $y - y_1 = m(x - x_1)$ **b.** $y = mx + b$ **c.** $ax + by + c = 0$, where a and b are not both zero.

3. a. $m_1 = m_2$ **b.** $m_2 = -\dfrac{1}{m_1}$

4. a. Solving the equation for y gives $By = -Ax - C$, so $y = -\dfrac{A}{B}x - \dfrac{C}{B}$. The slope of L is the coefficient of x, $-\dfrac{A}{B}$.

b. If $B = 0$, then the equation reduces to $Ax + C = 0$. Solving this equation for x, we obtain $x = -\dfrac{C}{A}$. This is an equation of a vertical line, and we conclude that the slope of L is undefined.

Exercises page 42

1. (e) **2.** (c) **3.** (a) **4.** (d) **5.** (f) **6.** (b)

7. Referring to the figure shown in the text, we see that $m = \dfrac{2 - 0}{0 - (-4)} = \dfrac{1}{2}$.

8. Referring to the figure shown in the text, we see that $m = \dfrac{4 - 0}{0 - 2} = -2$.

9. This is a vertical line, and hence its slope is undefined.

10. This is a horizontal line, and hence its slope is 0.

11. $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{8 - 3}{5 - 4} = 5$.

12. $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{8 - 5}{3 - 4} = \dfrac{3}{-1} = -3$.

13. $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{8 - 3}{4 - (-2)} = \dfrac{5}{6}$.

14. $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-4 - (-2)}{4 - (-2)} = \dfrac{-2}{6} = -\dfrac{1}{3}$.

15. $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{d - b}{c - a}$, provided $a \neq c$.

16. $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-b - (b - 1)}{a + 1 - (-a + 1)} = -\dfrac{-b - b + 1}{a + 1 + a - 1} = \dfrac{1 - 2b}{2a}$.

17. Because the equation is already in slope-intercept form, we read off the slope $m = 4$.

a. If x increases by 1 unit, then y increases by 4 units.

b. If x decreases by 2 units, then y decreases by $4(-2) = -8$ units.

18. Rewrite the given equation in slope-intercept form: $2x + 3y = 4$, $3y = 4 - 2x$, and so $y = \frac{4}{3} - \frac{2}{3}x$.

a. Because $m = -\frac{2}{3}$, we conclude that the slope is negative.

b. Because the slope is negative, y decreases as x increases.

c. If x decreases by 2 units, then y increases by $\left(-\frac{2}{3}\right)(-2) = \frac{4}{3}$ units.

19. The slope of the line through A and B is $\dfrac{-10 - (-2)}{-3 - 1} = \dfrac{-8}{-4} = 2$. The slope of the line through C and D is $\dfrac{1 - 5}{-1 - 1} = \dfrac{-4}{-2} = 2$. Because the slopes of these two lines are equal, the lines are parallel.

20. The slope of the line through A and B is $\dfrac{-2 - 3}{2 - 2}$. Because this slope is undefined, we see that the line is vertical. The slope of the line through C and D is $\dfrac{5 - 4}{-2 - (-2)}$. Because this slope is undefined, we see that this line is also vertical. Therefore, the lines are parallel.

21. The slope of the line through A and B is $\dfrac{2 - 5}{4 - (-2)} = -\dfrac{3}{6} = -\dfrac{1}{2}$. The slope of the line through C and D is $\dfrac{6 - (-2)}{3 - (-1)} = \dfrac{8}{4} = 2$. Because the slopes of these two lines are the negative reciprocals of each other, the lines are perpendicular.

22. The slope of the line through A and B is $\dfrac{-2 - 0}{1 - 2} = \dfrac{-2}{-1} = 2$. The slope of the line through C and D is $\dfrac{4 - 2}{-8 - 4} = \dfrac{2}{-12} = -\dfrac{1}{6}$. Because the slopes of these two lines are not the negative reciprocals of each other, the lines are not perpendicular.

23. The slope of the line through the point $(1, a)$ and $(4, -2)$ is $m_1 = \dfrac{-2 - a}{4 - 1}$ and the slope of the line through $(2, 8)$ and $(-7, a + 4)$ is $m_2 = \dfrac{a + 4 - 8}{-7 - 2}$. Because these two lines are parallel, m_1 is equal to m_2. Therefore, $\dfrac{-2 - a}{3} = \dfrac{a - 4}{-9}$, $-9(-2 - a) = 3(a - 4)$, $18 + 9a = 3a - 12$, and $6a = -30$, so $a = -5$.

24. The slope of the line through the point $(a, 1)$ and $(5, 8)$ is $m_1 = \dfrac{8 - 1}{5 - a}$ and the slope of the line through $(4, 9)$ and $(a + 2, 1)$ is $m_2 = \dfrac{1 - 9}{a + 2 - 4}$. Because these two lines are parallel, m_1 is equal to m_2. Therefore, $\dfrac{7}{5 - a} = \dfrac{-8}{a - 2}$, $7(a - 2) = -8(5 - a)$, $7a - 14 = -40 + 8a$, and $a = 26$.

25. An equation of a horizontal line is of the form $y = b$. In this case $b = -3$, so $y = -3$ is an equation of the line.

26. An equation of a vertical line is of the form $x = a$. In this case $a = 0$, so $x = 0$ is an equation of the line.

27. We use the point-slope form of an equation of a line with the point $(3, -4)$ and slope $m = 2$. Thus $y - y_1 = m(x - x_1)$ becomes $y - (-4) = 2(x - 3)$. Simplifying, we have $y + 4 = 2x - 6$, or $y = 2x - 10$.

28. We use the point-slope form of an equation of a line with the point $(2, 4)$ and slope $m = -1$. Thus $y - y_1 = m(x - x_1)$, giving $y - 4 = -1(x - 2)$, $y - 4 = -x + 2$, and finally $y = -x + 6$.

29. Because the slope $m = 0$, we know that the line is a horizontal line of the form $y = b$. Because the line passes through $(-3, 2)$, we see that $b = 2$, and an equation of the line is $y = 2$.

30. We use the point-slope form of an equation of a line with the point $(1, 2)$ and slope $m = -\frac{1}{2}$. Thus $y - y_1 = m(x - x_1)$ gives $y - 2 = -\frac{1}{2}(x - 1)$, $2y - 4 = -x + 1$, $2y = -x + 5$, and $y = -\frac{1}{2}x + \frac{5}{2}$.

31. We first compute the slope of the line joining the points $(2, 4)$ and $(3, 7)$ to be $m = \dfrac{7 - 4}{3 - 2} = 3$. Using the point-slope form of an equation of a line with the point $(2, 4)$ and slope $m = 3$, we find $y - 4 = 3(x - 2)$, or $y = 3x - 2$.

32. We first compute the slope of the line joining the points $(2, 1)$ and $(2, 5)$ to be $m = \dfrac{5 - 1}{2 - 2}$. Because this slope is undefined, we see that the line must be a vertical line of the form $x = a$. Because it passes through $(2, 5)$, we see that $x = 2$ is the equation of the line.

33. We first compute the slope of the line joining the points $(1, 2)$ and $(-3, -2)$ to be $m = \dfrac{-2 - 2}{-3 - 1} = \dfrac{-4}{-4} = 1$. Using the point-slope form of an equation of a line with the point $(1, 2)$ and slope $m = 1$, we find $y - 2 = x - 1$, or $y = x + 1$.

34. We first compute the slope of the line joining the points $(-1, -2)$ and $(3, -4)$ to be $m = \dfrac{-4 - (-2)}{3 - (-1)} = \dfrac{-2}{4} = -\dfrac{1}{2}$. Using the point-slope form of an equation of a line with the point $(-1, -2)$ and slope $m = -\frac{1}{2}$, we find $y - (-2) = -\frac{1}{2}[x - (-1)]$, $y + 2 = -\frac{1}{2}(x + 1)$, and finally $y = -\frac{1}{2}x - \frac{5}{2}$.

35. We use the slope-intercept form of an equation of a line: $y = mx + b$. Because $m = 3$ and $b = 4$, the equation is $y = 3x + 4$.

36. We use the slope-intercept form of an equation of a line: $y = mx + b$. Because $m = -2$ and $b = -1$, the equation is $y = -2x - 1$.

37. We use the slope-intercept form of an equation of a line: $y = mx + b$. Because $m = 0$ and $b = 5$, the equation is $y = 5$.

38. We use the slope-intercept form of an equation of a line:$y = mx + b$. Because $m = -\frac{1}{2}$, and $b = \frac{3}{4}$, the equation is $y = -\frac{1}{2}x + \frac{3}{4}$.

39. We first write the given equation in the slope-intercept form: $x - 2y = 0$, so $-2y = -x$, or $y = \frac{1}{2}x$. From this equation, we see that $m = \frac{1}{2}$ and $b = 0$.

40. We write the equation in slope-intercept form: $y - 2 = 0$, so $y = 2$. From this equation, we see that $m = 0$ and $b = 2$.

41. We write the equation in slope-intercept form: $2x - 3y - 9 = 0$, $-3y = -2x + 9$, and $y = \frac{2}{3}x - 3$. From this equation, we see that $m = \frac{2}{3}$ and $b = -3$.

42. We write the equation in slope-intercept form: $3x - 4y + 8 = 0$, $-4y = -3x - 8$, and $y = \frac{3}{4}x + 2$. From this equation, we see that $m = \frac{3}{4}$ and $b = 2$.

43. We write the equation in slope-intercept form: $2x + 4y = 14$, $4y = -2x + 14$, and $y = -\frac{2}{4}x + \frac{14}{4} = -\frac{1}{2}x + \frac{7}{2}$. From this equation, we see that $m = -\frac{1}{2}$ and $b = \frac{7}{2}$.

44. We write the equation in the slope-intercept form: $5x + 8y - 24 = 0$, $8y = -5x + 24$, and $y = -\frac{5}{8}x + 3$. From this equation, we conclude that $m = -\frac{5}{8}$ and $b = 3$.

45. We first write the equation $2x - 4y - 8 = 0$ in slope-intercept form: $2x - 4y - 8 = 0$, $4y = 2x - 8$, $y = \frac{1}{2}x - 2$. Now the required line is parallel to this line, and hence has the same slope. Using the point-slope form of an equation of a line with $m = \frac{1}{2}$ and the point $(-2, 2)$, we have $y - 2 = \frac{1}{2}[x - (-2)]$ or $y = \frac{1}{2}x + 3$.

46. We first write the equation $3x + 4y - 22 = 0$ in slope-intercept form: $3x + 4y - 22 = 0$, so $4y = -3x + 22$ and $y = -\frac{3}{4}x + \frac{11}{2}$ Now the required line is perpendicular to this line, and hence has slope $\frac{4}{3}$ (the negative reciprocal of $-\frac{3}{4}$). Using the point-slope form of an equation of a line with $m = \frac{4}{3}$ and the point $(2, 4)$, we have $y - 4 = \frac{4}{3}(x - 2)$, or $y = \frac{4}{3}x + \frac{4}{3}$.

47. The midpoint of the line segment joining $P_1(-2, -4)$ and $P_2(3, 6)$ is $M\left(\frac{-2+3}{2}, \frac{-4+6}{2}\right)$ or $M\left(\frac{1}{2}, 1\right)$.

Using the point-slope form of the equation of a line with $m = -2$, we have $y - 1 = -2\left(x - \frac{1}{2}\right)$ or $y = -2x + 2$.

48. The midpoint of the line segment joining $P_1(-1, -3)$ and $P_2(3, 3)$ is $M_1\left(\frac{-1+3}{2}, \frac{-3+3}{2}\right)$ or $M_1(1, 0)$.

The midpoint of the line segment joining $P_3(-2, 3)$ and $P_4(2, -3)$ is $M_2\left(\frac{-2+2}{2}, \frac{3-3}{2}\right)$ or $M_2(0, 0)$.

The slope of the required line is $m = \frac{0-0}{1-0} = 0$, so an equation of the line is $y - 0 = 0(x - 0)$ or $y = 0$.

49. A line parallel to the x-axis has slope 0 and is of the form $y = b$. Because the line is 6 units below the axis, it passes through $(0, -6)$ and its equation is $y = -6$.

50. Because the required line is parallel to the line joining $(2, 4)$ and $(4, 7)$, it has slope $m = \frac{7-4}{4-2} = \frac{3}{2}$. We also know that the required line passes through the origin $(0, 0)$. Using the point-slope form of an equation of a line, we find $y - 0 = \frac{3}{2}(x - 0)$, or $y = \frac{3}{2}x$.

51. We use the point-slope form of an equation of a line to obtain $y - b = 0(x - a)$, or $y = b$.

52. Because the line is parallel to the x-axis, its slope is 0 and its equation has the form $y = b$. We know that the line passes through $(-3, 4)$, so the required equation is $y = 4$.

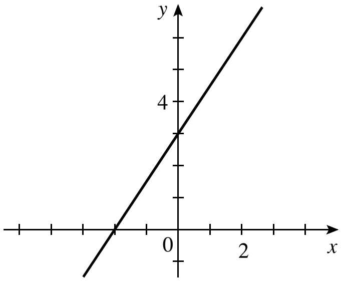
53. Because the required line is parallel to the line joining $(-3, 2)$ and $(6, 8)$, it has slope $m = \dfrac{8-2}{6-(-3)} = \dfrac{6}{9} = \dfrac{2}{3}$. We also know that the required line passes through $(-5, -4)$. Using the point-slope form of an equation of a line, we find $y - (-4) = \frac{2}{3}[x - (-5)]$, $y = \frac{2}{3}x + \frac{10}{3} - 4$, and finally $y = \frac{2}{3}x - \frac{2}{3}$.

54. Because the slope of the line is undefined, it has the form $x = a$. Furthermore, since the line passes through (a, b), the required equation is $x = a$.

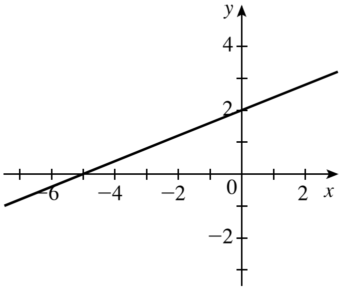
55. Because the point $(-3, 5)$ lies on the line $kx + 3y + 9 = 0$, it satisfies the equation. Substituting $x = -3$ and $y = 5$ into the equation gives $-3k + 15 + 9 = 0$, or $k = 8$.

56. Because the point $(2, -3)$ lies on the line $-2x + ky + 10 = 0$, it satisfies the equation. Substituting $x = 2$ and $y = -3$ into the equation gives $-2(2) + (-3)k + 10 = 0$, $-4 - 3k + 10 = 0$, $-3k = -6$, and finally $k = 2$.

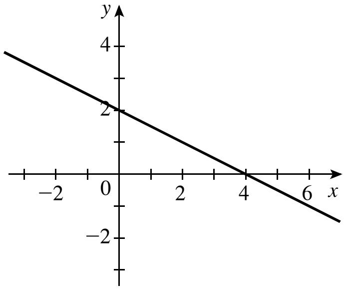
57. $3x - 2y + 6 = 0$. Setting $y = 0$, we have $3x + 6 = 0$ or $x = -2$, so the x-intercept is -2. Setting $x = 0$, we have $-2y + 6 = 0$ or $y = 3$, so the y-intercept is 3.



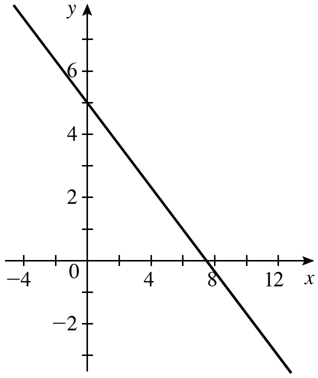
58. $2x - 5y + 10 = 0$. Setting $y = 0$, we have $2x + 10 = 0$ or $x = -5$, so the x-intercept is -5. Setting $x = 0$, we have $-5y + 10 = 0$ or $y = 2$, so the y-intercept is 2.



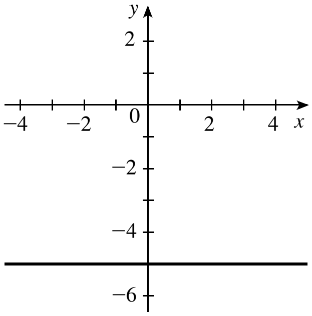
59. $x + 2y - 4 = 0$. Setting $y = 0$, we have $x - 4 = 0$ or $x = 4$, so the x-intercept is 4. Setting $x = 0$, we have $2y - 4 = 0$ or $y = 2$, so the y-intercept is 2.



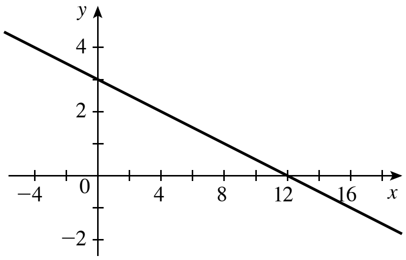
60. $2x + 3y - 15 = 0$. Setting $y = 0$, we have $2x - 15 = 0$, so the x-intercept is $\frac{15}{2}$. Setting $x = 0$, we have $3y - 15 = 0$, so the y-intercept is 5.



61. $y + 5 = 0$. Setting $y = 0$, we have $0 + 5 = 0$, which has no solution, so there is no x-intercept. Setting $x = 0$, we have $y + 5 = 0$ or $y = -5$, so the y-intercept is -5.



62. $-2x - 8y + 24 = 0$. Setting $y = 0$, we have $-2x + 24 = 0$ or $x = 12$, so the x-intercept is 12. Setting $x = 0$, we have $-8y + 24 = 0$ or $y = 3$, so the y-intercept is 3.



63. Because the line passes through the points $(a, 0)$ and $(0, b)$, its slope is $m = \frac{b-0}{0-a} = -\frac{b}{a}$. Then, using the point-slope form of an equation of a line with the point $(a, 0)$, we have $y - 0 = -\frac{b}{a}(x - a)$ or $y = -\frac{b}{a}x + b$, which may be written in the form $\frac{b}{a}x + y = b$. Multiplying this last equation by $\frac{1}{b}$, we have $\frac{x}{a} + \frac{y}{b} = 1$.

64. Using the equation $\frac{x}{a} + \frac{y}{b} = 1$ with $a = 3$ and $b = 4$, we have $\frac{x}{3} + \frac{y}{4} = 1$. Then $4x + 3y = 12$, so $3y = 12 - 4x$ and thus $y = -\frac{4}{3}x + 4$.

65. Using the equation $\frac{x}{a} + \frac{y}{b} = 1$ with $a = -2$ and $b = -4$, we have $-\frac{x}{2} - \frac{y}{4} = 1$. Then $-4x - 2y = 8$, $2y = -8 - 4x$, and finally $y = -2x - 4$.

66. Using the equation $\frac{x}{a} + \frac{y}{b} = 1$ with $a = -\frac{1}{2}$ and $b = \frac{3}{4}$, we have $\frac{x}{-1/2} + \frac{y}{3/4} = 1$, $\frac{3}{4}x - \frac{1}{2}y = \left(-\frac{1}{2}\right)\left(\frac{3}{4}\right)$, $\frac{1}{2}y = -\frac{3}{4}x - \frac{3}{8}$, and finally $y = 2\left(\frac{3}{4}x + \frac{3}{8}\right) = \frac{3}{2}x + \frac{3}{4}$.

67. Using the equation $\frac{x}{a} + \frac{y}{b} = 1$ with $a = 4$ and $b = -\frac{1}{2}$, we have $\frac{x}{4} + \frac{y}{-1/2} = 1$, $-\frac{1}{4}x + 2y = -1$, $2y = \frac{1}{4}x - 1$, and so $y = \frac{1}{8}x - \frac{1}{2}$.

68. The slope of the line passing through A and B is $m = \frac{-2-7}{2-(-1)} = -\frac{9}{3} = -3$, and the slope of the line passing through B and C is $m = \frac{-9-(-2)}{5-2} = -\frac{7}{3}$. Because the slopes are not equal, the points do not lie on the same line.

69. The slope of the line passing through A and B is $m = \frac{7-1}{1-(-2)} = \frac{6}{3} = 2$, and the slope of the line passing through B and C is $m = \frac{13-7}{4-1} = \frac{6}{3} = 2$. Because the slopes are equal, the points lie on the same line.

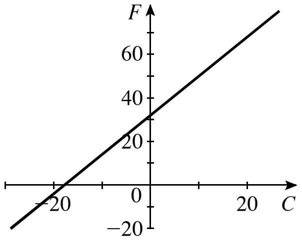
70. The slope of the line L passing through $P_1\ (1.2, -9.04)$ and $P_2\ (2.3, -5.96)$ is $m = \dfrac{-5.96 - (-9.04)}{2.3 - 1.2} = 2.8$, so an equation of L is $y - (-9.04) = 2.8\,(x - 1.2)$ or $y = 2.8x - 12.4$.

Substituting $x = 4.8$ into this equation gives $y = 2.8\,(4.8) - 12.4 = 1.04$. This shows that the point $P_3\ (4.8, 1.04)$ lies on L. Next, substituting $x = 7.2$ into the equation gives $y = 2.8\,(7.2) - 12.4 = 7.76$, which shows that the point $P_4\ (7.2, 7.76)$ also lies on L. We conclude that John's claim is valid.

71. The slope of the line L passing through $P_1\ (1.8, -6.44)$ and $P_2\ (2.4, -5.72)$ is $m = \dfrac{-5.72 - (-6.44)}{2.4 - 1.8} = 1.2$, so an equation of L is $y - (-6.44) = 1.2\,(x - 1.8)$ or $y = 1.2x - 8.6$.

Substituting $x = 5.0$ into this equation gives $y = 1.2\,(5) - 8.6 = -2.6$. This shows that the point $P_3\ (5.0, -2.72)$ does not lie on L, and we conclude that Alison's claim is not valid.

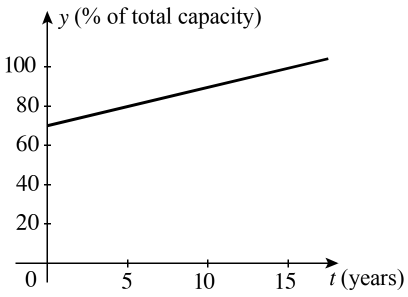
72. a.



b. The slope is $\frac{9}{5}$. It represents the change in °F per unit change in °C.

c. The F-intercept of the line is 32. It corresponds to 0°, so it is the freezing point in °F.

73. a.



b. The slope is 1.9467 and the y-intercept is 70.082.

c. The output is increasing at the rate of 1.9467% per year. The output at the beginning of 1990 was 70.082%.

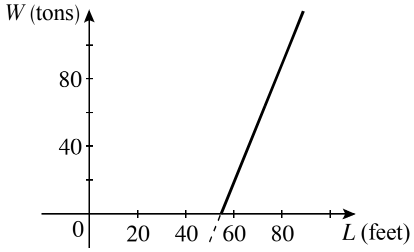
d. We solve the equation $1.9467t + 70.082 = 100$, obtaining $t \approx 15.37$. We conclude that the plants were generating at maximum capacity during April 2005.

74. a. $y = 0.0765x$ **b.** \$0.0765 **c.** $0.0765\,(65{,}000) = 4972.50$, or \$4972.50.

75. a. $y = 0.55x$ **b.** Solving the equation $1100 = 0.55x$ for x, we have $x = \dfrac{1100}{0.55} = 2000$.

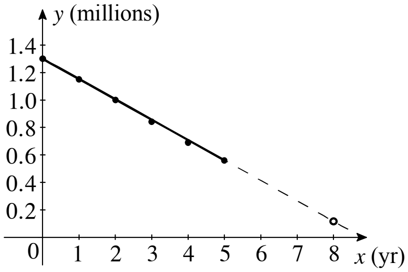
76. a. Substituting $L = 80$ into the given equation, we have $W = 3.51\,(80) - 192 = 280.8 - 192 = 88.8$, or 88.8 British tons.

b.



77. Using the points $(0, 0.68)$ and $(10, 0.80)$, we see that the slope of the required line is $m = \dfrac{0.80 - 0.68}{10 - 0} = \dfrac{0.12}{10} = 0.012$. Next, using the point-slope form of the equation of a line, we have $y - 0.68 = 0.012\,(t - 0)$ or $y = 0.012t + 0.68$. Therefore, when $t = 18$, we have $y = 0.012\,(18) + 0.68 = 0.896$, or 89.6%. That is, in 2008 women's wages were expected to be 89.6% of men's wages.

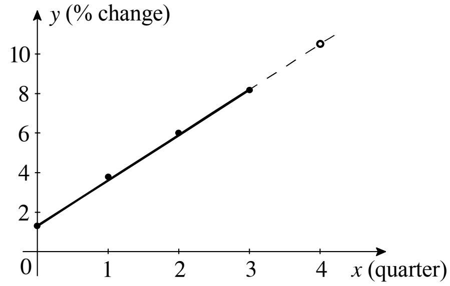
78. a, b.



c. The slope of L is $m = \dfrac{0.56 - 1.30}{5 - 0} = -0.148$, so an equation of L is $y - 1.3 = -0.148\,(x - 0)$ or $y = -0.148x + 1.3$.

d. The number of pay phones in 2012 is estimated to be $-0.148\,(8) + 1.3$, or approximately 116,000.

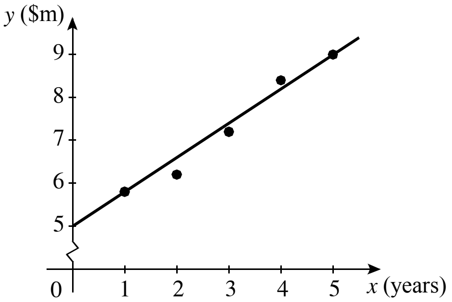
79. a, b.



c. The slope of L is $m = \dfrac{8.2 - 1.3}{3 - 0} = 2.3$, so an equation of L is $y - 1.3 = 2.3\,(x - 0)$ or $y = 2.3x + 1.3$.

d. The change in spending in the first quarter of 2014 is estimated to be $2.3\,(4) + 1.3$, or 10.5%.

80. a, b.

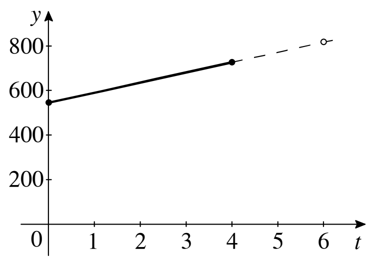


c. The slope of L is $m = \dfrac{9.0 - 5.8}{5 - 1} = \dfrac{3.2}{4} = 0.8$. Using the point-slope form of an equation of a line, we have $y - 5.8 = 0.8\,(x - 1) = 0.8x - 0.8$, or $y = 0.8x + 5$.

d. Using the equation from part (c) with $x = 9$, we have $y = 0.8\,(9) + 5 = 12.2$, or \$12.2 million.

81. a. The slope of the line L passing through $A\,(0, 545)$ and $B\,(4, 726)$ is $m = \dfrac{726 - 545}{4 - 0} = \dfrac{181}{4}$, so an equation of L is $y - 545 = \frac{181}{4}\,(t - 0)$ or $y = \frac{181}{4}t + 545$.

b.



c. The number of corporate fraud cases pending at the beginning of 2014 is estimated to be $\frac{181}{4}\,(6) + 545$, or approximately 817.

82. a. The slope of the line through $P_1\,(0, 27)$ and $P_2\,(1, 29)$ is $m_1 = \dfrac{29 - 27}{1 - 0} = 2$, which is equal to the slope of the line through $P_2\,(1, 29)$ and $P_3\,(2, 31)$, which is $m_2 = \dfrac{31 - 29}{1 - 0} = 2$. Thus, the three points lie on the line L.

b. The percentage is of moviegoers who use social media to chat about movies in 2014 is estimated to be $31 + 2\,(2)$, or 35%.

c. $y - 27 = 2\,(x - 0)$, so $y = 2x + 27$. The estimate for 2014 ($t = 4$) is $2\,(4) + 27 = 35$, as found in part (b).

83. True. The slope of the line is given by $-\frac{2}{4} = -\frac{1}{2}$.

84. True. The slope of the line $Ax + By + C = 0$ is $-\frac{A}{B}$. (Write it in slope-intercept form.) Similarly, the slope of the line $ax + by + c = 0$ is $-\frac{a}{b}$. They are parallel if and only if $-\frac{A}{B} = -\frac{a}{b}$, that is, if $Ab = aB$, or $Ab - aB = 0$.

85. False. Let the slope of L_1 be $m_1 > 0$. Then the slope of L_2 is $m_2 = -\frac{1}{m_1} < 0$.

86. True. The slope of the line $ax + by + c_1 = 0$ is $m_1 = -\frac{a}{b}$. The slope of the line $bx - ay + c_2 = 0$ is $m_2 = \frac{b}{a}$. Because $m_1 m_2 = -1$, the straight lines are indeed perpendicular.

87. True. Set $y = 0$ and we have $Ax + C = 0$ or $x = -C/A$, and this is where the line intersects the x-axis.

88. Yes. A straight line with slope zero ($m = 0$) is a horizontal line, whereas a straight line whose slope does not exist is a vertical line (m cannot be computed).

89. Writing each equation in the slope-intercept form, we have $y = -\frac{a_1}{b_1}x - \frac{c_1}{b_1}$ $(b_1 \neq 0)$ and $y = -\frac{a_2}{b_2}x - \frac{c_2}{b_2}$ $(b_2 \neq 0)$. Because two lines are parallel if and only if their slopes are equal, we see that the lines are parallel if and only if $-\frac{a_1}{b_1} = -\frac{a_2}{b_2}$, or $a_1 b_2 - b_1 a_2 = 0$.

90. The slope of L_1 is $m_1 = \frac{b - 0}{1 - 0} = b$. The slope of L_2 is $m_2 = \frac{c - 0}{1 - 0} = c$. Applying the Pythagorean theorem to ΔOAC and ΔOCB gives $(OA)^2 = 1^2 + b^2$ and $(OB)^2 = 1^2 + c^2$. Adding these equations and applying the Pythagorean theorem to ΔOBA gives $(AB)^2 = (OA)^2 + (OB)^2 = 1^2 + b^2 + 1^2 + c^2 = 2 + b^2 + c^2$. Also, $(AB)^2 = (b - c)^2$, so $(b - c)^2 = 2 + b^2 + c^2$, $b^2 - 2bc + c^2 = 2 + b^2 + c^2$, and $-2bc = 2$, $1 = -bc$. Finally, $m_1 m_2 = b \cdot c = bc = -1$, as was to be shown.

CHAPTER 1 Concept Review Questions page 48

1. ordered, abscissa or x-coordinate, ordinate or y-coordinate

2. **a.** x, y **b.** third

3. $\sqrt{(c-a)^2+(d-b)^2}$

4. $(x-a)^2+(y-b)^2=r^2$

5. **a.** $\dfrac{y_2-y_1}{x_2-x_1}$ **b.** undefined **c.** 0 **d.** positive

6. $m_1=m_2, m_1=-\dfrac{1}{m_2}$

7. **a.** $y-y_1=m\,(x-x_1)$, point-slope form **b.** $y=mx+b$, slope-intercept

8. **a.** $Ax+By+C=0$, where A and B are not both zero **b.** $-a/b$

CHAPTER 1 Review Exercises page 48

1. Adding x to both sides yields $3 \le 3x + 9$, $3x \ge -6$, or $x \ge -2$. We conclude that the solution set is $[-2, \infty)$.

2. $-2 \le 3x + 1 \le 7$ implies $-3 \le 3x \le 6$, or $-1 \le x \le 2$, and so the solution set is $[-1, 2]$.

3. The inequalities imply $x > 5$ or $x < -4$, so the solution set is $(-\infty, -4) \cup (5, \infty)$.

4. $2x^2 > 50$ is equivalent to $x^2 > 25$, so either $x > 5$ or $x < -5$ and the solution set is $(-\infty, -5) \cup (5, \infty)$.

5. $|-5 + 7| + |-2| = |2| + |-2| = 2 + 2 = 4$.

6. $\left|\dfrac{5-12}{-4-3}\right| = \dfrac{|5-12|}{|-7|} = \dfrac{|-7|}{7} = \dfrac{7}{7} = 1$.

7. $|2\pi - 6| - \pi = 2\pi - 6 - \pi = \pi - 6$.

8. $\left|\sqrt{3} - 4\right| + \left|4 - 2\sqrt{3}\right| = \left(4 - \sqrt{3}\right) + \left(4 - 2\sqrt{3}\right)$
$= 8 - 3\sqrt{3}$.

9. $\left(\dfrac{9}{4}\right)^{3/2} = \dfrac{9^{3/2}}{4^{3/2}} = \dfrac{27}{8}$.

10. $\dfrac{5^6}{5^4} = 5^{6-4} = 5^2 = 25$.

11. $(3 \cdot 4)^{-2} = 12^{-2} = \dfrac{1}{12^2} = \dfrac{1}{144}$.

12. $(-8)^{5/3} = \left(-8^{1/3}\right)^5 = (-2)^5 = -32$.

13. $\dfrac{\left(3 \cdot 2^{-3}\right)\left(4 \cdot 3^5\right)}{2 \cdot 9^3} = \dfrac{3 \cdot 2^{-3} \cdot 2^2 \cdot 3^5}{2 \cdot \left(3^2\right)^3} = \dfrac{2^{-1} \cdot 3^6}{2 \cdot 3^6} = \dfrac{1}{4}$.

14. $\dfrac{3\sqrt[3]{54}}{\sqrt[3]{18}} = \dfrac{3 \cdot \left(2 \cdot 3^3\right)^{1/3}}{\left(2 \cdot 3^2\right)^{1/3}} = \dfrac{3^2 \cdot 2^{1/3}}{2^{1/3} \cdot 3^{2/3}} = 3^{4/3} = 3\sqrt[3]{3}$.

15. $\dfrac{4\left(x^2 + y\right)^3}{x^2 + y} = 4\left(x^2 + y\right)^2$.

16. $\dfrac{a^6 b^{-5}}{\left(a^3 b^{-2}\right)^{-3}} = \dfrac{a^6 b^{-5}}{a^{-9} b^6} = \dfrac{a^{15}}{b^{11}}$.

17. $\dfrac{\sqrt[4]{16x^5yz}}{\sqrt[4]{81xyz^5}} = \dfrac{\left(2^4x^5yz\right)^{1/4}}{\left(3^4xyz^5\right)^{1/4}} = \dfrac{2x^{5/4}y^{1/4}z^{1/4}}{3x^{1/4}y^{1/4}z^{5/4}} = \dfrac{2x}{3z}$.

18. $\left(2x^3\right)\left(-3x^{-2}\right)\left(\frac{1}{6}x^{-1/2}\right) = -x^{1/2}$.

19. $\left(\dfrac{3xy^2}{4x^3y}\right)^{-2}\left(\dfrac{3xy^3}{2x^2}\right)^3 = \left(\dfrac{3y}{4x^2}\right)^{-2}\left(\dfrac{3y^3}{2x}\right)^3 = \left(\dfrac{4x^2}{3y}\right)^2\left(\dfrac{3y^3}{2x}\right)^3 = \dfrac{\left(16x^4\right)\left(27y^9\right)}{\left(9y^2\right)\left(8x^3\right)} = 6xy^7$.

20. $\sqrt[3]{81x^5y^{10}}\sqrt[3]{9xy^2} = \sqrt[3]{\left(3^4x^5y^{10}\right)\left(3^2xy^2\right)} = \left(3^6x^6y^{12}\right)^{1/3} = 3^2x^2y^4 = 9x^2y^4$.

21. $-2\pi^2r^3 + 100\pi r^2 = -2\pi r^2\left(\pi r - 50\right)$.

22. $2v^3w + 2vw^3 + 2u^2vw = 2vw\left(v^2 + w^2 + u^2\right)$.

23. $16 - x^2 = 4^2 - x^2 = (4 - x)(4 + x)$.

24. $12t^3 - 6t^2 - 18t = 6t\left(2t^2 - t - 3\right) = 6t(2t - 3)(t + 1)$.

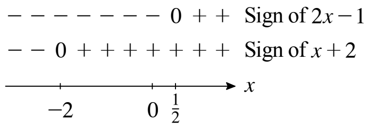
25. $8x^2 + 2x - 3 = (4x + 3)(2x - 1) = 0$, so $x = -\frac{3}{4}$ and $x = \frac{1}{2}$ are the roots of the equation.

26. $-6x^2 - 10x + 4 = 0$, $3x^2 + 5x - 2 = (3x - 1)(x + 2) = 0$, and so $x = -2$ or $x = \frac{1}{3}$.

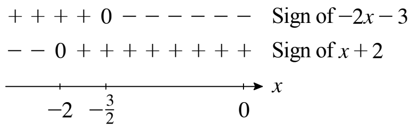
27. $-x^3 - 2x^2 + 3x = -x\left(x^2 + 2x - 3\right) = -x\,(x+3)\,(x-1) = 0$, and so the roots of the equation are $x = 0$, $x = -3$, and $x = 1$.

28. $2x^4 + x^2 = 1$. If we let $y = x^2$, we can write the equation as $2y^2 + y - 1 = (2y-1)\,(y+1) = 0$, giving $y = \frac{1}{2}$ or $y = -1$. We reject the second root since $y = x^2$ must be nonnegative. Therefore, $x^2 = \frac{1}{2}$, and so $x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$.

29. Factoring the given expression, we have $(2x-1)\,(x+2) \le 0$. From the sign diagram, we conclude that the given inequality is satisfied when $-2 \le x \le \frac{1}{2}$.



30. $\dfrac{1}{x+2} > 2$ gives $\dfrac{1}{x+2} - 2 > 0$, $\dfrac{1 - 2x - 4}{x+2} > 0$, and finally $\dfrac{-2x-3}{x+2} > 0$. From the sign diagram, we see that the given inequality is satisfied when $-2 < x < -\frac{3}{2}$.



31. The given inequality is equivalent to $|2x - 3| < 5$ or $-5 < 2x - 3 < 5$. Thus, $-2 < 2x < 8$, or $-1 < x < 4$.

32. The given equation implies that either $\dfrac{x+1}{x-1} = 5$ or $\dfrac{x+1}{x-1} = -5$. Solving the first equality, we have $x + 1 = 5\,(x-1) = 5x - 5$, $-4x = -6$, and $x = \frac{3}{2}$. Similarly, we solve the second equality and obtain $x + 1 = -5\,(x-1) = -5x + 5$, $6x = 4$, and $x = \frac{2}{3}$. Thus, the two values of x that satisfy the equation are $x = \frac{3}{2}$ and $x = \frac{2}{3}$.

33. We use the quadratic formula to solve the equation $x^2 - 2x - 5 = 0$. Here $a = 1$, $b = -2$, and $c = -5$, so

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4\,(1)\,(-5)}}{2\,(1)} = \frac{2 \pm \sqrt{24}}{2} = 1 \pm \sqrt{6}.$$

34. We use the quadratic formula to solve the equation $2x^2 + 8x + 7 = 0$. Here $a = 2$, $b = 8$, and $c = 7$, so

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{8^2 - 4\,(2)\,(7)}}{4} = \frac{-8 \pm 2\sqrt{2}}{4} = -2 \pm \tfrac{1}{2}\sqrt{2}.$$

35. $$\frac{(t+6)\,(60) - (60t + 180)}{(t+6)^2} = \frac{60t + 360 - 60t - 180}{(t+6)^2} = \frac{180}{(t+6)^2}.$$

36. $$\frac{6x}{2\left(3x^2+2\right)} + \frac{1}{4\,(x+2)} = \frac{(6x)\,(2)\,(x+2) + \left(3x^2+2\right)}{4\left(3x^2+2\right)(x+2)} = \frac{12x^2 + 24x + 3x^2 + 2}{4\left(3x^2+2\right)(x+2)} = \frac{15x^2 + 24x + 2}{4\left(3x^2+2\right)(x+2)}.$$

37. $$\frac{2}{3}\left(\frac{4x}{2x^2-1}\right) + 3\left(\frac{3}{3x-1}\right) = \frac{8x}{3\left(2x^2-1\right)} + \frac{9}{3x-1} = \frac{8x\,(3x-1) + 27\left(2x^2-1\right)}{3\left(2x^2-1\right)(3x-1)} = \frac{78x^2 - 8x - 27}{3\left(2x^2-1\right)(3x-1)}.$$

38. $$\frac{-2x}{\sqrt{x+1}} + 4\sqrt{x+1} = \frac{-2x + 4\,(x+1)}{\sqrt{x+1}} = \frac{2\,(x+2)}{\sqrt{x+1}}.$$

39. $$\frac{\sqrt{x}-1}{x-1} = \frac{\sqrt{x}-1}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \frac{\left(\sqrt{x}\right)^2 - 1}{(x-1)\left(\sqrt{x}+1\right)} = \frac{x-1}{(x-1)\left(\sqrt{x}+1\right)} = \frac{1}{\sqrt{x}+1}.$$

40. $\dfrac{\sqrt{x}-1}{2\sqrt{x}} = \dfrac{\sqrt{x}-1}{2\sqrt{x}} \cdot \dfrac{\sqrt{x}}{\sqrt{x}} = \dfrac{x-\sqrt{x}}{2x}$.

41. The distance is $d = \sqrt{[1-(-2)]^2 + [-7-(-3)]^2} = \sqrt{3^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$.

42. The distance is $d = \sqrt{(2-6)^2 + (6-9)^2} = \sqrt{16+9} = \sqrt{25} = 5$.

43. The distance is $d = \sqrt{\left(-\frac{1}{2}-\frac{1}{2}\right)^2 + \left(2\sqrt{3}-\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$.

44. An equation is $x = -2$.

45. An equation is $y = 4$.

46. The slope of L is $m = \dfrac{\frac{7}{2}-4}{3-(-2)} = -\dfrac{1}{10}$, and an equation of L is $y - 4 = -\frac{1}{10}[x-(-2)] = -\frac{1}{10}x - \frac{1}{5}$, or $y = -\frac{1}{10}x + \frac{19}{5}$. The general form of this equation is $x + 10y - 38 = 0$.

47. The line passes through the points $(-2, 4)$ and $(3, 0)$, so its slope is $m = \dfrac{4-0}{-2-3} = -\dfrac{4}{5}$. An equation is $y - 0 = -\frac{4}{5}(x-3)$, or $y = -\frac{4}{5}x + \frac{12}{5}$.

48. Writing the given equation in the form $y = \frac{5}{2}x - 3$, we see that the slope of the given line is $\frac{5}{2}$. Thus, an equation is $y - 4 = \frac{5}{2}(x+2)$, or $y = \frac{5}{2}x + 9$. The general form of this equation is $5x - 2y + 18 = 0$.

49. Writing the given equation in the form $y = -\frac{4}{3}x + 2$, we see that the slope of the given line is $-\frac{4}{3}$. Therefore, the slope of the required line is $\frac{3}{4}$ and an equation of the line is $y - 4 = \frac{3}{4}(x+2)$ or $y = \frac{3}{4}x + \frac{11}{2}$.

50. Rewriting the given equation in slope-intercept form, we have $4y = -3x + 8$ or $y = -\frac{3}{4}x + 2$. We conclude that the slope of the required line is $-\frac{3}{4}$. Using the point-slope form of the equation of a line with the point $(2, 3)$ and slope $-\frac{3}{4}$, we obtain $y - 3 = -\frac{3}{4}(x-2)$, and so $y = -\frac{3}{4}x + \frac{6}{4} + 3 = -\frac{3}{4}x + \frac{9}{2}$. The general form of this equation is $3x + 4y - 18 = 0$.

51. The slope of the line joining the points $(-3, 4)$ and $(2, 1)$ is $m = \dfrac{1-4}{2-(-3)} = -\dfrac{3}{5}$. Using the point-slope form of the equation of a line with the point $(-1, 3)$ and slope $-\frac{3}{5}$, we have $y - 3 = -\frac{3}{5}[x-(-1)]$. Therefore, $y = -\frac{3}{5}(x+1) + 3 = -\frac{3}{5}x + \frac{12}{5}$.

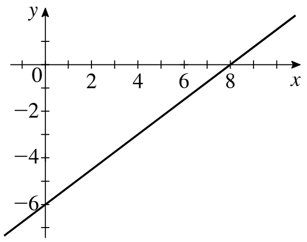
52. The slope of the line passing through $(-2, -4)$ and $(1, 5)$ is $m = \dfrac{5-(-4)}{1-(-2)} = \dfrac{9}{3} = 3$, so the required line is $y - (-2) = 3[x-(-3)]$. Simplifying, this is equivalent to $y + 2 = 3x + 9$, or $y = 3x + 7$.

53. Rewriting the given equation in the slope-intercept form $y = \frac{2}{3}x - 8$, we see that the slope of the line with this equation is $\frac{2}{3}$. The slope of a line perpendicular to this line is thus $-\frac{3}{2}$. Using the point-slope form of the equation of a line with the point $(-2, -4)$ and slope $-\frac{3}{2}$, we have $y - (-4) = -\frac{3}{2}[x-(-2)]$ or $y = -\frac{3}{2}x - 7$. The general form of this equation is $3x + 2y + 14 = 0$.

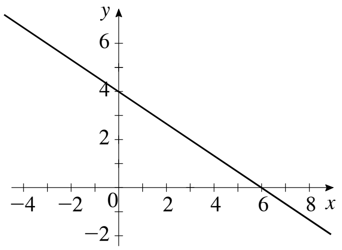
54. Substituting $x = -1$ and $y = -\frac{5}{4}$ into the left-hand side of the equation gives $6(-1) - 8\left(-\frac{5}{4}\right) - 16 = -12$. The equation is not satisfied, and so we conclude that the point $\left(-1, -\frac{5}{4}\right)$ does not lie on the line $6x - 8y - 16 = 0$.

55. Substituting $x = 2$ and $y = -4$ into the equation, we obtain $2(2) + k(-4) = -8$, so $-4k = -12$ and $k = 3$.

56. Setting $x = 0$ gives $y = -6$ as the y-intercept. Setting $y = 0$ gives $x = 8$ as the x-intercept. The graph of $3x - 4y = 24$ is shown.



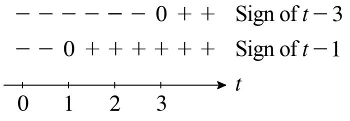
57. Using the point-slope form of an equation of a line, we have $y - 2 = -\frac{2}{3}(x - 3)$ or $y = -\frac{2}{3}x + 4$. If $y = 0$, then $x = 6$, and if $x = 0$, then $y = 4$. A sketch of the line is shown.



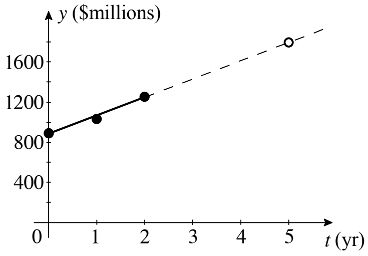
58. Simplifying $2(1.5C + 80) \leq 2(2.5C - 20)$, we obtain $1.5C + 80 \leq 2.5C - 20$, so $C \geq 100$ and the minimum cost is \$100.

59. $3(2R - 320) \leq 3R + 240$ gives $6R - 960 \leq 3R + 240$, $3R \leq 1200$ and finally $R \leq 400$. We conclude that the maximum revenue is \$400.

60. We solve the inequality $-16t^2 + 64t + 80 \geq 128$, obtaining $-16t^2 + 64t - 48 \geq 0$, $t^2 - 4t + 3 \leq 0$, and $(t - 3)(t - 1) \leq 0$. From the sign diagram, we see that the required solution is $[1, 3]$. Thus, the stone is 128 ft or higher off the ground between 1 and 3 seconds after it was thrown.



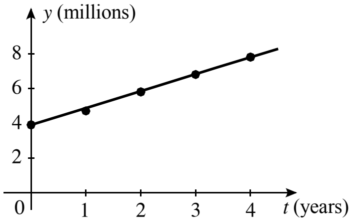
61. a, b.



c. The slope of L is $\dfrac{1251 - 887}{2 - 0} = 182$, so an equation of L is $y - 887 = 182(t - 0)$ or $y = 182t + 887$.

d. The amount consumers are projected to spend on Cyber Monday, 2014 ($t = 5$) is $182(5) + 887$, or \$1.797 billion.

62. a, b.



c. $P_1\ (0, 3.9)$ and $P_2\ (4, 7.8)$, so $m = \dfrac{7.8 - 3.9}{4 - 0} = \dfrac{3.9}{4} = 0.975$. Thus, $y - 3.9 = 0.975\,(t - 0)$, or $y = 0.975t + 3.9$.

d. If $t = 3$, then $y = 0.975\,(3) + 3.9 = 6.825$. Thus, the number of systems installed in 2005 (when $t = 3$) is 6,825,000, which is close to the projected value of 6.8 million.

CHAPTER 1 Before Moving On... page 50

1. a. $\left|\pi - 2\sqrt{3}\right| - \left|\sqrt{3} - \sqrt{2}\right| = -\left(\pi - 2\sqrt{3}\right) - \left(\sqrt{3} - \sqrt{2}\right) = \sqrt{3} + \sqrt{2} - \pi.$

b. $\left[\left(-\frac{1}{3}\right)^{-3}\right]^{1/3} = \left(-\frac{1}{3}\right)^{(-3)\left(\frac{1}{3}\right)} = \left(-\frac{1}{3}\right)^{-1} = -3.$

2. a. $\sqrt[3]{64x^6} \cdot \sqrt{9y^2x^6} = \left(4x^2\right)\left(3yx^3\right) = 12x^5y.$

b. $\left(\dfrac{a^{-3}}{b^{-4}}\right)^2 \left(\dfrac{b}{a}\right)^{-3} = \dfrac{a^{-6}}{b^{-8}} \cdot \dfrac{b^{-3}}{a^{-3}} = \dfrac{b^8}{a^6} \cdot \dfrac{a^3}{b^3} = \dfrac{b^5}{a^3}.$

3. a. $\dfrac{2x}{3\sqrt{y}} \cdot \dfrac{\sqrt{y}}{\sqrt{y}} = \dfrac{2x\sqrt{y}}{3y}.$

b. $\dfrac{x}{\sqrt{x} - 4} \cdot \dfrac{\sqrt{x} + 4}{\sqrt{x} + 4} = \dfrac{x\left(\sqrt{x} + 4\right)}{x - 16}.$

4. a. $\dfrac{\left(x^2 + 1\right)\left(\frac{1}{2}x^{-1/2}\right) - x^{1/2}\,(2x)}{\left(x^2 + 1\right)^2} = \dfrac{\frac{1}{2}x^{-1/2}\left[\left(x^2 + 1\right) - 4x^2\right]}{\left(x^2 + 1\right)^2} = \dfrac{1 - 3x^2}{2x^{1/2}\left(x^2 + 1\right)^2}.$

b. $-\dfrac{3x}{\sqrt{x + 2}} + 3\sqrt{x + 2} = \dfrac{-3x + 3\,(x + 2)}{\sqrt{x + 2}} = \dfrac{6}{\sqrt{x + 2}} = \dfrac{6\sqrt{x + 2}}{x + 2}.$

5. $\dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}} = \dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}} \cdot \dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} - \sqrt{y}} = \dfrac{x - y}{\left(\sqrt{x} - \sqrt{y}\right)^2}.$

6. a. $12x^3 - 10x^2 - 12x = 2x\left(6x^2 - 5x - 6\right) = 2x\,(2x - 3)\,(3x + 2).$

b. $2bx - 2by + 3cx - 3cy = 2b\,(x - y) + 3c\,(x - y) = (2b + 3c)\,(x - y)$

7. a. $12x^2 - 9x - 3 = 0$, so $3\left(4x^2 - 3x - 1\right) = 0$ and $3\,(4x + 1)\,(x - 1) = 0$. Thus, $x = -\frac{1}{4}$ or $x = 1$.

b. $3x^2 - 5x + 1 = 0$. Using the quadratic formula with $a = 3$, $b = -5$, and $c = 1$, we have

$$x = \frac{-(-5) \pm \sqrt{25 - 12}}{2\,(3)} = \frac{5 \pm \sqrt{13}}{6}.$$

8. $d = \sqrt{[6 - (-2)]^2 + (8 - 4)^2} = \sqrt{64 + 16} = \sqrt{80} = 4\sqrt{5}.$

9. $m = \dfrac{5 - (-2)}{4 - (-1)} = \dfrac{7}{5}$, so $y - (-2) = \frac{7}{5}\,[x - (-1)]$, $y + 2 = \frac{7}{5}x + \frac{7}{5}$, or $y = \frac{7}{5}x - \frac{3}{5}$.

10. $m = -\frac{1}{3}$ and $b = \frac{4}{3}$, so an equation is $y = -\frac{1}{3}x + \frac{4}{3}$.

CHAPTER 1 Explore & Discuss

Page 27

1. Let $P_1 = (2, 6)$ and $P_2 = (-4, 3)$. Then we have $x_1 = 2$, $y_1 = 6$, $x_2 = -4$, and $y_2 = 3$. Using Formula (1), we have $d = \sqrt{(-4-2)^2 + (3-6)^2} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$, as obtained in Example 1.

2. Let $P_1\,(x_1, y_1)$ and $P_2\,(x_2, y_2)$ be any two points in the plane. Then the result follows from the equality$\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$.

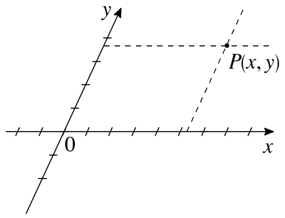
Page 28

1. **a.** All points on and inside the circle with center (h, k) and radius r.
   b. All points inside the circle with center (h, k) and radius r.
   c. All points on and outside the circle with center (h, k) and radius r.
   d. All points outside the circle with center (h, k) and radius r.

2. **a.** $y^2 = 4 - x^2$, and so $y = \pm\sqrt{4 - x^2}$.
   b. **(i)** The upper semicircle with center at the origin and radius 2.
   (ii) The lower semicircle with center at the origin and radius 2.

Page 29

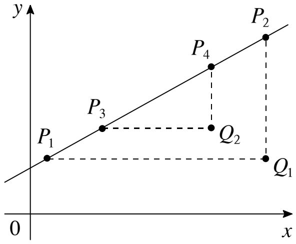
1. Let $P\,(x, y)$ be any point in the plane. Draw a line through P parallel to the y-axis and a line through P parallel to the x-axis (see the figure). The x-coordinate of P is the number corresponding to the point on the x-axis at which the line through P crosses the x-axis. Similarly, y is the number that corresponds to the point on the y-axis at which the line parallel to the x-axis crosses the y-axis. To show the converse, reverse the process.



2. You can use the Pythagorean Theorem in the Cartesian coordinate system. This greatly simplifies the computations.

Page 35

1. Refer to the accompanying figure. Observe that triangles $\Delta P_1Q_1P_2$ and $\Delta P_3Q_2P_4$ are similar. From this we conclude that $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_4 - y_3}{x_4 - x_3}$. Because P_3 and P_4 are arbitrary, the conclusion follows.



Page 39

1. We obtain a family of parallel lines each having slope m.

2. We obtain a family of straight lines all of which pass through the point $(0, b)$.

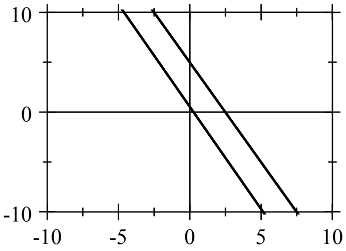
Page 40

1. In Example 11, we are told that the object is expected to appreciate in value at a given rate for the next five years, and the equation obtained in that example is based on this fact. Thus, the equation may not be used to predict the value of the object very much beyond five years from the date of purchase.

CHAPTER 1 Exploring with Technology

Page 38

1.

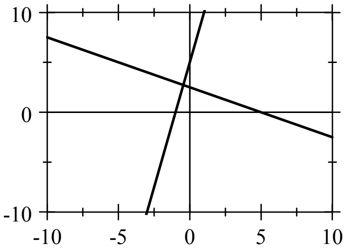


The straight lines L_1 and L_2 are shown in the figure.

a. L_1 and L_2 seem to be parallel.

b. Writing each equation in the slope-intercept form gives $y = -2x + 5$ and $y = -\frac{41}{20}x + \frac{11}{20}$, from which we see that the slopes of L_1 and L_2 are -2 and $-\frac{41}{20} = -2.05$, respectively. This shows that L_1 and L_2 are not parallel.

2.



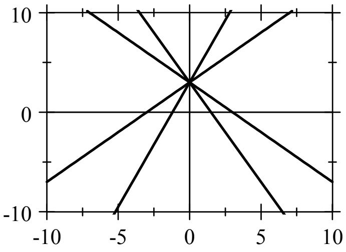
The straight lines L_1 and L_2 are shown in the figure.

a. L_1 and L_2 seem to be perpendicular.

b. The slopes of L_1 and L_2 are $m_1 = -\frac{1}{2}$ and $m_2 = 5$, respectively. Because $m_1 = -\dfrac{1}{2} \neq -\dfrac{1}{5} = -\dfrac{1}{m_2}$, we see that L_1 and L_2 are not perpendicular.

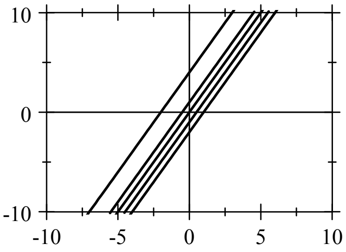
Page 39

1.



The straight lines with the given equations are shown in the figure. Changing the value of m in the equation $y = mx + b$ changes the slope of the line and thus rotates it.

2.



The straight lines of interest are shown in the figure. Changing the value of b in the equation $y = mx + b$ changes the y-intercept of the line and thus translates it (upward if $b > 0$ and downward if $b < 0$).

3. Changing both m and b in the equation $y = mx + b$ both rotates and translates the line.

2 FUNCTIONS, LIMITS, AND THE DERIVATIVE

2.1 Functions and Their Graphs

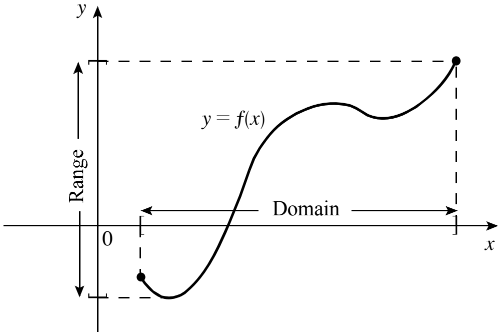
Concept Questions page 59

1. a. A function is a rule that associates with each element in a set A exactly one element in a set B.

b. The domain of a function f is the set of all elements x in the set such that $f(x)$ is an element in B. The range of f is the set of all elements $f(x)$ whenever x is an element in its domain.

c. An independent variable is a variable in the domain of a function f. The dependent variable is $y = f(x)$.

2. a. The graph of a function f is the set of all ordered pairs (x, y) such that $y = f(x)$, x being an element in the domain of f.



b. Use the vertical line test to determine if every vertical line intersects the curve in at most one point. If so, then the curve is the graph of a function.

3. a. Yes, every vertical line intersects the curve in at most one point.

b. No, a vertical line intersects the curve at more than one point.

c. No, a vertical line intersects the curve at more than one point.

d. Yes, every vertical line intersects the curve in at most one point.

4. The domain is $[1, 3) \cup [3, 5)$ and the range is $\left[\frac{1}{2}, 2\right) \cup (2, 4]$.

Exercises page 59

1. $f(x) = 5x + 6$. Therefore $f(3) = 5(3) + 6 = 21$, $f(-3) = 5(-3) + 6 = -9$, $f(a) = 5(a) + 6 = 5a + 6$, $f(-a) = 5(-a) + 6 = -5a + 6$, and $f(a + 3) = 5(a + 3) + 6 = 5a + 15 + 6 = 5a + 21$.

2. $f(x) = 4x - 3$. Therefore, $f(4) = 4(4) - 3 = 16 - 3 = 13$, $f\left(\frac{1}{4}\right) = 4\left(\frac{1}{4}\right) - 3 = 1 - 3 = -2$, $f(0) = 4(0) - 3 = -3$, $f(a) = 4(a) - 3 = 4a - 3$, $f(a + 1) = 4(a + 1) - 3 = 4a + 1$.

3. $g(x) = 3x^2 - 6x - 3$, so $g(0) = 3(0) - 6(0) - 3 = -3$, $g(-1) = 3(-1)^2 - 6(-1) - 3 = 3 + 6 - 3 = 6$, $g(a) = 3(a)^2 - 6(a) - 3 = 3a^2 - 6a - 3$, $g(-a) = 3(-a)^2 - 6(-a) - 3 = 3a^2 + 6a - 3$, and $g(x+1) = 3(x+1)^2 - 6(x+1) - 3 = 3\left(x^2 + 2x + 1\right) - 6x - 6 - 3 = 3x^2 + 6x + 3 - 6x - 9 = 3x^2 - 6$.

4. $h(x) = x^3 - x^2 + x + 1$, so $h(-5) = (-5)^3 - (-5)^2 + (-5) + 1 = -125 - 25 - 5 + 1 = -154$, $h(0) = (0)^3 - (0)^2 + 0 + 1 = 1$, $h(a) = a^3 - (a)^2 + a + 1 = a^3 - a^2 + a + 1$, and $h(-a) = (-a)^3 - (-a)^2 + (-a) + 1 = -a^3 - a^2 - a + 1$.

5. $f(x) = 2x + 5$, so $f(a+h) = 2(a+h) + 5 = 2a + 2h + 5$, $f(-a) = 2(-a) + 5 = -2a + 5$, $f\left(a^2\right) = 2\left(a^2\right) + 5 = 2a^2 + 5$, $f(a - 2h) = 2(a - 2h) + 5 = 2a - 4h + 5$, and $f(2a - h) = 2(2a - h) + 5 = 4a - 2h + 5$

6. $g(x) = -x^2 + 2x$, $g(a+h) = -(a+h)^2 + 2(a+h) = -a^2 - 2ah - h^2 + 2a + 2h$, $g(-a) = -(-a)^2 + 2(-a) = -a^2 - 2a = -a(a+2)$, $g\left(\sqrt{a}\right) = -\left(\sqrt{a}\right)^2 + 2\left(\sqrt{a}\right) = -a + 2\sqrt{a}$, $a + g(a) = a - a^2 + 2a = -a^2 + 3a = -a(a-3)$, and $\dfrac{1}{g(a)} = \dfrac{1}{-a^2 + 2a} = -\dfrac{1}{a(a-2)}$.

7. $s(t) = \dfrac{2t}{t^2 - 1}$. Therefore, $s(4) = \dfrac{2(4)}{(4)^2 - 1} = \dfrac{8}{15}$, $s(0) = \dfrac{2(0)}{0^2 - 1} = 0$, $s(a) = \dfrac{2(a)}{a^2 - 1} = \dfrac{2a}{a^2 - 1}$; $s(2+a) = \dfrac{2(2+a)}{(2+a)^2 - 1} = \dfrac{2(2+a)}{a^2 + 4a + 4 - 1} = \dfrac{2(2+a)}{a^2 + 4a + 3}$, and $s(t+1) = \dfrac{2(t+1)}{(t+1)^2 - 1} = \dfrac{2(t+1)}{t^2 + 2t + 1 - 1} = \dfrac{2(t+1)}{t(t+2)}$.

8. $g(u) = (3u - 2)^{3/2}$. Therefore, $g(1) = [3(1) - 2]^{3/2} = (1)^{3/2} = 1$, $g(6) = [3(6) - 2]^{3/2} = 16^{3/2} = 4^3 = 64$, $g\left(\frac{11}{3}\right) = \left[3\left(\frac{11}{3}\right) - 2\right]^{3/2} = (9)^{3/2} = 27$, and $g(u+1) = [3(u+1) - 2]^{3/2} = (3u+1)^{3/2}$.

9. $f(t) = \dfrac{2t^2}{\sqrt{t-1}}$. Therefore, $f(2) = \dfrac{2\left(2^2\right)}{\sqrt{2-1}} = 8$, $f(a) = \dfrac{2a^2}{\sqrt{a-1}}$, $f(x+1) = \dfrac{2(x+1)^2}{\sqrt{(x+1)-1}} = \dfrac{2(x+1)^2}{\sqrt{x}}$, and $f(x-1) = \dfrac{2(x-1)^2}{\sqrt{(x-1)-1}} = \dfrac{2(x-1)^2}{\sqrt{x-2}}$.

10. $f(x) = 2 + 2\sqrt{5-x}$. Therefore, $f(-4) = 2 + 2\sqrt{5-(-4)} = 2 + 2\sqrt{9} = 2 + 2(3) = 8$, $f(1) = 2 + 2\sqrt{5-1} = 2 + 2\sqrt{4} = 2 + 4 = 6$, $f\left(\frac{11}{4}\right) = 2 + 2\left(5 - \frac{11}{4}\right)^{1/2} = 2 + 2\left(\frac{9}{4}\right)^{1/2} = 2 + 2\left(\frac{3}{2}\right) = 5$, and $f(x+5) = 2 + 2\sqrt{5-(x+5)} = 2 + 2\sqrt{-x}$.

11. Because $x = -2 \le 0$, we calculate $f(-2) = (-2)^2 + 1 = 4 + 1 = 5$. Because $x = 0 \le 0$, we calculate $f(0) = (0)^2 + 1 = 1$. Because $x = 1 > 0$, we calculate $f(1) = \sqrt{1} = 1$.

12. Because $x = -2 < 2$, $g(-2) = -\frac{1}{2}(-2) + 1 = 1 + 1 = 2$. Because $x = 0 < 2$, $g(0) = -\frac{1}{2}(0) + 1 = 0 + 1 = 1$. Because $x = 2 \ge 2$, $g(2) = \sqrt{2-2} = 0$. Because $x = 4 \ge 2$, $g(4) = \sqrt{4-2} = \sqrt{2}$.

13. Because $x = -1 < 1$, $f(-1) = -\frac{1}{2}(-1)^2 + 3 = \frac{5}{2}$. Because $x = 0 < 1$, $f(0) = -\frac{1}{2}(0)^2 + 3 = 3$. Because $x = 1 \ge 1$, $f(1) = 2\left(1^2\right) + 1 = 3$. Because $x = 2 \ge 1$, $f(2) = 2\left(2^2\right) + 1 = 9$.

14. Because $x = 0 \le 1$, $f(0) = 2 + \sqrt{1-0} = 2 + 1 = 3$. Because $x = 1 \le 1$, $f(1) = 2 + \sqrt{1-1} = 2 + 0 = 2$. Because $x = 2 > 1$, $f(2) = \dfrac{1}{1-2} = \dfrac{1}{-1} = -1$.

15. **a.** $f(0) = -2$.

b. **(i)** $f(x) = 3$ when $x \approx 2$. **(ii)** $f(x) = 0$ when $x = 1$.

c. $[0, 6]$

d. $[-2, 6]$

16. **a.** $f(7) = 3$. **b.** $x = 4$ and $x = 6$. **c.** $x = 2$; 0. **d.** $[-1, 9]$; $[-2, 6]$.

17. $g(2) = \sqrt{2^2 - 1} = \sqrt{3}$, so the point $\left(2, \sqrt{3}\right)$ lies on the graph of g.

18. $f(3) = \dfrac{3+1}{\sqrt{3^2+7}} + 2 = \dfrac{4}{\sqrt{16}} + 2 = \dfrac{4}{4} + 2 = 3$, so the point $(3, 3)$ lies on the graph of f.

19. $f(-2) = \dfrac{|-2-1|}{-2+1} = \dfrac{|-3|}{-1} = -3$, so the point $(-2, -3)$ does lie on the graph of f.

20. $h(-3) = \dfrac{|-3+1|}{(-3)^3+1} = \dfrac{2}{-27+1} = -\dfrac{2}{26} = -\dfrac{1}{13}$, so the point $\left(-3, -\frac{1}{13}\right)$ does lie on the graph of h.

21. Because the point $(1, 5)$ lies on the graph of f it satisfies the equation defining f. Thus, $f(1) = 2(1)^2 - 4(1) + c = 5$, or $c = 7$.

22. Because the point $(2, 4)$ lies on the graph of f it satisfies the equation defining f. Thus, $f(2) = 2\sqrt{9 - (2)^2} + c = 4$, or $c = 4 - 2\sqrt{5}$.

23. Because $f(x)$ is a real number for any value of x, the domain of f is $(-\infty, \infty)$.

24. Because $f(x)$ is a real number for any value of x, the domain of f is $(-\infty, \infty)$.

25. $f(x)$ is not defined at $x = 0$ and so the domain of f is $(-\infty, 0) \cup (0, \infty)$.

26. $g(x)$ is not defined at $x = 1$ and so the domain of g is $(-\infty, 1) \cup (1, \infty)$.

27. $f(x)$ is a real number for all values of x. Note that $x^2 + 1 \ge 1$ for all x. Therefore, the domain of f is $(-\infty, \infty)$.

28. Because the square root of a number is defined for all real numbers greater than or equal to zero, we have $x - 5 \ge 0$ or $x \ge 5$, and the domain is $[5, \infty)$.

29. Because the square root of a number is defined for all real numbers greater than or equal to zero, we have $5 - x \ge 0$, or $-x \ge -5$ and so $x \le 5$. (Recall that multiplying by -1 reverses the sign of an inequality.) Therefore, the domain of f is $(-\infty, 5]$.

30. Because $2x^2 + 3$ is always greater than zero, the domain of g is $(-\infty, \infty)$.

31. The denominator of f is zero when $x^2 - 1 = 0$, or $x = \pm 1$. Therefore, the domain of f is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

32. The denominator of f is equal to zero when $x^2 + x - 2 = (x+2)(x-1) = 0$; that is, when $x = -2$ or $x = 1$. Therefore, the domain of f is $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$.

33. f is defined when $x + 3 \geq 0$, that is, when $x \geq -3$. Therefore, the domain of f is $[-3, \infty)$.

34. g is defined when $x - 1 \geq 0$; that is when $x \geq 1$. Therefore, the domain of f is $[1, \infty)$.

35. The numerator is defined when $1 - x \geq 0$, $-x \geq -1$ or $x \leq 1$. Furthermore, the denominator is zero when $x = \pm 2$. Therefore, the domain is the set of all real numbers in $(-\infty, -2) \cup (-2, 1]$.

36. The numerator is defined when $x - 1 \geq 0$, or $x \geq 1$, and the denominator is zero when $x = -2$ and when $x = 3$. So the domain is $[1, 3) \cup (3, \infty)$.

37. **a.** The domain of f is the set of all real numbers.

b. $f(x) = x^2 - x - 6$, so

$f(-3) = (-3)^2 - (-3) - 6 = 9 + 3 - 6 = 6$,

$f(-2) = (-2)^2 - (-2) - 6 = 4 + 2 - 6 = 0$,

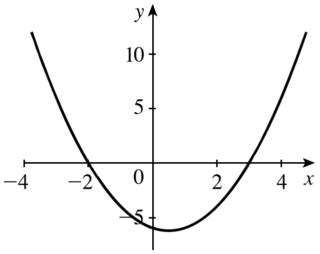
$f(-1) = (-1)^2 - (-1) - 6 = 1 + 1 - 6 = -4$,

$f(0) = (0)^2 - (0) - 6 = -6$,

$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 6 = \frac{1}{4} - \frac{2}{4} - \frac{24}{4} = -\frac{25}{4}$, $f(1) = (1)^2 - 1 - 6 = -6$,

$f(2) = (2)^2 - 2 - 6 = 4 - 2 - 6 = -4$, and $f(3) = (3)^2 - 3 - 6 = 9 - 3 - 6 = 0$.

c.



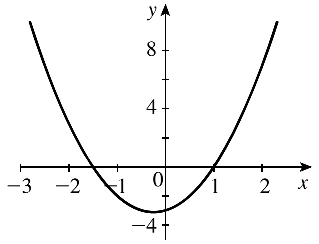
38. $f(x) = 2x^2 + x - 3$.

a. Because $f(x)$ is a real number for all values of x, the domain of f is $(-\infty, \infty)$.

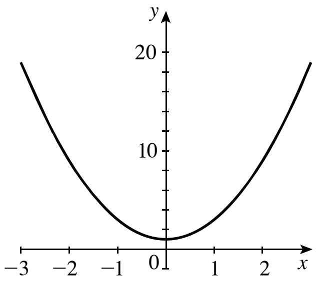
b.

x	-3	-2	-1	$-\frac{1}{2}$	0	1	2	3
y	12	3	-2	-3	-3	0	7	18

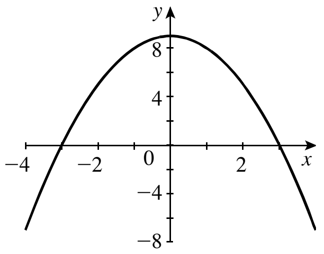
c.



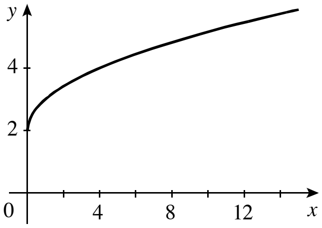
39. $f(x) = 2x^2 + 1$ has domain $(-\infty, \infty)$ and range $[1, \infty)$.



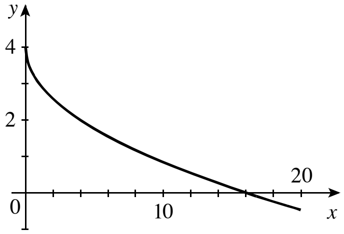
40. $f(x) = 9 - x^2$ has domain $(-\infty, \infty)$ and range $(-\infty, 9]$.



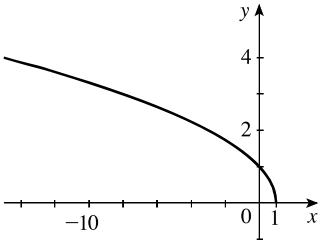
41. $f(x) = 2 + \sqrt{x}$ has domain $[0, \infty)$ and range $[2, \infty)$.



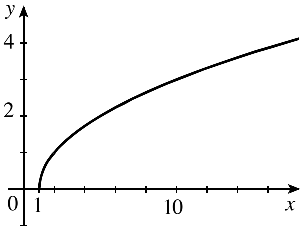
42. $g(x) = 4 - \sqrt{x}$ has domain $[0, \infty)$ and range $(-\infty, 4]$.



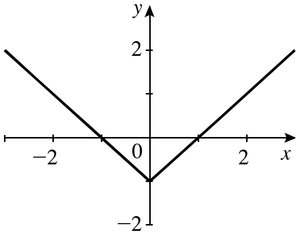
43. $f(x) = \sqrt{1 - x}$ has domain $(-\infty, 1]$ and range $[0, \infty)$



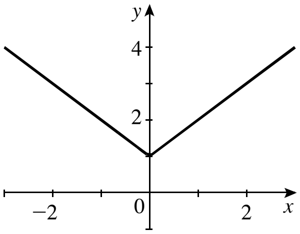
44. $f(x) = \sqrt{x - 1}$ has domain $(1, \infty)$ and range $[0, \infty)$.



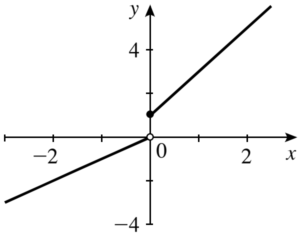
45. $f(x) = |x| - 1$ has domain $(-\infty, \infty)$ and range $[-1, \infty)$.



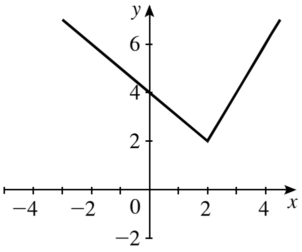
46. $f(x) = |x| + 1$ has domain $(-\infty, \infty)$ and range $[1, \infty)$



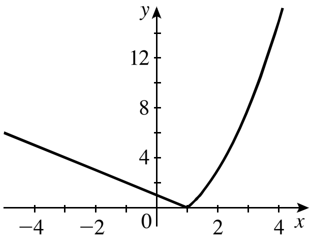
47. $f(x) = \begin{cases} x & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$ has domain $(-\infty, \infty)$ and range $(-\infty, 0) \cup [1, \infty)$.



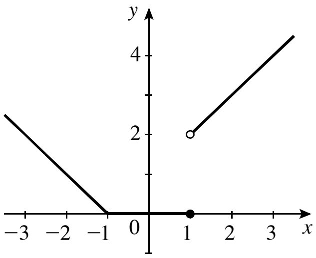
48. For $x < 2$, the graph of f is the half-line $y = 4 - x$. For $x \geq 2$, the graph of f is the half-line $y = 2x - 2$. f has domain $(-\infty, \infty)$ and range $[2, \infty)$.



49. If $x \leq 1$, the graph of f is the half-line $y = -x + 1$. For $x > 1$, we calculate a few points: $f(2) = 3$, $f(3) = 8$, and $f(4) = 15$. f has domain $(-\infty, \infty)$ and range $[0, \infty)$.



50. If $x < -1$ the graph of f is the half-line $y = -x - 1$. For $-1 \leq x \leq 1$, the graph consists of the line segment $y = 0$. For $x > 1$, the graph is the half-line $y = x + 1$. f has domain $(-\infty, \infty)$ and range $[0, \infty)$.



51. Each vertical line cuts the given graph at exactly one point, and so the graph represents y as a function of x.

52. Because the y-axis, which is a vertical line, intersects the graph at two points, the graph does not represent y as a function of x.

53. Because there is a vertical line that intersects the graph at three points, the graph does not represent y as a function of x.

54. Each vertical line intersects the graph of f at exactly one point, and so the graph represents y as a function of x.

55. Each vertical line intersects the graph of f at exactly one point, and so the graph represents y as a function of x.

56. The y-axis intersects the circle at *two* points, and this shows that the circle is not the graph of a function of x.

57. Each vertical line intersects the graph of f at exactly one point, and so the graph represents y as a function of x.

58. A vertical line containing a line segment comprising the graph cuts it at infinitely many points and so the graph does not define y as a function of x.

59. The circumference of a circle with a 5-inch radius is given by $C(5) = 2\pi(5) = 10\pi$, or 10π inches.

60. $V(2.1) = \frac{4}{3}\pi(2.1)^3 \approx 38.79$, $V(2) = \frac{4}{3}\pi(8) \approx 33.51$, and so $V(2.1) - V(2) = 38.79 - 33.51 = 5.28$ is the amount by which the volume of a sphere of radius 2.1 exceeds the volume of a sphere of radius 2.

61. $C(0) = 6$, or 6 billion dollars; $C(50) = 0.75(50) + 6 = 43.5$, or 43.5 billion dollars; and $C(100) = 0.75(100) + 6 = 81$, or 81 billion dollars.

62. The child should receive $D(4) = \frac{2}{25}(500)(4) = 160$, or 160 mg.

63. **a.** From $t = 0$ through $t = 5$, that is, from the beginning of 2001 until the end of 2005.

b. From $t = 5$ through $t = 9$, that is, from the beginning of 2006 until the end of 2010.

c. The average expenditures were the same at approximately $t = 5.2$, that is, in the year 2006. The level of expenditure on each service was approximately \$900.

64. a. The slope of the straight line passing through $(0, 0.61)$ and $(10, 0.59)$ is $m_1 = \frac{0.59 - 0.61}{10 - 0} = -0.002$. Therefore, an equation of the straight line passing through the two points is $y - 0.61 = -0.002\,(t - 0)$ or $y = -0.002t + 0.61$. Next, the slope of the straight line passing through $(10, 0.59)$ and $(20, 0.60)$ is $m_2 = \frac{0.60 - 0.59}{20 - 10} = 0.001$, and so an equation of the straight line passing through the two points is $y - 0.59 = 0.001\,(t - 10)$ or $y = 0.001t + 0.58$. The slope of the straight line passing through $(20, 0.60)$ and $(30, 0.66)$ is $m_3 = \frac{0.66 - 0.60}{30 - 20} = 0.006$, and so an equation of the straight line passing through the two points is $y - 0.60 = 0.006\,(t - 20)$ or $y = 0.006t + 0.48$. The slope of the straight line passing through $(30, 0.66)$ and $(40.0, 0.78)$ is $m_4 = \frac{0.78 - 0.66}{40 - 30} = 0.012$, and so an equation of the straight line passing through the two points is $y = 0.012t + 0.30$. Therefore, a rule for f is $f\,(t) = \begin{cases} -0.002t + 0.61 & \text{if } 0 \le t \le 10 \\ 0.001t + 0.58 & \text{if } 10 < t \le 20 \\ 0.006t + 0.48 & \text{if } 20 < t \le 30 \\ 0.012t + 0.30 & \text{if } 30 < t \le 40 \end{cases}$

b. The gender gap was expanding between 1960 and 1970 and shrinking between 1970 and 2000.

c. The gender gap was expanding at the rate of 0.002/yr between 1960 and 1970, shrinking at the rate of 0.001/yr between 1970 and 1980, shrinking at the rate of 0.006/yr between 1980 and 1990, and shrinking at the rate of 0.012/yr between 1990 and 2000.

65. a. The slope of the straight line passing through the points $(0, 0.58)$ and $(20, 0.95)$ is $m_1 = \frac{0.95 - 0.58}{20 - 0} = 0.0185$, so an equation of the straight line passing through these two points is $y - 0.58 = 0.0185\,(t - 0)$ or $y = 0.0185t + 0.58$. Next, the slope of the straight line passing through the points $(20, 0.95)$ and $(30, 1.1)$ is $m_2 = \frac{1.1 - 0.95}{30 - 20} = 0.015$, so an equation of the straight line passing through the two points is $y - 0.95 = 0.015\,(t - 20)$ or $y = 0.015t + 0.65$. Therefore, a rule for f is

$$f\,(t) = \begin{cases} 0.0185t + 0.58 & \text{if } 0 \le t \le 20 \\ 0.015t + 0.65 & \text{if } 20 < t \le 30 \end{cases}$$

b. The ratios were changing at the rates of 0.0185/yr from 1960 through 1980 and 0.015/yr from 1980 through 1990.

c. The ratio was 1 when $t \approx 20.3$. This shows that the number of bachelor's degrees earned by women equaled the number earned by men for the first time around 1983.

66. a. $T\,(x) = 0.06x$

b. $T\,(200) = 0.06\,(200) = 12$, or \$12.00 and $T\,(5.65) = 0.06\,(5.65) = 0.34$, or \$0.34.

67. a. $I\,(x) = 1.053x$

b. $I\,(1520) = 1.053\,(1520) = 1600.56$, or \$1600.56.

68. a. The function is linear with y-intercept 1.44 and slope 0.058, so we have $f\,(t) = 0.058t + 1.44$, $0 \le t \le 9$.

b. The projected spending in 2018 will be $f\,(9) = 0.058\,(9) + 1.44 = 1.962$, or \$1.962 trillion.

69. $S\,(r) = 4\pi r^2$.

70. $\frac{4}{3}(\pi)(2r)^3 = \frac{4}{3}\pi 8r^3 = 8\left(\frac{4}{3}\pi r^3\right)$. Therefore, the volume of the tumor is increased by a factor of 8.

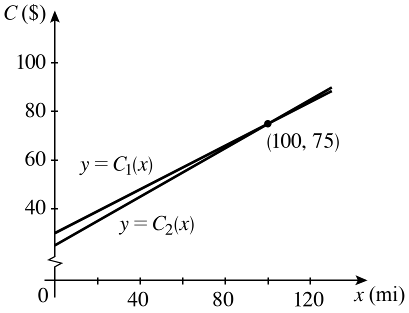
71. **a.** The median age was changing at the rate of 0.3 years/year.

b. The median age in 2011 was $M(11) = 0.3(11) + 37.9 = 41.2$ (years).

c. The median age in 2015 is projected to be $M(5) = 0.3(15) + 37.9 = 42.4$ (years).

72. **a.** The daily cost of leasing from Ace is $C_1(x) = 30 + 0.45x$, while the daily cost of leasing from Acme is $C_2(x) = 25 + 0.50x$, where x is the number of miles driven.

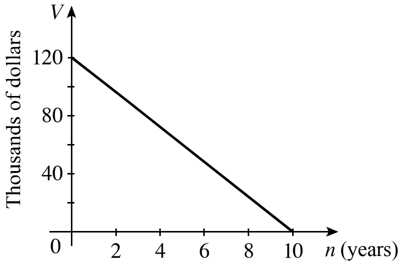
b.



c. The costs are the same when $C_1(x) = C_2(x)$, that is, when $30 + 0.45x = 25 + 0.50x$, $-0.05x = -5$, or $x = 100$. Because $C_1(70) = 30 + 0.45(70) = 61.5$ and $C_2(70) = 25 + 0.50(70) = 60$, and the customer plans to drive less than 70 miles, she should rent from Acme.

73. **a.** The graph of the function is a straight line passing through (0, 120000) and (10, 0). Its slope is $m = \dfrac{0 - 120{,}000}{10 - 0} = -12{,}000$. The required equation is $V = -12{,}000n + 120{,}000$.

b.



c. $V = -12{,}000(6) + 120{,}000 = 48{,}000$, or \$48,000.

d. This is given by the slope, that is, \$12,000 per year.

74. Here $V = -20{,}000n + 1{,}000{,}000$. The book value in 2010 is given by $V = -20{,}000(15) + 1{,}000{,}000$, or \$700,000. The book value in 2014 is given by $V = -20{,}000(19) + 1{,}000{,}000$, or \$620,000. The book value in 2019 is $V = -20{,}000(24) + 1{,}000{,}000$, or \$520,000.

75. **a.** The number of incidents in 2009 was $f(0) = 0.46$ (million).

b. The number of incidents in 2013 was $f(4) = 0.2\left(4^2\right) - 0.14(4) + 0.46 = 3.1$ (million).

76. **a.** The number of passengers in 1995 was $N(0) = 4.6$ (million).

b. The number of passengers in 2010 was $N(15) = 0.011(15)^2 + 0.521(15) + 4.6 = 14.89$ (million).

77. **a.** The life expectancy of a male whose current age is 65 is $f(65) = 0.0069502(65)^2 - 1.6357(65) + 93.76 \approx 16.80$, or approximately 16.8 years.

b. The life expectancy of a male whose current age is 75 is $f(75) = 0.0069502(75)^2 - 1.6357(75) + 93.76 \approx 10.18$, or approximately 10.18 years.

78. **a.** $N(t) = 0.00445t^2 + 0.2903t + 9.564$. $N(0) = 9.564$, or 9.6 million people; $N(12) = 0.00445(12)^2 + 0.2903(12) + 9.564 \approx 13.6884$, or approximately 13.7 million people.

b. $N(14) = 0.00445(14)^2 + 0.2903(14) + 9.564 \approx 14.5004$, or approximately 14.5 million people.

79. The projected number in 2030 is $P(20) = -0.0002083(20)^3 + 0.0157(20)^2 - 0.093(20) + 5.2 = 7.9536$, or approximately 8 million.

The projected number in 2050 is $P(40) = -0.0002083(40)^3 + 0.0157(40)^2 - 0.093(40) + 5.2 = 13.2688$, or approximately 13.3 million.

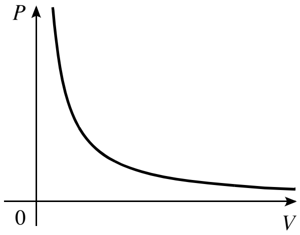
80. $N(t) = -t^3 + 6t^2 + 15t$. Between 8 a.m. and 9 a.m., the average worker can be expected to assemble $N(1) - N(0) = (-1 + 6 + 15) - 0 = 20$, or 20 walkie-talkies. Between 9 a.m. and 10 a.m., we expect that $N(2) - N(1) = \left[-2^3 + 6\left(2^2\right) + 15(2)\right] - (-1 + 6 + 15) = 46 - 20 = 26$, or 26 walkie-talkies can be assembled by the average worker.

81. When the proportion of popular votes won by the Democratic presidential candidate is 0.60, the proportion of seats in the House of Representatives won by Democratic candidates is given by

$$s(0.6) = \frac{(0.6)^3}{(0.6)^3 + (1 - 0.6)^3} = \frac{0.216}{0.216 + 0.064} = \frac{0.216}{0.280} \approx 0.77.$$

82. The amount spent in 2004 was $S(0) = 5.6$, or \$5.6 billion. The amount spent in 2008 was $S(4) = -0.03(4)^3 + 0.2(4)^2 + 0.23(4) + 5.6 = 7.8$, or \$7.8 billion.

83. The domain of the function f is the set of all real positive numbers where $V \neq 0$; that is, $(0, \infty)$.

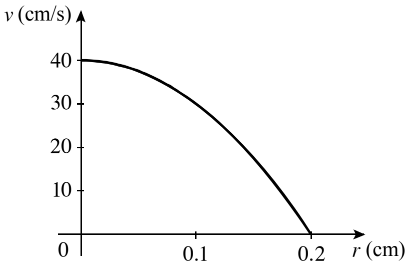


84. **a.** We require that $0.04 - r^2 \geq 0$ and $r \geq 0$. This is true if $0 \leq r \leq 0.2$. Therefore, the domain of v is $[0, 0.2]$.

b. We compute $v(0) = 1000\left[0.04 - (0)^2\right] = 1000(0.04) = 40$,

$$v(0.1) = 1000\left[0.04 - (0.1)^2\right] = 1000(0.04 - 0.01)$$
$$= 1000(0.03) = 30, \text{ and}$$
$$v(0.2) = 1000\left[0.04 - (0.2)^2\right] = 1000(0.04 - 0.04) = 0.$$

c.



d. As the distance r increases, the velocity of the blood decreases.

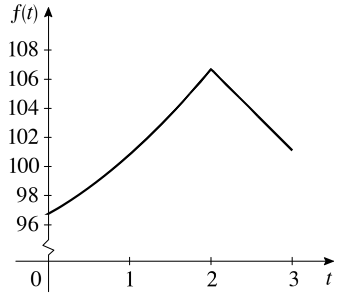
85. **a.** The assets at the beginning of 2002 were \$0.6 trillion. At the beginning of 2003, they were $f(1) = 0.6$, or \$0.6 trillion.

b. The assets at the beginning of 2005 were $f(3) = 0.6(3)^{0.43} \approx 0.96$, or \$0.96 trillion. At the beginning of 2007, they were $f(5) = 0.6(5)^{0.43} \approx 1.20$, or \$1.2 trillion.

86. a.

Year	2006	2007	2008
Rate	96.75	100.84	106.69

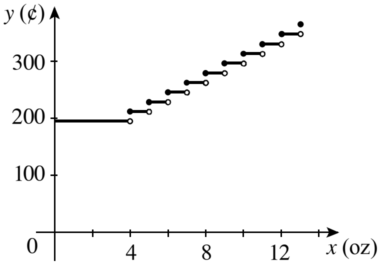
b.



87. a. The domain of f is $(0, 13]$.

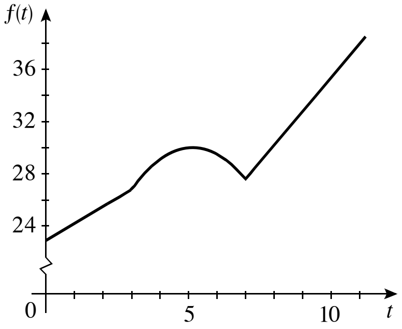
$$f(x)=\begin{cases}1.95 & \text{if } 0<x<4\\ 2.12 & \text{if } 4\leq x<5\\ 2.29 & \text{if } 5\leq x<6\\ 2.46 & \text{if } 6\leq x<7\\ 2.63 & \text{if } 7\leq x<8\\ 2.80 & \text{if } 8\leq x<9\\ 2.97 & \text{if } 9\leq x<10\\ 3.14 & \text{if } 10\leq x\leq 11\\ 3.31 & \text{if } 11\leq x<12\\ 3.48 & \text{if } 12\leq x<13\\ 3.65 & \text{if } x=13\end{cases}$$

b.



88. a. The median age of the U.S. population at the beginning of 1900 was $f(0)=22.9$, or 22.9 years; at the beginning of 1950 it was $f(5)=-0.7(5)^2+7.2(5)+11.5=30$, or 30 years; and at the beginning of 2000 it was $f(10)=2.6(10)+9.4=35.4$, or 35.4 years.

b.



89. a. The passenger ship travels a distance given by $14t$ miles east and the cargo ship travels a distance of $10(t-2)$ miles north. After two hours have passed, the distance between the two ships is given by $\sqrt{[10(t-2)]^2+(14t)^2}=\sqrt{296t^2-400t+400}$ miles, so $D(t)=\begin{cases}14t & \text{if } 0\leq t\leq 2\\ 2\sqrt{74t^2-100t+100} & \text{if } t>2\end{cases}$

b. Three hours after the cargo ship leaves port the value of t is 5. Therefore, $D=2\sqrt{74(5)^2-100(5)+100}\approx 76.16$, or 76.16 miles.

90. True, by definition of a function (page 52).

91. False. Take $f(x)=x^2$, $a=1$, and $b=-1$. Then $f(1)=1=f(-1)$, but $a\neq b$.

92. False. Let $f(x)=x^2$, then take $a=1$ and $b=2$. Then $f(a)=f(1)=1$, $f(b)=f(2)=4$, and $f(a)+f(b)=1+4\neq f(a+b)=f(3)=9$.

93. False. It intersects the graph of a function in at most one point.

94. True. We have $x+2\geq 0$ and $2-x\geq 0$ simultaneously; that is $x\geq -2$ and $x\leq 2$. These inequalities are satisfied if $-2\leq x\leq 2$.

95. False. Take $f(x) = x^2$ and $k = 2$. Then $f(x) = (2x)^2 = 4x^2 \neq 2x^2 = 2f(x)$.

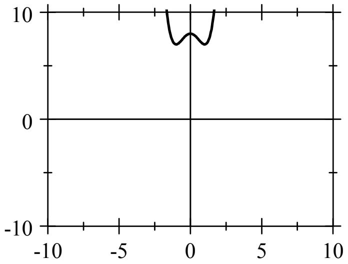
96. False. Take $f(x) = 2x + 3$ and $c = 2$. Then $f(2x + y) = 2(2x + y) + 3 = 4x + 2y + 3$, but $cf(x) + f(y) = 2(2x + 3) + (2y + 3) = 4x + 2y + 9 \neq f(2x + y)$.

97. False. They are equal everywhere except at $x = 0$, where g is not defined.

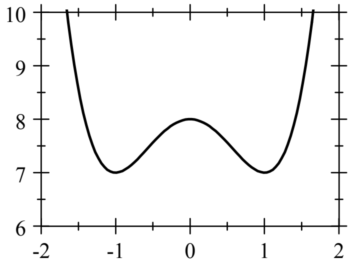
98. False. The rule suggests that R takes on the values 0 and 1 when $x = 1$. This violates the uniqueness property that a function must possess.

Using Technology page 68

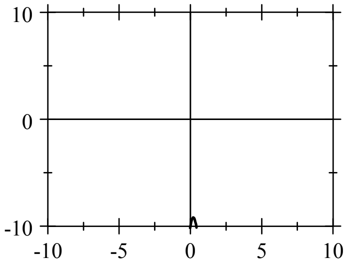
1. a.



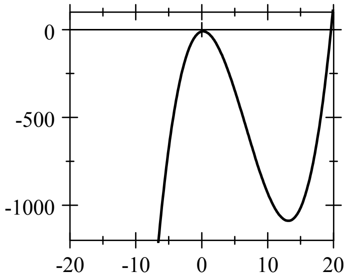
b.



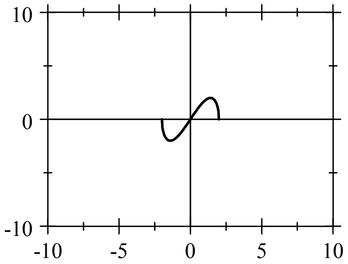
2. a.



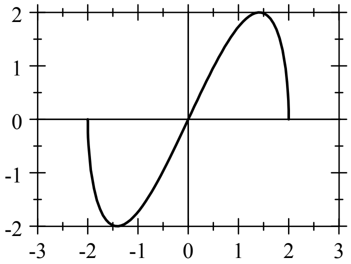
b.



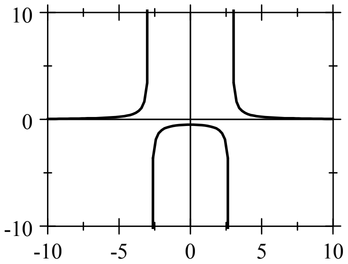
3. a.



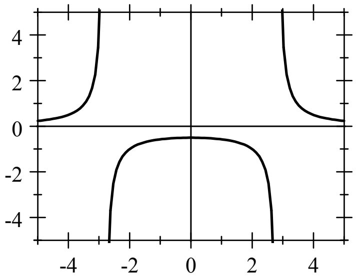
b.



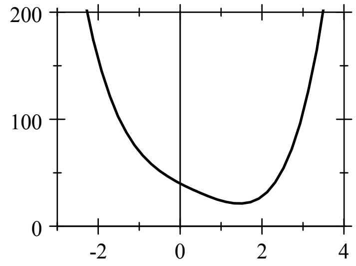
4. a.



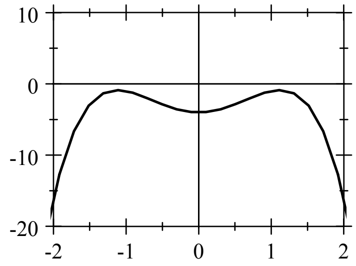
b.



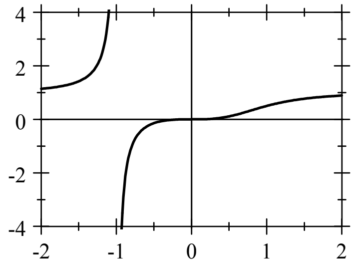
5.



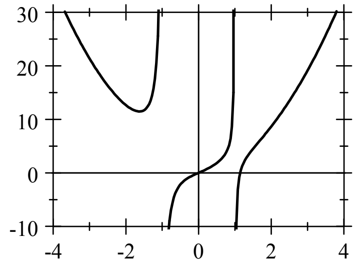
6.



7.



8.



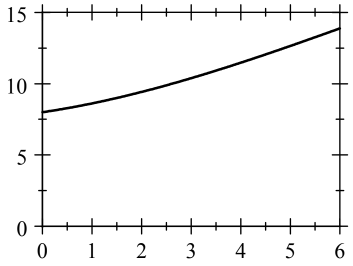
9. $f(2.145) \approx 18.5505$.

10. $f(1.28) \approx 17.3850$.

11. $f(2.41) \approx 4.1616$.

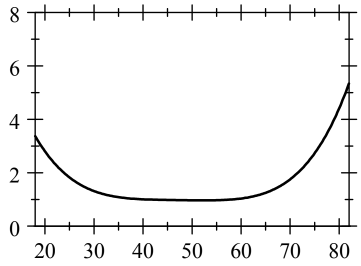
12. $f(0.62) \approx 1.7214$.

13. a.



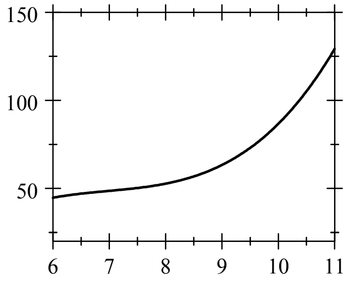
b. The amount spent in the year 2005 was $f(2) \approx 9.42$, or approximately \$9.4 billion. In 2009, it was $f(6) \approx 13.88$, or approximately \$13.9 billion.

14. a.



b. $f(18) = 3.3709$, $f(50) = 0.971$, and $f(80) = 4.4078$.

15. a.



b. $f(6) = 44.7$, $f(8) = 52.7$, and $f(11) = 129.2$.

2.2 The Algebra of Functions

Concept Questions page 73

1. a. $P(x_1) = R(x_1) - C(x_1)$ gives the profit if x_1 units are sold.

b. $P(x_2) = R(x_2) - C(x_2)$. Because $P(x_2) < 0$, $|R(x_2) - C(x_2)| = -[R(x_2) - C(x_2)]$ gives the loss sustained if x_2 units are sold.

2. a. $(f+g)(x) = f(x) + g(x)$, $(f-g)(x) = f(x) - g(x)$, and $(fg)(x) = f(x)g(x)$; all have domain $A \cap B$. $(f/g)(x) = \dfrac{f(x)}{g(x)}$ has domain $A \cap B$ excluding $x \in A \cap B$ such that $g(x) = 0$.

b. $(f+g)(2) = f(2) + g(2) = 3 + (-2) = 1$, $(f-g)(2) = f(2) - g(2) = 3 - (-2) = 5$, $(fg)(2) = f(2)g(2) = 3(-2) = -6$, and $(f/g)(2) = \dfrac{f(2)}{g(2)} = \dfrac{3}{-2} = -\dfrac{3}{2}$

3. a. $y = (f+g)(x) = f(x) + g(x)$

b. $y = (f-g)(x) = f(x) - g(x)$

c. $y = (fg)(x) = f(x)g(x)$

d. $y = \left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$

4. a. The domain of $(f \circ g)(x) = f(g(x))$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. The domain of $(g \circ f)(x) = g(f(x))$ is the set of all x in the domain of f such that $f(x)$ is in the domain of g.

b. $(g \circ f)(2) = g(f(2)) = g(3) = 8$. We cannot calculate $(f \circ g)(3)$ because $(f \circ g)(3) = f(g(3)) = f(8)$, and we don't know the value of $f(8)$.

5. No. Let $A = (-\infty, \infty)$, $f(x) = x$, and $g(x) = \sqrt{x}$. Then $a = -1$ is in A, but $(g \circ f)(-1) = g(f(-1)) = g(-1) = \sqrt{-1}$ is not defined.

6. The required expression is $P = g(f(p))$.

Exercises page 74

1. $(f+g)(x) = f(x) + g(x) = (x^3+5) + (x^2-2) = x^3 + x^2 + 3$.

2. $(f-g)(x) = f(x) - g(x) = (x^3+5) - (x^2-2) = x^3 - x^2 + 7$.

3. $fg(x) = f(x)g(x) = (x^3+5)(x^2-2) = x^5 - 2x^3 + 5x^2 - 10$.

4. $gf(x) = g(x)f(x) = (x^2-2)(x^3+5) = x^5 - 2x^3 + 5x^2 - 10$.

5. $\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)} = \dfrac{x^3+5}{x^2-2}$.

6. $\dfrac{f-g}{h}(x) = \dfrac{f(x)-g(x)}{h(x)} = \dfrac{x^3+5-(x^2-2)}{2x+4} = \dfrac{x^3-x^2+7}{2x+4}$.

7. $\dfrac{fg}{h}(x) = \dfrac{f(x)g(x)}{h(x)} = \dfrac{(x^3+5)(x^2-2)}{2x+4} = \dfrac{x^5-2x^3+5x^2-10}{2x+4}$.

8. $fgh(x) = f(x)\,g(x)\,h(x) = \left(x^3+5\right)\left(x^2-2\right)(2x+4) = \left(x^5-2x^3+5x^2-10\right)(2x+4)$

$= 2x^6 - 4x^4 + 10x^3 - 20x + 4x^5 - 8x^3 + 20x^2 - 40 = 2x^6 + 4x^5 - 4x^4 + 2x^3 + 20x^2 - 20x - 40.$

9. $(f+g)(x) = f(x) + g(x) = x - 1 + \sqrt{x+1}.$

10. $(g-f)(x) = g(x) - f(x) = \sqrt{x+1} - (x-1) = \sqrt{x+1} - x + 1.$

11. $(fg)(x) = f(x)\,g(x) = (x-1)\sqrt{x+1}.$

12. $(gf)(x) = g(x)\,f(x) = \sqrt{x+1}\,(x-1).$

13. $\dfrac{g}{h}(x) = \dfrac{g(x)}{h(x)} = \dfrac{\sqrt{x+1}}{2x^3-1}.$

14. $\dfrac{h}{g}(x) = \dfrac{h(x)}{g(x)} = \dfrac{2x^3-1}{\sqrt{x+1}}.$

15. $\dfrac{fg}{h}(x) = \dfrac{(x-1)\left(\sqrt{x+1}\right)}{2x^3-1}.$

16. $\dfrac{fh}{g}(x) = \dfrac{(x-1)\left(2x^3-1\right)}{\sqrt{x+1}} = \dfrac{2x^4-2x^3-x+1}{\sqrt{x+1}}.$

17. $\dfrac{f-h}{g}(x) = \dfrac{x-1-\left(2x^3-1\right)}{\sqrt{x+1}} = \dfrac{x-2x^3}{\sqrt{x+1}}.$

18. $\dfrac{gh}{g-f}(x) = \dfrac{\sqrt{x+1}\left(2x^3-1\right)}{\sqrt{x+1}-(x-1)} = \dfrac{\sqrt{x+1}\left(2x^3-1\right)}{\sqrt{x+1}-x+1}.$

19. $(f+g)(x) = x^2+5+\sqrt{x}-2 = x^2+\sqrt{x}+3$, $(f-g)(x) = x^2+5-\left(\sqrt{x}-2\right) = x^2-\sqrt{x}+7$, $(fg)(x) = \left(x^2+5\right)\left(\sqrt{x}-2\right)$, and $\left(\dfrac{f}{g}\right)(x) = \dfrac{x^2+5}{\sqrt{x}-2}.$

20. $(f+g)(x) = \sqrt{x-1}+x^3+1$, $(f-g)(x) = \sqrt{x-1}-x^3-1$, $(fg)(x) = \sqrt{x-1}\left(x^3+1\right)$, and $\left(\dfrac{f}{g}\right)(x) = \dfrac{\sqrt{x-1}}{x^3+1}.$

21. $(f+g)(x) = \sqrt{x+3}+\dfrac{1}{x-1} = \dfrac{(x-1)\sqrt{x+3}+1}{x-1}$, $(f-g)(x) = \sqrt{x+3}-\dfrac{1}{x-1} = \dfrac{(x-1)\sqrt{x+3}-1}{x-1}$, $(fg)(x) = \sqrt{x+3}\left(\dfrac{1}{x-1}\right) = \dfrac{\sqrt{x+3}}{x-1}$, and $\left(\dfrac{f}{g}\right) = \sqrt{x+3}\,(x-1).$

22. $(f+g)(x) = \dfrac{1}{x^2+1}+\dfrac{1}{x^2-1} = \dfrac{x^2-1+x^2+1}{\left(x^2+1\right)\left(x^2-1\right)} = \dfrac{2x^2}{\left(x^2+1\right)\left(x^2-1\right)}$,

$(f-g)(x) = \dfrac{1}{x^2+1}-\dfrac{1}{x^2-1} = \dfrac{x^2-1-x^2-1}{\left(x^2+1\right)\left(x^2-1\right)} = -\dfrac{2}{\left(x^2+1\right)\left(x^2-1\right)}$, $(fg)(x) = \dfrac{1}{\left(x^2+1\right)\left(x^2-1\right)}$, and $\left(\dfrac{f}{g}\right)(x) = \dfrac{x^2-1}{x^2+1}.$

23. $$(f+g)(x) = \frac{x+1}{x-1} + \frac{x+2}{x-2} = \frac{(x+1)(x-2)+(x+2)(x-1)}{(x-1)(x-2)} = \frac{x^2-x-2+x^2+x-2}{(x-1)(x-2)} = \frac{2x^2-4}{(x-1)(x-2)} = \frac{2\left(x^2-2\right)}{(x-1)(x-2)},$$

$$(f-g)(x) = \frac{x+1}{x-1} - \frac{x+2}{x-2} = \frac{(x+1)(x-2)-(x+2)(x-1)}{(x-1)(x-2)} = \frac{x^2-x-2-x^2-x+2}{(x-1)(x-2)} = \frac{-2x}{(x-1)(x-2)},$$

$(fg)(x) = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$, and $\left(\dfrac{f}{g}\right)(x) = \dfrac{(x+1)(x-2)}{(x-1)(x+2)}$.

24. $(f+g)(x) = x^2+1+\sqrt{x+1}$, $(f-g)(x) = x^2+1-\sqrt{x+1}$, $(fg)(x) = \left(x^2+1\right)\sqrt{x+1}$, and $\left(\dfrac{f}{g}\right)(x) = \dfrac{x^2+1}{\sqrt{x+1}}$.

25. $(f\circ g)(x) = f(g(x)) = f\left(x^2\right) = \left(x^2\right)^2 + x^2 + 1 = x^4+x^2+1$ and $(g\circ f)(x) = g(f(x)) = g\left(x^2+x+1\right) = \left(x^2+x+1\right)^2$.

26. $(f\circ g)(x) = f(g(x)) = 3\left[g(x)\right]^2 + 2g(x) + 1 = 3(x+3)^2 + 2(x+3) + 1 = 3x^2+20x+34$ and $(g\circ f)(x) = g(f(x)) = f(x)+3 = 3x^2+2x+1+3 = 3x^2+2x+4$.

27. $(f\circ g)(x) = f(g(x)) = f\left(x^2-1\right) = \sqrt{x^2-1}+1$ and $(g\circ f)(x) = g(f(x)) = g\left(\sqrt{x}+1\right) = \left(\sqrt{x}+1\right)^2 - 1 = x+2\sqrt{x}+1-1 = x+2\sqrt{x}$.

28. $(f\circ g)(x) = f(g(x)) = 2\sqrt{g(x)}+3 = 2\sqrt{x^2+1}+3$ and $(g\circ f)(x) = g(f(x)) = \left[f(x)\right]^2+1 = \left(2\sqrt{x}+3\right)^2+1 = 4x+12\sqrt{x}+10$.

29. $(f\circ g)(x) = f(g(x)) = f\left(\dfrac{1}{x}\right) = \dfrac{1}{x} \div \left(\dfrac{1}{x^2}+1\right) = \dfrac{1}{x}\cdot\dfrac{x^2}{x^2+1} = \dfrac{x}{x^2+1}$ and $(g\circ f)(x) = g(f(x)) = g\left(\dfrac{x}{x^2+1}\right) = \dfrac{x^2+1}{x}$.

30. $(f\circ g)(x) = f(g(x)) = f\left(\dfrac{1}{x-1}\right) = \sqrt{\dfrac{x}{x-1}}$ and $(g\circ f)(x) = g(f(x)) = g\left(\sqrt{x+1}\right) = \dfrac{1}{\sqrt{x+1}-1}\cdot\dfrac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \dfrac{\sqrt{x+1}+1}{x}$.

31. $h(2) = g(f(2))$. But $f(2) = 2^2+2+1 = 7$, so $h(2) = g(7) = 49$.

32. $h(2) = g(f(2))$. But $f(2) = \left(2^2-1\right)^{1/3} = 3^{1/3}$, so $h(2) = g\left(3^{1/3}\right) = 3\left(3^{1/3}\right)^3 + 1 = 3(3)+1 = 10$.

33. $h(2) = g(f(2))$. But $f(2) = \dfrac{1}{2(2)+1} = \dfrac{1}{5}$, so $h(2) = g\left(\dfrac{1}{5}\right) = \dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$.

34. $h(2) = g(f(2))$. But $f(2) = \dfrac{1}{2-1} = 1$, so $g(1) = 1^2+1 = 2$.

35. $f(x) = 2x^3 + x^2 + 1, g(x) = x^5.$

36. $f(x) = 3x^2 - 4, g(x) = x^{-3}.$

37. $f(x) = x^2 - 1, g(x) = \sqrt{x}.$

38. $f(x) = (2x - 3), g(x) = x^{3/2}.$

39. $f(x) = x^2 - 1, g(x) = \dfrac{1}{x}.$

40. $f(x) = x^2 - 4, g(x) = \dfrac{1}{\sqrt{x}}.$

41. $f(x) = 3x^2 + 2, g(x) = \dfrac{1}{x^{3/2}}.$

42. $f(x) = \sqrt{2x+1}, g(x) = \dfrac{1}{x} + x.$

43. $f(a+h) - f(a) = [3(a+h) + 4] - (3a+4) = 3a + 3h + 4 - 3a - 4 = 3h.$

44. $f(a+h) - f(a) = -\frac{1}{2}(a+h) + 3 - \left(-\frac{1}{2}a + 3\right) = -\frac{1}{2}a - \frac{1}{2}h + 3 + \frac{1}{2}a - 3 = -\frac{1}{2}h.$

45. $f(a+h) - f(a) = 4 - (a+h)^2 - \left(4 - a^2\right) = 4 - a^2 - 2ah - h^2 - 4 + a^2 = -2ah - h^2 = -h(2a+h).$

46.
$$\begin{aligned} f(a+h) - f(a) &= \left[(a+h)^2 - 2(a+h) + 1\right] - \left(a^2 - 2a + 1\right) \\ &= a^2 + 2ah + h^2 - 2a - 2h + 1 - a^2 + 2a - 1 = h(2a + h - 2). \end{aligned}$$

47.
$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{\left[(a+h)^2 + 1\right] - \left(a^2 + 1\right)}{h} = \frac{a^2 + 2ah + h^2 + 1 - a^2 - 1}{h} = \frac{2ah + h^2}{h} \\ &= \frac{h(2a+h)}{h} = 2a + h. \end{aligned}$$

48.
$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{\left[2(a+h)^2 - (a+h) + 1\right] - \left(2a^2 - a + 1\right)}{h} \\ &= \frac{2a^2 + 4ah + 2h^2 - a - h + 1 - 2a^2 + a - 1}{h} = \frac{4ah + 2h^2 - h}{h} = 4a + 2h - 1. \end{aligned}$$

49.
$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{\left[(a+h)^3 - (a+h)\right] - \left(a^3 - a\right)}{h} = \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a - h - a^3 + a}{h} \\ &= \frac{3a^2h + 3ah^2 + h^3 - h}{h} = 3a^2 + 3ah + h^2 - 1. \end{aligned}$$

50.
$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{\left[2(a+h)^3 - (a+h)^2 + 1\right] - \left(2a^3 - a^2 + 1\right)}{h} \\ &= \frac{2a^3 + 6a^2h + 6ah^2 + 2h^3 - a^2 - 2ah - h^2 + 1 - 2a^3 + a^2 - 1}{h} \\ &= \frac{6a^2h + 6ah^2 + 2h^3 - 2ah - h^2}{h} = 6a^2 + 6ah + 2h^2 - 2a - h. \end{aligned}$$

51.
$$\frac{f(a+h) - f(a)}{h} = \frac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h} = \frac{\dfrac{a - (a+h)}{a(a+h)}}{h} = -\frac{1}{a(a+h)}.$$

52.
$$\frac{f(a+h) - f(a)}{h} = \frac{\sqrt{a+h} - \sqrt{a}}{h} \cdot \frac{\sqrt{a+h} + \sqrt{a}}{\sqrt{a+h} + \sqrt{a}} = \frac{(a+h) - a}{h\left(\sqrt{a+h} + \sqrt{a}\right)} = \frac{1}{\sqrt{a+h} + \sqrt{a}}.$$

53. $F(t)$ represents the total revenue for the two restaurants at time t.

54. $F(t)$ represents the net rate of growth of the species of whales in year t.

55. $f(t)g(t)$ represents the dollar value of Nancy's holdings at time t.

56. $f(t)/g(t)$ represents the unit cost of the commodity at time t.

57. $g \circ f$ is the function giving the amount of carbon monoxide pollution from cars in parts per million at time t.

58. $f \circ g$ is the function giving the revenue at time t.

59. $C(x) = 0.6x + 12{,}100$.

60. **a.** $h(t) = f(t) - g(t) = (3t + 69) - (-0.2t + 13.8) = 3.2t + 55.2,\ 0 \le t \le 5$.

b. $f(5) = 3(5) + 69 = 84$, $g(5) = -0.2(5) + 13.8 = 12.8$, and $h(5) = 3.2(5) + 55.2 = 71.2$. Since $f(5) - g(5) = 84 - 12.8 = 71.2$, we see that $h(5)$ is indeed equal to $f(5) - g(5)$.

61. $D(t) = (D_2 - D_1)(t) = D_2(t) - D_1(t) = \left(0.035t^2 + 0.21t + 0.24\right) - \left(0.0275t^2 + 0.081t + 0.07\right)$

$\approx 0.0075t^2 + 0.129t + 0.17$.

The function D gives the difference in year t between the deficit without the \$160 million rescue package and the deficit with the rescue package.

62. **a.** $(g \circ f)(0) = g(f(0)) = g(0.64) = 26$, so the mortality rate of motorcyclists in the year 2000 was 26 per 100 million miles traveled.

b. $(g \circ f)(6) = g(f(6)) = g(0.51) = 42$, so the mortality rate of motorcyclists in 2006 was 42 per 100 million miles traveled.

c. Between 2000 and 2006, the percentage of motorcyclists wearing helmets had dropped from 64 to 51, and as a consequence, the mortality rate of motorcyclists had increased from 26 million miles traveled to 42 million miles traveled.

63. **a.** $(g \circ f)(1) = g(f(1)) = g(406) = 23$. So in 2002, the percentage of reported serious crimes that end in arrests or in the identification of suspects was 23.

b. $(g \circ f)(6) = g(f(6)) = g(326) = 18$. In 2007, 18% of reported serious crimes ended in arrests or in the identification of suspects.

c. Between 2002 and 2007, the total number of detectives had dropped from 406 to 326 and as a result, the percentage of reported serious crimes that ended in arrests or in the identification of suspects dropped from 23 to 18.

64. **a.** $C(x) = 0.000003x^3 - 0.03x^2 + 200x + 100{,}000$, so $C(2000) = 0.000003(2000)^3 - 0.03(2000)^2 + 200(2000) + 100{,}000 = 404{,}000$, or \$404,000.

b. $P(x) = R(x) - C(x) = -0.1x^2 + 500x - \left(0.000003x^3 - 0.03x^2 + 200x + 100{,}000\right)$

$= -0.000003x^3 - 0.07x^2 + 300x - 100{,}000$.

c. $P(1500) = -0.000003(1500)^3 - 0.07(1500)^2 + 300(1500) - 100{,}000 = 182{,}375$, or \$182,375.

65. **a.** $C(x) = V(x) + 20000 = 0.000001x^3 - 0.01x^2 + 50x + 20000 = 0.000001x^3 - 0.01x^2 + 50x + 20{,}000$.

b. $P(x) = R(x) - C(x) = -0.02x^2 + 150x - 0.000001x^3 + 0.01x^2 - 50x - 20{,}000$

$= -0.000001x^3 - 0.01x^2 + 100x - 20{,}000$.

c. $P(2000) = -0.000001(2000)^3 - 0.01(2000)^2 + 100(2000) - 20{,}000 = 132{,}000$, or \$132,000.

66. **a.**
$$\begin{aligned} D(t) &= R(t) - S(t) \\ &= \left(0.023611t^3 - 0.19679t^2 + 0.34365t + 2.42\right) - \left(-0.015278t^3 + 0.11179t^2 + 0.02516t + 2.64\right) \\ &= 0.038889t^3 - 0.30858t^2 + 0.31849t - 0.22,\ 0 \le t \le 6. \end{aligned}$$

b. $S(3) = 3.309084$, $R(3) = 2.317337$, and $D(3) = -0.991747$, so the spending, revenue, and deficit are approximately \$3.31 trillion, \$2.32 trillion, and \$0.99 trillion, respectively.

c. Yes: $R(3) - S(3) = 2.317337 - 3.308841 = -0.991504 = D(3)$.

67. **a.**
$$\begin{aligned} h(t) = f(t) + g(t) &= \left(4.389t^3 - 47.833t^2 + 374.49t + 2390\right) + \left(13.222t^3 - 132.524t^2 + 757.9t + 7481\right) \\ &= 17.611t^3 - 180.357t^2 + 1132.39t + 9871,\ 1 \le t \le 7. \end{aligned}$$

b. $f(6) = 3862.976$ and $g(6) = 10{,}113.488$, so $f(6) + g(6) = 13{,}976.464$. The worker's contribution was approximately \$3862.98, the employer's contribution was approximately \$10,113.49, and the total contributions were approximately \$13,976.46.

c. $h(6) = 13{,}976 = f(6) + g(6)$, as expected.

68. **a.**
$$N(r(t)) = \frac{7}{1 + 0.02\left(\dfrac{5t + 75}{t + 10}\right)^2}.$$

b.
$$N(r(0)) = \frac{7}{1 + 0.02\left(\dfrac{5 \cdot 0 + 75}{0 + 10}\right)^2} = \frac{7}{1 + 0.02\left(\dfrac{75}{10}\right)^2} \approx 3.29, \text{ or } 3.29 \text{ million units.}$$
$$N(r(12)) = \frac{7}{1 + 0.02\left(\dfrac{5 \cdot 12 + 75}{12 + 10}\right)^2} = \frac{7}{1 + 0.02\left(\dfrac{135}{22}\right)^2} \approx 3.99, \text{ or } 3.99 \text{ million units.}$$
$$N(r(18)) = \frac{7}{1 + 0.02\left(\dfrac{5 \cdot 18 + 75}{18 + 10}\right)^2} = \frac{7}{1 + 0.02\left(\dfrac{165}{28}\right)^2} \approx 4.13, \text{ or } 4.13 \text{ million units.}$$

69. **a.** The occupancy rate at the beginning of January is $r(0) = \frac{10}{81}(0)^3 - \frac{10}{3}(0)^2 + \frac{200}{9}(0) + 55 = 55$, or 55%. $r(5) = \frac{10}{81}(5)^3 - \frac{10}{3}(5)^2 + \frac{200}{9}(5) + 55 \approx 98.2$, or approximately 98.2%.

b. The monthly revenue at the beginning of January is $R(55) = -\frac{3}{5000}(55)^3 + \frac{9}{50}(55)^2 \approx 444.68$, or approximately \$444,700.

The monthly revenue at the beginning of June is $R(98.2) = -\frac{3}{5000}(98.2)^3 + \frac{9}{50}(98.2)^2 \approx 1167.6$, or approximately \$1,167,600.

70. $N(t) = 1.42 \cdot x(t) = \dfrac{1.42 \cdot 7(t+10)^2}{(t+10)^2 + 2(t+15)^2} = \dfrac{9.94(t+10)^2}{(t+10)^2 + 2(t+15)^2}$. The number of jobs created 6 months from now will be $N(6) = \dfrac{9.94(16)^2}{(16)^2 + 2(21)^2} \approx 2.24$, or approximately 2.24 million jobs. The number of jobs created 12 months from now will be $N(12) = \dfrac{9.94(22)^2}{(22)^2 + 2(27)^2} \approx 2.48$, or approximately 2.48 million jobs.

71. a. $s = f + g + h = (f + g) + h = f + (g + h)$. This suggests we define the sum s by $s(x) = (f + g + h)(x) = f(x) + g(x) + h(x)$.

b. Let f, g, and h define the revenue (in dollars) in week t of three branches of a store. Then its total revenue (in dollars) in week t is $s(t) = (f + g + h)(t) = f(t) + g(t) + h(t)$.

72. a. $(h \circ g \circ f)(x) = h(g(f(x)))$

b. Let t denote time. Suppose f gives the number of people at time t in a town, g gives the number of cars as a function of the number of people in the town, and H gives the amount of carbon monoxide in the atmosphere. Then $(h \circ g \circ f)(t) = h(g(f(t)))$ gives the amount of carbon monoxide in the atmosphere at time t.

73. True. $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$.

74. False. Let $f(x) = x + 2$ and $g(x) = \sqrt{x}$. Then $(g \circ f)(x) = \sqrt{x+2}$ is defined at $x = -1$, But $(f \circ g)(x) = \sqrt{x} + 2$ is not defined at $x = -1$.

75. False. Take $f(x) = \sqrt{x}$ and $g(x) = x + 1$. Then $(g \circ f)(x) = \sqrt{x} + 1$, but $(f \circ g)(x) = \sqrt{x+1}$.

76. False. Take $f(x) = x + 1$. Then $(f \circ f)(x) = f(f(x)) = x + 2$, but $f^2(x) = \left[f(x)\right]^2 = (x+1)^2 = x^2 + 2x + 1$.

77. True. $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$ and $((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$.

78. False. Take $h(x) = \sqrt{x}$, $g(x) = x$, and $f(x) = x^2$. Then $(h \circ (g + f))(x) = h\left(x + x^2\right) = \sqrt{x + x^2} \neq ((h \circ g) + (h \circ f))(x) = h(g(x)) + h(f(x)) = \sqrt{x} + \sqrt{x^2}$.

2.3 Functions and Mathematical Models

Concept Questions

page 88

1. See page 78 of the text. Answers will vary.

2. a. $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, where $a_n \neq 0$ and n is a positive integer. An example is $P(x) = 4x^3 - 3x^2 + 2$.

b. $R(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials with $Q(x) \neq 0$. An example is $R(x) = \dfrac{3x^4 - 2x^2 + 1}{x^2 + 3x + 5}$.

3. a. A demand function $p = D(x)$ gives the relationship between the unit price of a commodity p and the quantity x demanded. A supply function $p = S(x)$ gives the relationship between the unit price of a commodity p and the quantity x the supplier will make available in the marketplace.

b. Market equilibrium occurs when the quantity produced is equal to the quantity demanded. To find the market equilibrium, we solve the equations $p = D(x)$ and $p = S(x)$ simultaneously.

Exercises page 88

1. Yes. $2x + 3y = 6$ and so $y = -\frac{2}{3}x + 2$.

2. Yes. $4y = 2x + 7$ and so $y = \frac{1}{2}x + \frac{7}{4}$.

3. Yes. $2y = x + 4$ and so $y = \frac{1}{2}x + 2$.

4. Yes. $3y = 2x - 8$ and so $y = \frac{2}{3}x - \frac{8}{3}$.

5. Yes. $4y = 2x + 9$ and so $y = \frac{1}{2}x + \frac{9}{4}$.

6. Yes. $6y = 3x + 7$ and so $y = \frac{1}{2}x + \frac{7}{6}$.

7. No, because of the term x^2.

8. No, because of the term $\sqrt{x}$.

9. f is a polynomial function in x of degree 6.

10. f is a rational function.

11. Expanding $G(x) = 2(x^2 - 3)^3$, we have $G(x) = 2x^6 - 18x^4 + 54x^2 - 54$, and we conclude that G is a polynomial function of degree 6 in x.

12. We can write $H(x) = \dfrac{2}{x^3} + \dfrac{5}{x^2} + 6 = \dfrac{2 + 5x + 6x^3}{x^3}$ and conclude that H is a rational function.

13. f is neither a polynomial nor a rational function.

14. f is a rational function.

15. $f(0) = 2$ gives $f(0) = m(0) + b = b = 2$. Next, $f(3) = -1$ gives $f(3) = m(3) + b = -1$. Substituting $b = 2$ in this last equation, we have $3m + 2 = -1$, or $3m = -3$, and therefore, $m = -1$ and $b = 2$.

16. $f(2) = 4$ gives $f(2) = 2m + b = 4$. We also know that $m = -1$. Therefore, we have $2(-1) + b = 4$ and so $b = 6$.

17. **a.** $C(x) = 8x + 40{,}000$.

b. $R(x) = 12x$.

c. $P(x) = R(x) - C(x) = 12x - (8x + 40{,}000) = 4x - 40{,}000$.

d. $P(8000) = 4(8000) - 40{,}000 = -8000$, or a loss of \$8000. $P(12{,}000) = 4(12{,}000) - 40{,}000 = 8000$, or a profit of \$8000.

18. **a.** $C(x) = 14x + 100{,}000$.

b. $R(x) = 20x$.

c. $P(x) = R(x) - C(x) = 20x - (14x + 100{,}000) = 6x - 100{,}000$.

d. $P(12{,}000) = 6(12{,}000) - 100{,}000 = -28{,}000$, or a loss of \$28,000.
$P(20{,}000) = 6(20{,}000) - 100{,}000 = 20{,}000$, or a profit of \$20,000.

19. The individual's disposable income is $D = (1 - 0.28) \cdot 60{,}000 = 43{,}200$, or \$43,200.

20. The child should receive $D(0.4) = \dfrac{(0.4)(500)}{1.7} \approx 117.65$, or approximately 118 mg.

21. The child should receive $D(4) = \left(\dfrac{4+1}{24}\right)(500) \approx 104.17$, or approximately 104 mg.

22. **a.** The graph of f passes through the points $P_1(0, 17.5)$ and $P_2(10, 10.3)$. Its slope is $\dfrac{10.3 - 17.5}{10 - 0} = -0.72$. An equation of the line is $y - 17.5 = -0.72(t - 0)$ or $y = -0.72t + 17.5$, so the linear function is $f(t) = -0.72t + 17.5$.

b. The rate was decreasing at 0.72% per year.

c. The percentage of high school students who drink and drive at the beginning of 2014 is projected to be $f(13) = -0.72(13) + 17.5 = 8.14$, or 8.14%.

23. a. The slope of the graph of f is a line with slope -13.2 passing through the point $(0, 400)$, so an equation of the line is $y - 400 = -13.2(t - 0)$ or $y = -13.2t + 400$, and the required function is $f(t) = -13.2t + 400$.

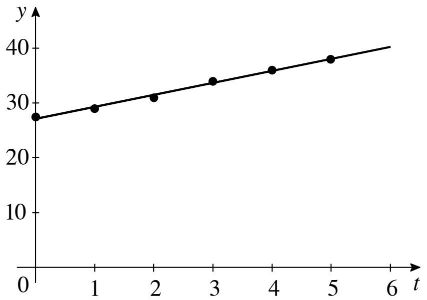
b. The emissions cap is projected to be $f(2) = -13.2(2) + 400 = 373.6$, or 373.6 million metric tons of carbon dioxide equivalent.

24. a. The graph of f is a line through the points $P_1(0, 0.7)$ and $P_2(20, 1.2)$, so it has slope $\frac{1.2 - 0.7}{20 - 0} = 0.025$. Its equation is $y - 0.7 = 0.025(t - 0)$ or $y = 0.025t + 0.7$. The required function is thus $f(t) = 0.025t + 0.7$.

b. The projected annual rate of growth is the slope of the graph of f, that is, 0.025 billion per year, or 25 million per year.

c. The projected number of boardings per year in 2022 is $f(10) = 0.025(10) + 0.7 = 0.95$, or 950 million boardings per year.

25. a.



b. The projected revenue in 2010 is projected to be $f(6) = 2.19(6) + 27.12 = 40.26$, or \$40.26 billion.

c. The rate of increase is the slope of the graph of f, that is, 2.19 (billion dollars per year).

26. Two hours after starting work, the average worker will be assembling at the rate of $f(2) = -\frac{3}{2}(2)^2 + 6(2) + 10 = 16$, or 16 phones per hour.

27. $P(28) = -\frac{1}{8}(28)^2 + 7(28) + 30 = 128$, or \$128,000.

28. a. The amount paid out in 2010 was $S(0) = 0.72$, or \$0.72 trillion (or \$720 billion).

b. The amount paid out in 2030 is projected to be $S(3) = 0.1375(3)^2 + 0.5185(3) + 0.72 = 3.513$, or \$3.513 trillion.

29. a. The average time spent per day in 2009 was $f(0) = 21.76$ (minutes).

b. The average time spent per day in 2013 is projected to be $f(4) = 2.25(4)^2 + 13.41(4) + 21.76 = 111.4$ (minutes).

30. a. The GDP in 2011 was $G(0) = 15$, or \$15 trillion.

b. The projected GDP in 2015 is $G(4) = 0.064(4)^2 + 0.473(4) + 15.0 = 17.916$, or \$17.196 trillion.

31. a. The GDP per capita in 2000 was $f(10) = 1.86251(10)^2 - 28.08043(10) + 884 = 789.4467$, or \$789.45.

b. The GDP per capita in 2030 is projected to be $f(40) = 1.86251(40)^2 - 28.08043(40) + 884 = 2740.7988$, or \$2740.80.

32. **a.** The number of enterprise IM accounts in 2006 is given by $N(0) = 59.7$, or 59.7 million.

b. The number of enterprise IM accounts in 2010, assuming a continuing trend, is given by $N(4) = 2.96(4)^2 + 11.37(4) + 59.7 = 152.54$ million.

33. $S(6) = 0.73(6)^2 + 15.8(6) + 2.7 = 123.78$ million kilowatt-hr.
$S(8) = 0.73(8)^2 + 15.8(8) + 2.7 = 175.82$ million kilowatt-hr.

34. The U.S. public debt in 2005 was $f(0) = 8.246$, or \$8.246 trillion. The public debt in 2008 was $f(3) = -0.03817(3)^3 + 0.4571(3)^2 - 0.1976(3) + 8.246 = 10.73651$, or approximately \$10.74 trillion.

35. The percentage who expected to work past age 65 in 1991 was $f(0) = 11$, or 11%. The percentage in 2013 was $f(22) = 0.004545(22)^3 - 0.1113(22)^2 + 1.385(22) + 11 = 35.99596$, or approximately 36%.

36. $N(0) = 0.7$ per 100 million vehicle miles driven. $N(7) = 0.0336(7)^3 - 0.118(7)^2 + 0.215(7) + 0.7 = 7.9478$ per 100 million vehicle miles driven.

37. **a.** Total global mobile data traffic in 2009 was $f(0) = 0.06$, or 60,000 terabytes.

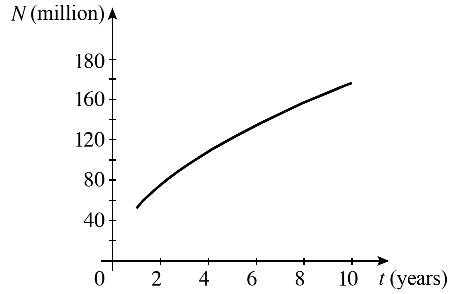
b. The total in 2014 will be $f(5) = 0.021(5)^3 + 0.015(5)^2 + 0.12(5) + 0.06 = 3.66$, or 3.66 million terabytes.

38. Here $Y = 0.06$, $D = 0.2$, and $R = 0.05$, so the leveraged return is $L = \dfrac{0.06 - (1 - 0.2)(0.05)}{0.2} = 0.1$, or 10%.

39. **a.** We first construct a table.

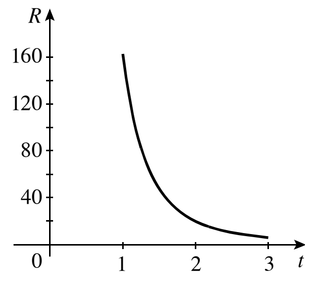
t	$N(t)$
1	52
2	75
3	93
4	109
5	122

t	$N(t)$
6	135
7	146
8	157
9	167
10	177



b. The number of viewers in 2012 is given by $N(10) = 52(10)^{0.531} \approx 176.61$, or approximately 177 million viewers.

40. **a.**



$R(1) = 162.8(1)^{-3.025} = 162.8$, $R(2) = 162.8(2)^{-3.025} \approx 20.0$, and $R(3) = 162.8(3)^{-3.025} \approx 5.9$.

b. The infant mortality rates in 1900, 1950, and 2000 are 162.8, 20.0, and 5.9 per 1000 live births, respectively.

41. $N(5) = 0.0018425(10)^{2.5} \approx 0.58265$, or approximately 0.583 million. $N(13) = 0.0018425(18)^{2.5} \approx 2.5327$, or approximately 2.5327 million.

42. **a.** $S(0) = 4.3(0 + 2)^{0.94} \approx 8.24967$, or approximately \$8.25 billion.

b. $S(8) = 4.3(8 + 2)^{0.94} \approx 37.45$, or approximately \$37.45 billion.

43. a. We are given that $f(1) = 5240$ and $f(4) = 8680$. This leads to the system of equations $a + b = 5240$, $11a + b = 8680$. Solving, we find $a = 344$ and $b = 4896$.

b. From part (a), we have $f(t) = 344t + 4896$, so the approximate per capita costs in 2005 were $f(5) = 344(5) + 4896 = 6616$, or \$6616.

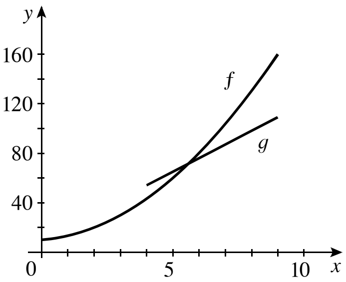
44. a. The given data imply that $R(40) = 50$, that is, $\dfrac{100(40)}{b+40} = 50$, so $50(b + 40) = 4000$, or $b = 40$. Therefore, the required response function is $R(x) = \dfrac{100x}{40+x}$.

b. The response will be $R(60) = \dfrac{100(60)}{40+60} = 60$, or approximately 60 percent.

45. a. $f(0) = 6.85$, $g(0) = 16.58$. Because $g(0) > f(0)$, we see that more film cameras were sold in 2001 (when $t = 0$).

b. We solve the equation $f(t) = g(t)$, that is, $3.05t + 6.85 = -1.85t + 16.58$, so $4.9t = 9.73$ and $t = 1.99 \approx 2$. So sales of digital cameras first exceed those of film cameras in approximately 2003.

46. a.



b. $5x^2 + 5x + 30 = 33x + 30$, so $5x^2 - 28x = 0$, $x(5x - 28) = 0$, and $x = 0$ or $x = \frac{28}{5} = 5.6$, representing 5.6 mi/h.
$g(x) = 11(5.6) + 10 = 71.6$, or 71.6 mL/lb/min.

c. The oxygen consumption of the walker is greater than that of the runner.

47. a. We are given that $T = aN + b$ where a and b are constants to be determined. The given conditions imply that $70 = 120a + b$ and $80 = 160a + b$. Subtracting the first equation from the second gives $10 = 40a$, or $a = \frac{1}{4}$. Substituting this value of a into the first equation gives $70 = 120\left(\frac{1}{4}\right) + b$, or $b = 40$. Therefore, $T = \frac{1}{4}N + 40$.

b. Solving the equation in part (a) for N, we find $\frac{1}{4}N = T - 40$, or $N = f(t) = 4T - 160$. When $T = 102$, we find $N = 4(102) - 160 = 248$, or 248 times per minute.

48. a. $f(0) = 3173$ gives $c = 3173$, $f(4) = 6132$ gives $16a + 4b + c = 6132$, and $f(6) = 7864$ gives $36a + 6b + c = 1864$. Solving, we find $a \approx 21.0417$, $b \approx 655.5833$, and $c = 3173$.

b. From part (a), we have $f(t) = 21.0417t^2 + 655.5833t + 3173$, so the number of farmers' markets in 2014 is projected to be $f(8) = 21.0417(8)^2 + 655.5833(8) + 3173 = 9764.3352$, or approximately 9764.

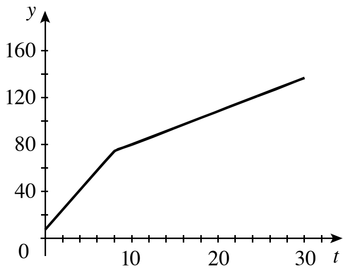
49. a. We have $f(0) = c = 1547$, $f(2) = 4a + 2b + c = 1802$, and $f(4) = 16a + 4b + c = 2403$. Solving this system of equations gives $a = 43.25$, $b = 41$, and $c = 1547$.

b. From part (a), we have $f(t) = 43.25t^2 + 41t + 1547$, so the number of craft-beer breweries in 2014 is projected to be $f(6) = 43.25(6)^2 + 41(6) + 1547 = 3350$.

50. The slope of the line is $m = \dfrac{S-C}{n}$. Therefore, an equation of the line is $y - C = \dfrac{S-C}{n}(t - 0)$. Letting $y = V(t)$, we have $V(t) = C - \dfrac{C-S}{n}t$.

51. Using the formula given in Exercise 50, we have $V(2) = 100{,}000 - \frac{100{,}000 - 30{,}000}{5}(2) = 100{,}000 - \frac{70{,}000}{5}(2) = 72{,}000$, or \$72,000.

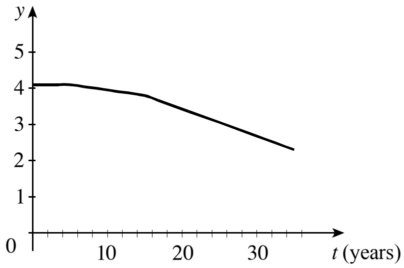
52. a.



b. $f(0) = 8.37(0) + 7.44 = 7.44$, or \$7.44/kilo.
$f(20) = 2.84(20) + 51.68 = 108.48$, or \$108.48/kilo.

53. The total cost by 2011 is given by $f(1) = 5$, or \$5 billion. The total cost by 2015 is given by $f(5) = -0.5278(5^3) + 3.012(5^2) + 49.23(5) - 103.29 = 152.185$, or approximately \$152 billion.

54. a.

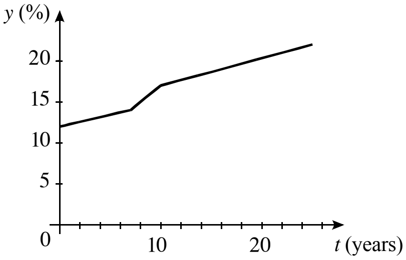


b. At the beginning of 2005, the ratio will be $f(10) = -0.03(10) + 4.25 = 3.95$. At the beginning of 2020, the ratio will be $f(25) = -0.075(25) + 4.925 = 3.05$.

c. The ratio is constant from 1995 to 2000.

d. The decline of the ratio is greatest from 2010 through 2030. It is $\frac{f(35) - f(15)}{35 - 15} = \frac{2.3 - 3.8}{20} = -0.075$.

55. a.

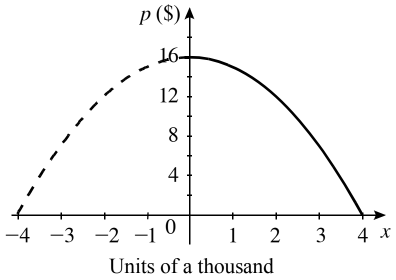


b. $f(5) = \frac{2}{7}(5) + 12 = \frac{10}{7} + 12 \approx 13.43$, or approximately 13.43%. $f(25) = \frac{1}{3}(25) + \frac{41}{3} = 22$, or 22%.

56. a. $f(0) = 5.6$ and $g(0) = 22.5$. Because $g(0) > f(0)$, we conclude that more VCRs than DVD players were sold in 2001.

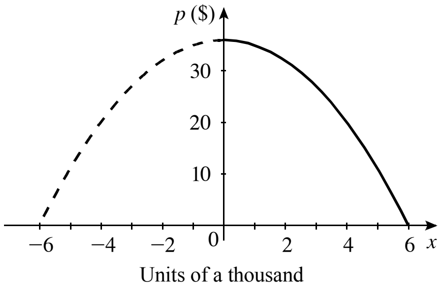
b. We solve the equations $f(t) = g(t)$ over each of the subintervals. $5.6 + 5.6t = -9.6t + 22.5$ for $0 \le t \le 1$. We solve to find $15.2t = 16.9$, so $t \approx 1.11$. This is outside the range for t, so we reject it. $5.6 + 5.6t = -0.5t + 13.4$ for $1 < t \le 2$, so $6.1t = 7.8$, and thus $t \approx 1.28$. So sales of DVD players first exceed those of VCRs at $t \approx 1.3$, or in early 2002.

57. a.



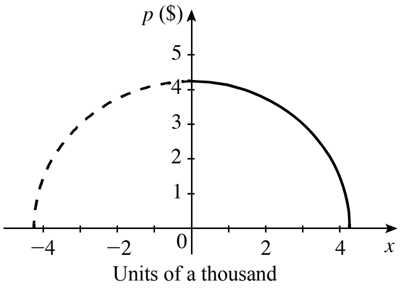
b. If $p = 7$, we have $7 = -x^2 + 16$, or $x^2 = 9$, so that $x = \pm 3$. Therefore, the quantity demanded when the unit price is \$7 is 3000 units.

58. a.



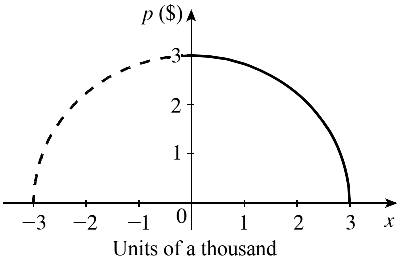
b. If $p = 11$, we have $11 = -x^2 + 36$, or $x^2 = 25$, so that $x = \pm 5$. Therefore, the quantity demanded when the unit price is \$11 is 5000 units.

59. a.



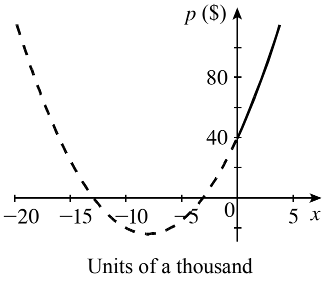
b. If $p = 3$, then $3 = \sqrt{18 - x^2}$, and $9 = 18 - x^2$, so that $x^2 = 9$ and $x = \pm\, 3$. Therefore, the quantity demanded when the unit price is \$3 is 3000 units.

60. a.



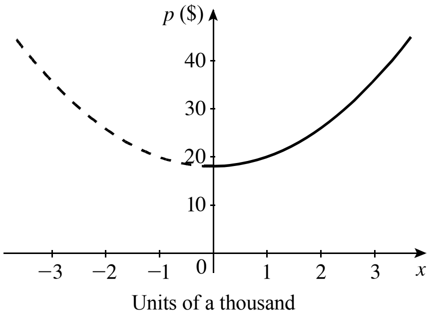
b. If $p = 2$, then $2 = \sqrt{9 - x^2}$, and $4 = 9 - x^2$, so that $x^2 = 5$ and and $x = \pm\sqrt{5}$, or $x \approx \pm 2.236$, Therefore, the quantity demanded when the unit price is \$2 is approximately 2236 units.

61. a.



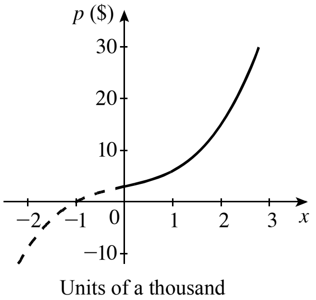
b. If $x = 2$, then $p = 2^2 + 16\,(2) + 40 = 76$, or \$76.

62. a.



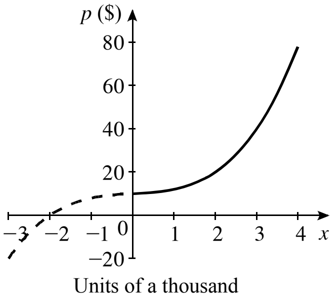
b. If $x = 2$, then $p = 2\,(2)^2 + 18 = 26$, or \$26.

63. a.



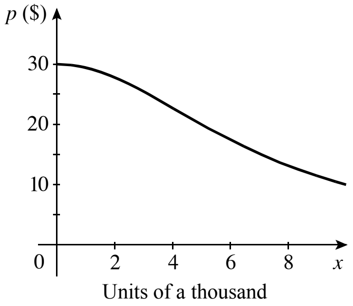
b. $p = 2^3 + 2(2) + 3 = 15$, or \$15.

64. a.



b. $p = 2^3 + 2 + 10 = 20$, or \$20.

65. a.

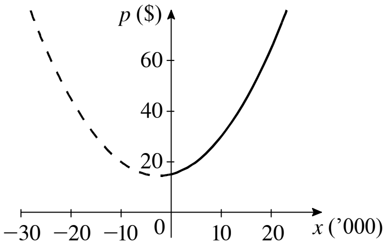


b. Substituting $x = 10$ into the demand function, we have

$$p = \frac{30}{0.02(10)^2 + 1} = \frac{30}{3} = 10, \text{ or } \$10.$$

66. Substituting $x = 6$ and $p = 8$ into the given equation gives $8 = \sqrt{-36a + b}$, or $-36a + b = 64$. Next, substituting $x = 8$ and $p = 6$ into the equation gives $6 = \sqrt{-64a + b}$, or $-64a + b = 36$. Solving the system $\begin{cases} -36a + b = 64 \\ -64a + b = 36 \end{cases}$ for a and b, we find $a = 1$ and $b = 100$. Therefore the demand equation is $p = \sqrt{-x^2 + 100}$. When the unit price is set at \$7.50, we have $7.5 = \sqrt{-x^2 + 100}$, or $56.25 = -x^2 + 100$ from which we deduce that $x \approx \pm 6.614$. Thus, the quantity demanded is approximately 6614 units.

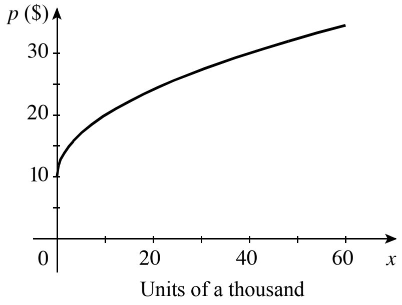
67. a.



b. If $x = 5$, then

$p = 0.1(5)^2 + 0.5(5) + 15 = 20$, or \$20.

68. Substituting $x = 10{,}000$ and $p = 20$ into the given equation yields $20 = a\sqrt{10{,}000} + b = 100a + b$. Next, substituting $x = 62{,}500$ and $p = 35$ into the equation yields $35 = a\sqrt{62{,}500} + b = 250a + b$. Subtracting the first equation from the second yields $15 = 150a$, or $a = \frac{1}{10}$. Substituting this value of a into the first equation gives $b = 10$. Therefore, the required equation is $p = \frac{1}{10}\sqrt{x} + 10$. Substituting $x = 40{,}000$ into the supply equation yields $p = \frac{1}{10}\sqrt{40{,}000} + 10 = 30$, or \$30.



69. **a.** We solve the system of equations $p = cx + d$ and $p = ax + b$. Substituting the first equation into the second gives $cx + d = ax + d$, so $(c - a)x = b - d$ and $x = \frac{b - d}{c - a}$. Because $a < 0$ and $c > 0$, $c - a \neq 0$ and x is well-defined. Substituting this value of x into the second equation, we obtain $p = a\left(\frac{b - d}{c - a}\right) + b = \frac{ab - ad + bc - ab}{c - a} = \frac{bc - ad}{c - a}$. Therefore, the equilibrium quantity is $\frac{b - d}{c - a}$ and the equilibrium price is $\frac{bc - ad}{c - a}$.

b. If c is increased, the denominator in the expression for x increases and so x gets smaller. At the same time, the first term in the first equation for p decreases and so p gets larger. This analysis shows that if the unit price for producing the product is increased then the equilibrium quantity decreases while the equilibrium price increases.

c. If b is decreased, the numerator of the expression for x decreases while the denominator stays the same. Therefore, x decreases. The expression for p also shows that p decreases. This analysis shows that if the (theoretical) upper bound for the unit price of a commodity is lowered, then both the equilibrium quantity and the equilibrium price drop.

70. We solve the system of equations $p = -x^2 - 2x + 100$ and $p = 8x + 25$. Thus, $-x^2 - 2x + 100 = 8x + 25$, or $x^2 + 10x - 75 = 0$. Factoring this equation, we have $(x + 15)(x - 5) = 0$. Therefore, $x = -15$ or $x = 5$. Rejecting the negative root, we have $x = 5$, and the corresponding value of p is $p = 8(5) + 25 = 65$. We conclude that the equilibrium quantity is 5000 and the equilibrium price is \$65.

71. We solve the equation $-2x^2 + 80 = 15x + 30$, or $2x^2 + 15x - 50 = 0$ for x. Thus, $(2x - 5)(x + 10) = 0$, and so $x = \frac{5}{2}$ or $x = -10$. Rejecting the negative root, we have $x = \frac{5}{2}$. The corresponding value of p is $p = -2\left(\frac{5}{2}\right)^2 + 80 = 67.5$. We conclude that the equilibrium quantity is 2500 and the equilibrium price is \$67.50.

72. We solve the system $\begin{cases} p = 60 - 2x^2 \\ p = x^2 + 9x + 30 \end{cases}$ Equating the right-hand sides, we have $x^2 + 9x + 30 = 60 - 2x^2$, so $3x^2 + 9x - 30 = 0$, $x^2 + 3x - 10 = 0$, and $(x + 5)(x - 2) = 0$, giving $x = -5$ or $x = 2$. We take $x = 2$. The corresponding value of p is 52, so the equilibrium quantity is 2000 and the equilibrium price is \$52.

73. Solving both equations for x, we have $x = -\frac{11}{3}p + 22$ and $x = 2p^2 + p - 10$. Equating the right-hand sides, we have $-\frac{11}{3}p + 22 = 2p^2 + p - 10$, or $-11p + 66 = 6p^2 + 3p - 30$, and so $6p^2 + 14p - 96 = 0$. Dividing this last equation by 2 and then factoring, we have $(3p + 16)(p - 3) = 0$, so $p = 3$ is the only valid solution. The corresponding value of x is $2(3)^2 + 3 - 10 = 11$. We conclude that the equilibrium quantity is 11,000 and the equilibrium price is \$3.

74. Equating the right-hand sides of the two equations, we have $0.1x^2 + 2x + 20 = -0.1x^2 - x + 40$, so $0.2x^2 + 3x - 20 = 0$, $2x^2 + 30x - 200 = 0$, $x^2 + 15x - 100 = 0$, and $(x + 20)(x - 5) = 0$. Therefore the only valid solution is $x = 5$. Substituting $x = 5$ into the first equation gives $p = -0.1(25) - 5 + 40 = 32.5$. Therefore, the equilibrium quantity is 500 tents (x is measured in hundreds) and the equilibrium price is \$32.50.

75. Equating the right-hand sides of the two equations, we have $144 - x^2 = 48 + \frac{1}{2}x^2$, so $288 - 2x^2 = 96 + x^2$, $3x^2 = 192$, and $x^2 = 64$. Therefore, $x = \pm 8$. We take $x = 8$, and the corresponding value of p is $144 - 8^2 = 80$. We conclude that the equilibrium quantity is 8000 tires and the equilibrium price is \$80.

76. Because there is 80 feet of fencing available, $2x + 2y = 80$, so $x + y = 40$ and $y = 40 - x$. Then the area of the garden is given by $f = xy = x(40 - x) = 40x - x^2$. The domain of f is $[0, 40]$.

77. The area of Juanita's garden is 250 ft^2. Therefore $xy = 250$ and $y = \dfrac{250}{x}$. The amount of fencing needed is given by $2x + 2y$. Therefore, $f = 2x + 2\left(\dfrac{250}{x}\right) = 2x + \dfrac{500}{x}$. The domain of f is $x > 0$.

78. The volume of the box is given by area of the base times the height of the box. Thus, $V = f(x) = (15 - 2x)(8 - 2x)x$.

79. Because the volume of the box is the area of the base times the height of the box, we have $V = x^2 y = 20$. Thus, we have $y = \dfrac{20}{x^2}$. Next, the amount of material used in constructing the box is given by the area of the base of the box, plus the area of the four sides, plus the area of the top of the box; that is, $A = x^2 + 4xy + x^2$. Then, the cost of constructing the box is given by $f(x) = 0.30x^2 + 0.40x \cdot \dfrac{20}{x^2} + 0.20x^2 = 0.5x^2 + \dfrac{8}{x}$, where $f(x)$ is measured in dollars and $f(x) > 0$.

80. Because the perimeter of a circle is $2\pi r$, we know that the perimeter of the semicircle is πx. Next, the perimeter of the rectangular portion of the window is given by $2y + 2x$, so the perimeter of the Norman window is $\pi x + 2y + 2x$ and $\pi x + 2y + 2x = 28$, or $y = \frac{1}{2}(28 - \pi x - 2x)$. Because the area of the window is given by $2xy + \frac{1}{2}\pi x^2$, we see that $A = 2xy + \frac{1}{2}\pi x^2$. Substituting the value of y found earlier, we see that

$$A = f(x) = x(28 - \pi x - 2x) + \tfrac{1}{2}\pi x^2 = \tfrac{1}{2}\pi x^2 + 28x - \pi x^2 - 2x^2 = 28x - \tfrac{\pi}{2}x^2 - 2x^2$$
$$= 28x - \left(\frac{\pi}{2} + 2\right)x^2.$$

81. The average yield of the apple orchard is 36 bushels/tree when the density is 22 trees/acre. Let x be the unit increase in tree density beyond 22. Then the yield of the apple orchard in bushels/acre is given by $(22 + x)(36 - 2x)$.

82. $xy = 50$ and so $y = \dfrac{50}{x}$. The area of the printed page is $A = (x - 1)(y - 2) = (x - 1)\left(\dfrac{50}{x} - 2\right) = -2x + 52 - \dfrac{50}{x}$, so the required function is $f(x) = -2x + 52 - \dfrac{50}{x}$. We must have $x > 0$, $x - 1 \geq 0$, and and $\dfrac{50}{x} - 2 \geq 2$. The last inequality is solved as follows: $\dfrac{50}{x} \geq 4$, so $\dfrac{x}{50} \leq \dfrac{1}{4}$, so $x \leq \frac{50}{4} = \frac{25}{2}$. Thus, the domain is $\left[1, \frac{25}{2}\right]$.

83. **a.** Let x denote the number of bottles sold beyond 10,000 bottles. Then $P(x) = (10{,}000 + x)(5 - 0.0002x) = -0.0002x^2 + 3x + 50{,}000$.

b. He can expect a profit of $P(6000) = -0.0002\left(6000^2\right) + 3(6000) + 50{,}000 = 60{,}800$, or \$60,800.

84. **a.** Let x denote the number of people beyond 20 who sign up for the cruise. Then the revenue is $R(x) = (20 + x)(600 - 4x) = -4x^2 + 520x + 12{,}000$.

b. $R(40) = -4\left(40^2\right) + 520(40) + 12{,}000 = 26{,}400$, or \$26,400.

c. $R(60) = -4\left(60^2\right) + 520(60) + 12{,}000 = 28{,}800$, or \$28,800.

85. False. $f(x) = 3x^{3/4} + x^{1/2} + 1$ is not a polynomial function. The powers of x must be nonnegative integers.

86. True. If $P(x)$ is a polynomial function, then $P(x) = \dfrac{P(x)}{1}$ and so it is a rational function. The converse is false. For example, $R(x) = \dfrac{x+1}{x-1}$ is a rational function that is not a polynomial.

87. False. $f(x) = x^{1/2}$ is not defined for negative values of x.

88. False. A power function has the form x^r, where r is a real number.

Using Technology

page 98

1. $(-3.0414, 0.1503)$, $(3.0414, 7.4497)$.

2. $(-5.3852, 9.8007)$, $(5.3852, -4.2007)$.

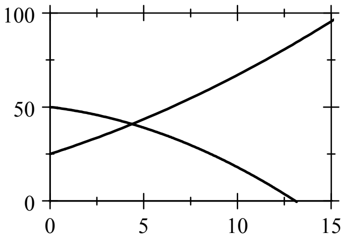
3. $(-2.3371, 2.4117)$, $(6.0514, -2.5015)$.

4. $(-2.5863, -0.3585)$, $(6.1863, -4.5694)$.

5. $(-1.0219, -6.3461)$, $(1.2414, -1.5931)$, and $(5.7805, 7.9391)$.

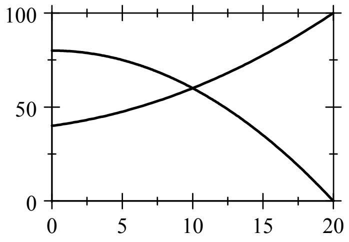
6. $(-0.0484, 2.0609)$, $(2.0823, 2.8986)$, and $(4.9661, 1.1405)$.

7. a.



b. 438 wall clocks; \$40.92.

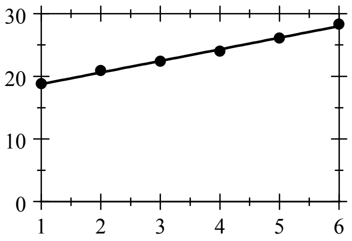
8. a.



b. 1000 cameras; \$60.00.

9. a. $f(t) = 1.85t + 16.9$.

b.



c.

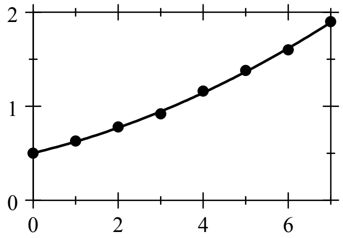
t	y
1	18.8
2	20.6
3	22.5
4	24.3
5	26.2
6	28.0

These values are close to the given data.

d. $f(8) = 1.85(8) + 16.9 = 31.7$ gallons.

10. **a.** $f(t) = 0.0128t^2 + 0.109t + 0.50.$

b.



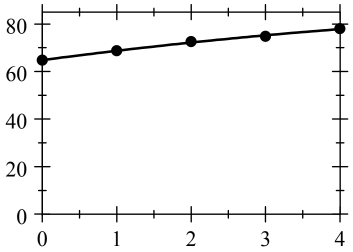
c.

t	y
0	0.50
3	0.94
6	1.61
7	1.89

These values are close to the given data.

11. **a.** $f(t) = -0.221t^2 + 4.14t + 64.8.$

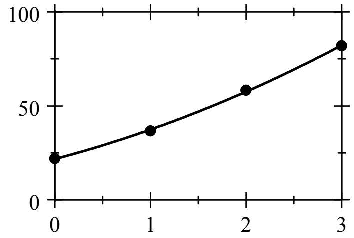
b.



c. 77.8 million

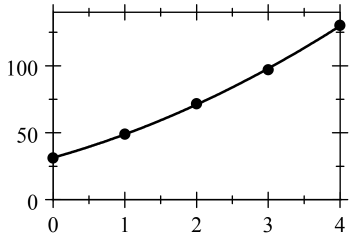
12. **a.** $f(t) = 2.25x^2 + 13.41x + 21.76.$

b.



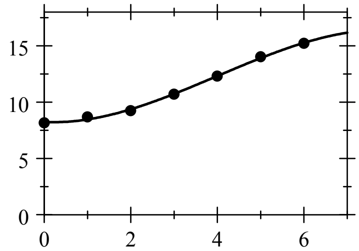
13. **a.** $f(t) = 2.4t^2 + 15t + 31.4.$

b.



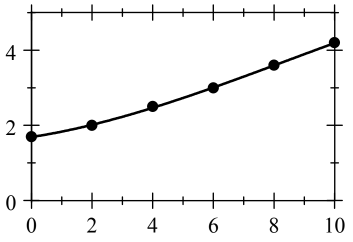
14. **a.** $f(t) = -0.038167t^3 + 0.45713t^2 - 0.19758t + 8.2457.$

b.



15. **a.** $f(t) = -0.00081t^3 + 0.0206t^2 + 0.125t + 1.69.$

b.



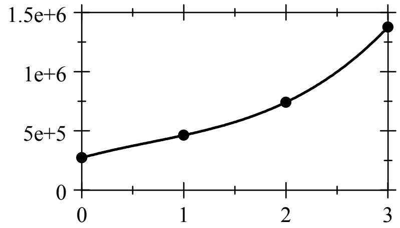
c.

t	y
1	1.8
5	2.7
10	4.2

The revenues were \$1.8 trillion in 2001, \$2.7 trillion in 2005, and \$4.2 trillion in 2010.

16. a. $y = 44{,}560t^3 - 89{,}394t^2 + 234{,}633t + 273{,}288.$

b.

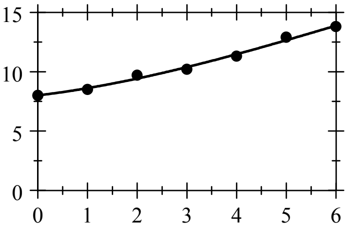


c.

t	$f(t)$
0	273,288
1	463,087
2	741,458
3	1,375,761

17. a. $f(t) = -0.0056t^3 + 0.112t^2 + 0.51t + 8.$

b.

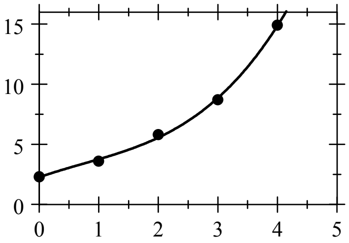


c.

t	0	3	6
$f(t)$	8	10.4	13.9

18. a. $f(t) = 0.2t^3 - 0.45t^2 + 1.75t + 2.26.$

b.

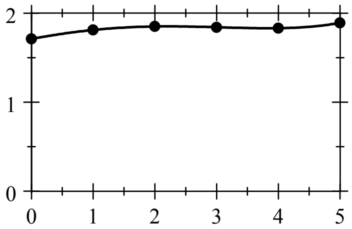


c.

t	0	1	2	3	4
$f(t)$	2.3	3.8	5.6	8.9	14.9

19. a. $f(t) = 0.00125t^4 + 0.0051t^3 - 0.0243t^2 + 0.129t + 1.71.$

b.



c.

t	0	1	2	3	4	5
$f(t)$	1.71	1.81	1.85	1.84	1.83	1.89

d. The average amount of nicotine in 2005 is $f(6) = 2.128$, or approximately 2.13 mg/cigarette.

20. $A(t) = 0.000008140t^4 - 0.00043833t^3 - 0.0001305t^2 + 0.02202t + 2.612.$

2.4 Limits

Concept Questions page 115

1. The values of $f(x)$ can be made as close to 3 as we please by taking x sufficiently close to $x = 2$.

2. a. Nothing. Whether $f(3)$ is defined or not does not depend on $\lim_{x \to 3} f(x)$.

b. Nothing. $\lim_{x \to 2} f(x)$ has nothing to do with the value of f at $x = 2$.

3. a. $$\lim_{x\to4}\sqrt{x}\left(2x^2+1\right)=\lim_{x\to4}\left(\sqrt{x}\right)\lim_{x\to4}\left(2x^2+1\right) \qquad \text{(Rule 4)}$$
$$=\sqrt{4}\left[2\left(4\right)^2+1\right] \qquad \text{(Rules 1 and 3)}$$
$$=66$$

b. $$\lim_{x\to1}\left(\frac{2x^2+x+5}{x^4+1}\right)^{3/2}=\left(\lim_{x\to1}\frac{2x^2+x+5}{x^4+1}\right)^{3/2} \qquad \text{(Rule 1)}$$
$$=\left(\frac{2+1+5}{1+1}\right)^{3/2} \qquad \text{(Rules 2, 3, and 5)}$$
$$=4^{3/2}=8$$

4. A limit that has the form $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$. For example, $\lim_{x\to3}\frac{x^2-9}{x-3}$.

5. $\lim_{x\to\infty}f(x)=L$ means $f(x)$ can be made as close to L as we please by taking x sufficiently large.
$\lim_{x\to-\infty}f(x)=M$ means $f(x)$ can be made as close to M as we please by taking negative x as large as we please in absolute value.

Exercises page 115

1. $\lim_{x\to-2}f(x)=3$.

2. $\lim_{x\to1}f(x)=2$.

3. $\lim_{x\to3}f(x)=3$.

4. $\lim_{x\to1}f(x)$ does not exist. If we consider any value of x to the right of $x=1$, we find that $f(x)=3$. On the other hand, if we consider values of x to the left of $x=1$, $f(x)\le1.5$, so that $f(x)$ does not approach a fixed number as x approaches 1.

5. $\lim_{x\to-2}f(x)=3$.

6. $\lim_{x\to-2}f(x)=3$.

7. The limit does not exist. If we consider any value of x to the right of $x=-2$, $f(x)\le2$. If we consider values of x to the left of $x=-2$, $f(x)\ge-2$. Because $f(x)$ does not approach any one number as x approaches $x=-2$, we conclude that the limit does not exist.

8. The limit does not exist.

9.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	4.61	4.9601	4.9960	5.004	5.0401	5.41

$\lim_{x\to2}\left(x^2+1\right)=5$.

10.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.62	0.9602	0.996002	1.004002	1.0402	1.42

$\lim_{x\to 1}\left(2x^2-1\right)=1.$

11.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	−1	−1	−1	1	1	1

The limit does not exist.

12.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	−1	−1	−1	1	1	1

The limit does not exist.

13.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	100	10,000	1,000,000	1,000,000	10,000	100

The limit does not exist.

14.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	−10	−100	−1000	1000	100	10

The limit does not exist.

15.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	2.9	2.99	2.999	3.001	3.01	3.1

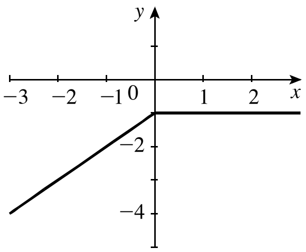
$\lim_{x\to 1}\frac{x^2+x-2}{x-1}=3.$

16.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	1	1	1	1	1	1

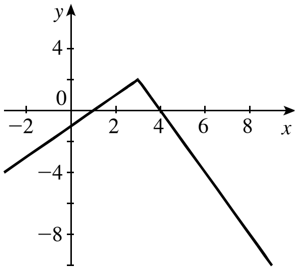
$\lim_{x\to 1}\frac{x-1}{x-1}=1.$

17.



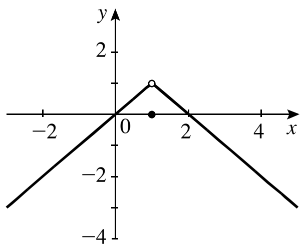
$\lim_{x\to 0}f(x)=-1.$

18.



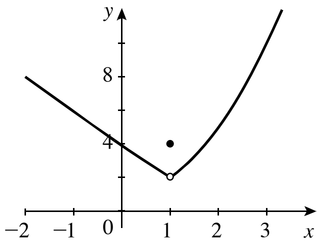
$\lim_{x\to 3}f(x)=2.$

19.



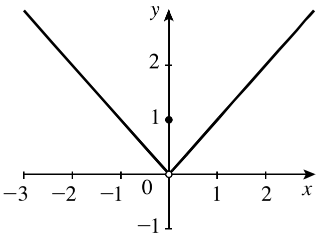
$\lim_{x\to 1} f(x) = 1.$

20.



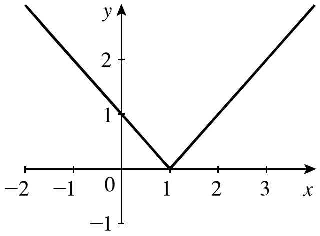
$\lim_{x\to 1} f(x) = 2.$

21.



$\lim_{x\to 0} f(x) = 0.$

22.



$\lim_{x\to 1} f(x) = 0.$

23. $\lim_{x\to 2} 3 = 3.$

24. $\lim_{x\to -2} -3 = -3.$

25. $\lim_{x\to 3} x = 3.$

26. $\lim_{x\to -2} -3x = -3(-2) = 6.$

27. $\lim_{x\to 1} \left(1 - 2x^2\right) = 1 - 2(1)^2 = -1.$

28. $\lim_{t\to 3} \left(4t^2 - 2t + 1\right) = 4(3)^2 - 2(3) + 1 = 31.$

29.
$$\lim_{x\to 1} \left(2x^3 - 3x^2 + x + 2\right) = 2(1)^3 - 3(1)^2 + 1 + 2 = 2.$$

30.
$$\lim_{x\to 0} \left(4x^5 - 20x^2 + 2x + 1\right) = 4(0)^5 - 20(0)^2 + 2(0) + 1 = 1.$$

31. $\lim_{s\to 0} \left(2s^2 - 1\right)(2s + 4) = (-1)(4) = -4.$

32. $\lim_{x\to 2} \left(x^2 + 1\right)\left(x^2 - 4\right) = \left(2^2 + 1\right)\left(2^2 - 4\right) = 0.$

33. $\lim_{x\to 2} \dfrac{2x + 1}{x + 2} = \dfrac{2(2) + 1}{2 + 2} = \dfrac{5}{4}.$

34. $\lim_{x\to 1} \dfrac{x^3 + 1}{2x^3 + 2} = \dfrac{1^3 + 1}{2\left(1^3\right) + 2} = \dfrac{2}{4} = \dfrac{1}{2}.$

35. $\lim_{x\to 2} \sqrt{x + 2} = \sqrt{2 + 2} = 2.$

36. $\lim_{x\to -2} \sqrt[3]{5x + 2} = \sqrt[3]{5(-2) + 2} = \sqrt[3]{-8} = -2.$

37.
$$\lim_{x\to -3} \sqrt{2x^4 + x^2} = \sqrt{2(-3)^4 + (-3)^2} = \sqrt{162 + 9} = \sqrt{171} = 3\sqrt{19}.$$

38. $\lim_{x\to 2} \sqrt{\dfrac{2x^3 + 4}{x^2 + 1}} = \sqrt{\dfrac{2(8) + 4}{4 + 1}} = 2.$

39. $\lim_{x\to -1} \dfrac{\sqrt{x^2 + 8}}{2x + 4} = \dfrac{\sqrt{(-1)^2 + 8}}{2(-1) + 4} = \dfrac{\sqrt{9}}{2} = \dfrac{3}{2}.$

40. $\lim_{x\to 3} \dfrac{x\sqrt{x^2 + 7}}{2x - \sqrt{2x + 3}} = \dfrac{3\sqrt{3^2 + 7}}{2(3) - \sqrt{2(3) + 3}} = \dfrac{12}{3} = 4.$

41. $\lim_{x\to a}\left[f(x)-g(x)\right]=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)$

$= 3-4=-1.$

42. $\lim_{x\to a}2f(x)=2(3)=6.$

43. $\lim_{x\to a}\left[2f(x)-3g(x)\right]=\lim_{x\to a}2f(x)-\lim_{x\to a}3g(x)$

$=2(3)-3(4)=-6.$

44. $\lim_{x\to a}\left[f(x)g(x)\right]=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)=3\cdot 4$

$=12.$

45. $\lim_{x\to a}\sqrt{g(x)}=\lim_{x\to a}\sqrt{4}=2.$

46. $\lim_{x\to a}\sqrt[3]{5f(x)+3g(x)}=\sqrt[3]{5(3)+3(4)}=\sqrt[3]{27}=3.$

47. $\lim_{x\to a}\dfrac{2f(x)-g(x)}{f(x)g(x)}=\dfrac{2(3)-(4)}{(3)(4)}=\dfrac{2}{12}=\dfrac{1}{6}.$

48. $\lim_{x\to a}\dfrac{g(x)-f(x)}{f(x)+\sqrt{g(x)}}=\dfrac{4-3}{3+2}=\dfrac{1}{5}.$

49. $\lim_{x\to 1}\dfrac{x^2-1}{x-1}=\lim_{x\to 1}\dfrac{(x-1)(x+1)}{x-1}=\lim_{x\to 1}(x+1)$

$=1+1=2.$

50. $\lim_{x\to -2}\dfrac{x^2-4}{x+2}=\lim_{x\to -2}\dfrac{(x-2)(x+2)}{x+2}$

$=\lim_{x\to -2}(x-2)=-2-2=-4.$

51. $\lim_{x\to 0}\dfrac{x^2-x}{x}=\lim_{x\to 0}\dfrac{x(x-1)}{x}=\lim_{x\to 0}(x-1)$

$=0-1=-1.$

52. $\lim_{x\to 0}\dfrac{2x^2-3x}{x}=\lim_{x\to 0}\dfrac{x(2x-3)}{x}=\lim_{x\to 0}(2x-3)$

$=-3.$

53. $\lim_{x\to -5}\dfrac{x^2-25}{x+5}=\lim_{x\to -5}\dfrac{(x+5)(x-5)}{x+5}$

$=\lim_{x\to -5}(x-5)=-10.$

54. $\lim_{b\to -3}\dfrac{b+1}{b+3}$ does not exist.

55. $\lim_{x\to 1}\dfrac{x}{x-1}$ does not exist.

56. $\lim_{x\to 2}\dfrac{x+2}{x-2}$ does not exist.

57. $\lim_{x\to -2}\dfrac{x^2-x-6}{x^2+x-2}=\lim_{x\to -2}\dfrac{(x-3)(x+2)}{(x+2)(x-1)}=\lim_{x\to -2}\dfrac{x-3}{x-1}=\dfrac{-2-3}{-2-1}=\dfrac{5}{3}.$

58. $\lim_{z\to 2}\dfrac{z^3-8}{z-2}=\lim_{z\to 2}\dfrac{(z-2)\left(z^2+2z+4\right)}{z-2}=\lim_{z\to 2}\left(z^2+2z+4\right)=2^2+2(2)+4=12.$

59. $\lim_{x\to 1}\dfrac{\sqrt{x}-1}{x-1}=\lim_{x\to 1}\dfrac{\sqrt{x}-1}{x-1}\cdot\dfrac{\sqrt{x}+1}{\sqrt{x}+1}=\lim_{x\to 1}\dfrac{x-1}{(x-1)\left(\sqrt{x}+1\right)}=\lim_{x\to 1}\dfrac{1}{\sqrt{x}+1}=\dfrac{1}{2}.$

60. $\lim_{x\to 4}\dfrac{x-4}{\sqrt{x}-2}=\lim_{x\to 4}\dfrac{x-4}{\sqrt{x}-2}\cdot\dfrac{\sqrt{x}+2}{\sqrt{x}+2}=\lim_{x\to 4}\sqrt{x}+2=2+2=4.$

61. $\lim_{x\to 1}\dfrac{x-1}{x^3+x^2-2x}=\lim_{x\to 1}\dfrac{x-1}{x(x-1)(x+2)}=\lim_{x\to 1}\dfrac{1}{x(x+2)}=\dfrac{1}{3}.$

62. $\lim_{x\to -2}\dfrac{4-x^2}{2x^2+x^3}=\lim_{x\to -2}\dfrac{(2-x)(2+x)}{x^2(2+x)}=\lim_{x\to -2}\dfrac{2-x}{x^2}=\dfrac{2-(-2)}{(-2)^2}=1.$

63. $\lim_{x\to\infty}f(x)=\infty$ (does not exist) and $\lim_{x\to-\infty}f(x)=\infty$ (does not exist).

64. $\lim_{x\to\infty} f(x) = \infty$ (does not exist) and $\lim_{x\to-\infty} f(x) = -\infty$ (does not exist).

65. $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.

66. $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = 1$.

67. $\lim_{x\to\infty} f(x) = -\infty$ (does not exist) and $\lim_{x\to-\infty} f(x) = -\infty$ (does not exist).

68. $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = \infty$ (does not exist).

69. $f(x) = \dfrac{1}{x^2+1}$.

x	1	10	100	1000
$f(x)$	0.5	0.009901	0.0001	0.000001

x	-1	-10	-100	-1000
$f(x)$	0.5	0.009901	0.0001	0.000001

$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$.

70. $f(x) = \dfrac{2x}{x+1}$.

x	1	10	100	1000
$f(x)$	1	1.818	1.980	1.998

x	-5	-10	-100	-1000
$f(x)$	2.5	2.222	2.020	2.002

$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 2$.

71. $f(x) = 3x^3 - x^2 + 10$.

x	1	5	10	100	1000
$f(x)$	12	360	2910	2.99×10^6	2.999×10^9

x	-1	-5	-10	-100	-1000
$f(x)$	6	-390	-3090	-3.01×10^6	-3.0×10^9

$\lim_{x\to\infty} f(x) = \infty$ (does not exist) and $\lim_{x\to-\infty} f(x) = -\infty$ (does not exist).

72. $f(x) = \dfrac{|x|}{x}$

x	1	10	100
$f(x)$	1	1	1

x	-1	-10	-100
$f(x)$	-1	-1	-1

$\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = -1$.

73. $\lim_{x\to\infty} \dfrac{3x+2}{x-5} = \lim_{x\to\infty} \dfrac{3+\dfrac{2}{x}}{1-\dfrac{5}{x}} = \dfrac{3}{1} = 3$.

74. $\lim\limits_{x\to-\infty} \dfrac{4x^2-1}{x+2} = \lim\limits_{x\to-\infty} \dfrac{4x-\dfrac{1}{x}}{1+\dfrac{2}{x}} = -\infty$; that is, the limit does not exist.

75. $\lim\limits_{x\to-\infty} \dfrac{3x^3+x^2+1}{x^3+1} = \lim\limits_{x\to-\infty} \dfrac{3+\dfrac{1}{x}+\dfrac{1}{x^3}}{1+\dfrac{1}{x^3}} = 3.$

76. $\lim\limits_{x\to\infty} \dfrac{2x^2+3x+1}{x^4-x^2} = \lim\limits_{x\to\infty} \dfrac{\dfrac{2}{x^2}+\dfrac{3}{x^3}+\dfrac{1}{x^4}}{1-\dfrac{1}{x^2}} = 0.$

77. $\lim\limits_{x\to-\infty} \dfrac{x^4+1}{x^3-1} = \lim\limits_{x\to-\infty} \dfrac{x+\dfrac{1}{x^3}}{1-\dfrac{1}{x^3}} = -\infty$; that is, the limit does not exist.

78. $\lim\limits_{x\to\infty} \dfrac{4x^4-3x^2+1}{2x^4+x^3+x^2+x+1} = \lim\limits_{x\to\infty} \dfrac{4-\dfrac{3}{x^2}+\dfrac{1}{x^4}}{2+\dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{1}{x^3}+\dfrac{1}{x^4}} = 2.$

79. $\lim\limits_{x\to\infty} \dfrac{x^5-x^3+x-1}{x^6+2x^2+1} = \lim\limits_{x\to\infty} \dfrac{\dfrac{1}{x}-\dfrac{1}{x^3}+\dfrac{1}{x^5}-\dfrac{1}{x^6}}{1+\dfrac{2}{x^4}+\dfrac{1}{x^6}} = 0.$

80. $\lim\limits_{x\to\infty} \dfrac{2x^2-1}{x^3+x^2+1} = \lim\limits_{x\to\infty} \dfrac{\dfrac{2}{x}-\dfrac{1}{x^3}}{1+\dfrac{1}{x}+\dfrac{1}{x^3}} = 0.$

81. a. The cost of removing 50% of the pollutant is $C(50) = \dfrac{0.5(50)}{100-50} = 0.5$, or \$500,000. Similarly, we find that the cost of removing 60%, 70%, 80%, 90%, and 95% of the pollutant is \$750,000, \$1,166,667, \$2,000,000, \$4,500,000, and \$9,500,000, respectively.

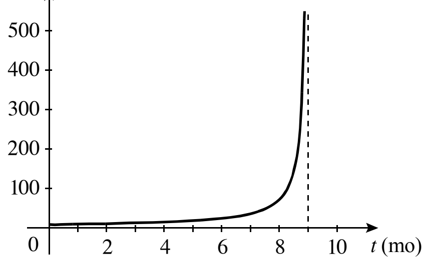
b. $\lim\limits_{x\to100} \dfrac{0.5x}{100-x} = \infty$, which means that the cost of removing the pollutant increases without bound if we wish to remove almost all of the pollutant.

82. a. The number present initially is given by $P(0) = \dfrac{72}{9-0} = 8.$

b. As t approaches 9 (remember that $0 < t < 9$), the denominator approaches 0 while the numerator remains constant at 72. Therefore, $P(t)$ gets larger and larger. Thus,

$$\lim_{t\to9} P(t) = \lim_{t\to9} \frac{72}{9-t} = \infty.$$

c.



83. $\lim_{x\to\infty} \overline{C}(x) = \lim_{x\to\infty} 2.2 + \frac{2500}{x} = 2.2$, or \$2.20 per DVD. In the long run, the average cost of producing x DVDs approaches \$2.20/disc.

84. $\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \frac{0.2t}{t^2+1} = \lim_{t\to\infty} \frac{\frac{0.2}{t}}{1+\frac{1}{t^2}} = 0$, which says that the concentration of drug in the bloodstream eventually decreases to zero.

85. a. $T(1) = \frac{120}{1+4} = 24$, or \$24 million. $T(2) = \frac{120(2)^2}{8} = 60$, or \$60 million. $T(3) = \frac{120(3)^2}{13} = 83.1$, or \$83.1 million.

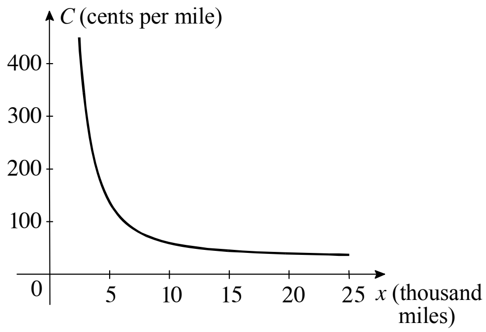
b. In the long run, the movie will gross $\lim_{x\to\infty} \frac{120x^2}{x^2+4} = \lim_{x\to\infty} \frac{120}{1+\frac{4}{x^2}} = 120$, or \$120 million.

86. a. The current population is $P(0) = \frac{200}{40} = 5$, or 5000.

b. The population in the long run will be $\lim_{t\to\infty} \frac{25t^2+125t+200}{t^2+5t+40} = \lim_{t\to\infty} \frac{25+\frac{125}{t}+\frac{200}{t^2}}{1+\frac{5}{t}+\frac{40}{t^2}} = 25$, or 25,000.

87. a. The average cost of driving 5000 miles per year is $C(5) = \frac{2410}{5^{1.95}} + 32.8 \approx 137.28$, or 137.3 cents per mile. Similarly, we see that the average costs of driving 10, 15, 20, and 25 thousand miles per year are 59.8, 45.1, 39.8, and 37.3 cents per mile, respectively.

b.



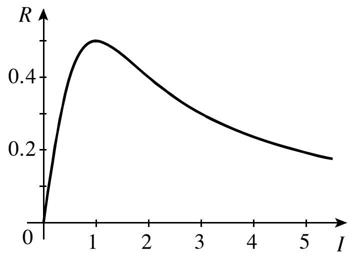
c. It approaches 32.8 cents per mile.

88. a. $R(I) = \frac{I}{1+I^2}$

I	0	1	2	3	4	5
$R(I)$	0	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{4}{17}$	$\frac{5}{26}$

b. $\lim_{I\to\infty} R(I) = \lim_{I\to\infty} \frac{I}{1+I^2} = 0.$

c.



89. False. Let $f(x) = \begin{cases} -1 & \text{if} x < 0 \\ 1 & \text{if} x \geq 0 \end{cases}$ Then $\lim_{x\to 0} f(x) = 1$, but $f(1)$ is not defined.

90. True.

91. True. Division by zero is not permitted.

92. False. Let $f(x) = (x-3)^2$ and $g(x) = x-3$. Then $\lim_{x\to3} f(x) = 0$ and $\lim_{x\to3} g(x) = 0$, but $\lim_{x\to3} \frac{f(x)}{g(x)} = \lim_{x\to3} \frac{(x-3)^2}{x-3} = \lim_{x\to3} (x-3) = 0$.

93. True. Each limit in the sum exists. Therefore, $\lim_{x\to2} \left(\frac{x}{x+1} + \frac{3}{x-1}\right) = \lim_{x\to2} \frac{x}{x+1} + \lim_{x\to2} \frac{3}{x-1} = \frac{2}{3} + \frac{3}{1} = \frac{11}{3}$.

94. False. Neither of the limits $\lim_{x\to1} \frac{2x}{x-1}$ and $\lim_{x\to1} \frac{2}{x-1}$ exists.

95. $\lim_{x\to\infty} \frac{ax}{x+b} = \lim_{x\to\infty} \frac{a}{1+\frac{b}{x}} = a$. As the amount of substrate becomes very large, the initial speed approaches the constant a moles per liter per second.

96. Consider the functions $f(x) = 1/x$ and $g(x) = -1/x$. Observe that $\lim_{x\to0} f(x)$ and $\lim_{x\to0} g(x)$ do not exist, but $\lim_{x\to0} [f(x) + g(x)] = \lim_{x\to0} 0 = 0$. This example does not contradict Theorem 1 because the hypothesis of Theorem 1 is that $\lim_{x\to0} f(x)$ and $\lim_{x\to0} g(x)$ both exist. It does not say anything about the situation where one or both of these limits fails to exist.

97. Consider the functions $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ and $g(x) = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x \geq 0 \end{cases}$ Then $\lim_{x\to0} f(x)$ and $\lim_{x\to0} g(x)$ do not exist, but $\lim_{x\to0} [f(x)g(x)] = \lim_{x\to0} (-1) = -1$. This example does not contradict Theorem 1 because the hypothesis of Theorem 1 is that $\lim_{x\to0} f(x)$ and $\lim_{x\to0} g(x)$ both exist. It does not say anything about the situation where one or both of these limits fails to exist.

98. Take $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{x^2}$, and $a = 0$. Then $\lim_{x\to0} f(x)$ and $\lim_{x\to0} g(x)$ do not exist, but $\lim_{x\to0} \frac{f(x)}{g(x)} = \lim_{x\to0} \frac{1}{x} \cdot \frac{x^2}{1} = \lim_{x\to0} x = a$ exists. This example does not contradict Theorem 1 because the hypothesis of Theorem 1 is that $\lim_{x\to0} f(x)$ and $\lim_{x\to0} g(x)$ both exist. It does not say anything about the situation where one or both of these limits fails to exist.

Using Technology page 121

1. 5

2. 11

3. 3

4. 0

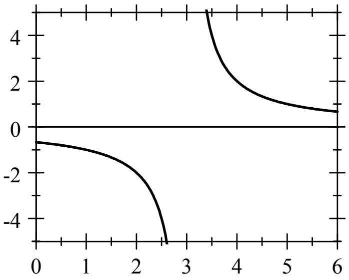
5. $\frac{2}{3}$

6. $\frac{10}{11}$

7. $e^2 \approx 7.38906$

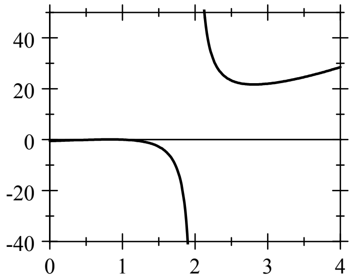
8. $\ln 2 \approx 0.693147$

9.



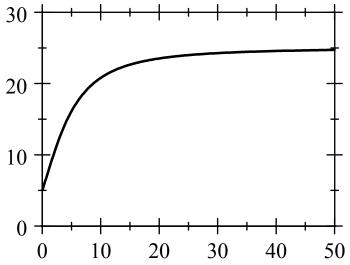
From the graph we see that $f(x)$ does not approach any finite number as x approaches 3.

10.



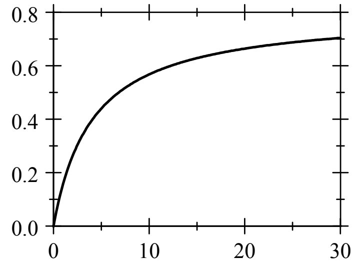
From the graph, we see that $f(x)$ does not approach any finite number as x approaches 2.

11. a.



b. $\lim\limits_{t\to\infty} \dfrac{25t^2 + 125t + 200}{t^2 + 5t + 40} = 25$, so in the long run the population will approach 25,000.

12. a.



b. $\lim\limits_{t\to\infty} \dfrac{0.8t}{t + 4.1} = \lim\limits_{t\to\infty} \dfrac{0.8}{1 + \dfrac{4.1}{t}} = 0.8.$

2.5 One-Sided Limits and Continuity

Concept Questions page 129

1. $\lim\limits_{x\to 3^-} f(x) = 2$ means $f(x)$ can be made as close to 2 as we please by taking x sufficiently close to but to the left of $x = 3$. $\lim\limits_{x\to 3^+} f(x) = 4$ means $f(x)$ can be made as close to 4 as we please by taking x sufficiently close to but to the right of $x = 3$.

2. a. $\lim\limits_{x\to 1} f(x)$ does not exist because the left- and right-hand limits at $x = 1$ are different.

b. Nothing, because the existence or value of f at $x = 1$ does not depend on the existence (or nonexistence) of the left- or right-hand, or two-sided, limits of f.

3. a. f is continuous at a if $\lim\limits_{x\to a} f(x) = f(a)$.

b. f is continuous on an interval I if f is continuous at each point in I.

4. $f(a) = L = M$.

5. a. f is continuous because the plane does not suddenly jump from one point to another.

b. f is continuous.

c. f is discontinuous because the fare “jumps”after the cab has covered a certain distance or after a certain amount of time has elapsed.

d. f is discontinuous because the rates "jump"by a certain amount (up or down) when it is adjusted at certain times.

6. Refer to page 127 in the text. Answers will vary.

Exercises page 130

1. $\lim_{x\to2^-} f(x) = 3$ and $\lim_{x\to2^+} f(x) = 2$, so $\lim_{x\to2} f(x)$ does not exist.

2. $\lim_{x\to3^-} f(x) = 3$ and $\lim_{x\to3^+} f(x) = 5$, so $\lim_{x\to3} f(x)$ does not exist.

3. $\lim_{x\to-1^-} f(x) = \infty$ and $\lim_{x\to-1^+} f(x) = 2$, so $\lim_{x\to-1} f(x)$ does not exist.

4. $\lim_{x\to1^-} f(x) = 3$ and $\lim_{x\to1^+} f(x) = 3$, so $\lim_{x\to1} f(x) = 3$.

5. $\lim_{x\to1^-} f(x) = 0$ and $\lim_{x\to1^+} f(x) = 2$, so $\lim_{x\to1} f(x)$ does not exist.

6. $\lim_{x\to0^-} f(x) = 2$ and $\lim_{x\to0^+} f(x) = \infty$, so $\lim_{x\to0} f(x)$ does not exist.

7. $\lim_{x\to0^-} f(x) = -2$ and $\lim_{x\to0^+} f(x) = 2$, so $\lim_{x\to0} f(x)$ does not exist.

8. $\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = \lim_{x\to0} f(x) = 2$.

9. True.
10. True.
11. True.
12. True.
13. False.
14. True.

15. True.
16. True.
17. False.
18. True.
19. True.
20. False

21. $\lim_{x\to1^+} (2x+4) = 6$.

22. $\lim_{x\to1^-} (3x-4) = -1$.

23. $\lim_{x\to2^-} \frac{x-3}{x+2} = \frac{2-3}{2+2} = -\frac{1}{4}$.

24. $\lim_{x\to1^+} \frac{x+2}{x+1} = \frac{1+2}{1+1} = \frac{3}{2}$.

25. $\lim_{x\to0^+} \frac{1}{x}$ does not exist because $\frac{1}{x} \to \infty$ as $x \to 0$ from the right.

26. $\lim_{x\to0^-} \frac{1}{x} = \infty$; that is, the limit does not exist.

27. $\lim_{x\to0^+} \frac{x-1}{x^2+1} = \frac{-1}{1} = -1$.

28. $\lim_{x\to2^+} \frac{x+1}{x^2-2x+3} = \frac{2+1}{4-4+3} = 1$.

29. $\lim_{x\to0^+} \sqrt{x} = \sqrt{\lim_{x\to0^+} x} = 0$.

30. $\lim_{x\to2^+} 2\sqrt{x-2} = 2\cdot0 = 0$.

31. $\lim_{x\to-2^+} \left(2x+\sqrt{2+x}\right) = \lim_{x\to-2^+} 2x + \lim_{x\to-2^+} \sqrt{2+x} = -4+0 = -4$.

32. $\lim_{x\to-5^+} x\left(1+\sqrt{5+x}\right) = -5\left[1+\sqrt{5+(-5)}\right] = -5$.

33. $\lim_{x\to1^-} \frac{1+x}{1-x} = \infty$, that is, the limit does not exist.

34. $\lim\limits_{x\to1^+} \dfrac{1+x}{1-x} = -\infty$.

35. $\lim\limits_{x\to2^-} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to2^-} \dfrac{(x+2)(x-2)}{x-2} = \lim\limits_{x\to2^-} (x+2) = 4$.

36. $\lim\limits_{x\to-3^+} \dfrac{\sqrt{x+3}}{x^2+1} = \dfrac{0}{10} = 0$.

37. $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} x^2 = 0$ and $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} 2x = 0$.

38. $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} (2x+3) = 3$ and $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} (-x+1) = 1$.

39. The function is discontinuous at $x = 0$. Conditions 2 and 3 are violated.

40. The function is not continuous because condition 3 for continuity is not satisfied.

41. The function is continuous everywhere.

42. The function is continuous everywhere.

43. The function is discontinuous at $x = 0$. Condition 3 is violated.

44. The function is not continuous at $x = -1$ because condition 3 for continuity is violated.

45. f is continuous for all values of x.

46. f is continuous for all values of x.

47. f is continuous for all values of x. Note that $x^2 + 1 \geq 1 > 0$.

48. f is continuous for all values of x. Note that $2x^2 + 1 \geq 1 > 0$.

49. f is discontinuous at $x = \frac{1}{2}$, where the denominator is 0. Thus, f is continuous on $\left(-\infty, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \infty\right)$.

50. f is discontinuous at $x = 1$, where the denominator is 0. Thus, f is continuous on $(-\infty, 1)$ and $(1, \infty)$.

51. Observe that $x^2 + x - 2 = (x+2)(x-1) = 0$ if $x = -2$ or $x = 1$, so f is discontinuous at these values of x. Thus, f is continuous on $(-\infty, -2)$, $(-2, 1)$, and $(1, \infty)$.

52. Observe that $x^2 + 2x - 3 = (x+3)(x-1) = 0$ if $x = -3$ or $x = 1$, so, f is discontinuous at these values of x. Thus, f is continuous on $(-\infty, -3)$, $(-3, 1)$, and $(1, \infty)$.

53. f is continuous everywhere since all three conditions are satisfied.

54. f is continuous everywhere since all three conditions are satisfied.

55. f is continuous everywhere since all three conditions are satisfied.

56. f is not defined at $x = 1$ and is discontinuous there. It is continuous everywhere else.

57. Because the denominator $x^2 - 1 = (x-1)(x+1) = 0$ if $x = -1$ or 1, we see that f is discontinuous at -1 and 1.

58. The function f is not defined at $x = 1$ and $x = 2$. Therefore, f is discontinuous at 1 and 2.

59. Because $x^2 - 3x + 2 = (x - 2)(x - 1) = 0$ if $x = 1$ or 2, we see that the denominator is zero at these points and so f is discontinuous at these numbers.

60. The denominator of the function f is equal to zero when $x^2 - 2x = x(x - 2) = 0$; that is, when $x = 0$ or $x = 2$. Therefore, f is discontinuous at $x = 0$ and $x = 2$.

61. The function f is discontinuous at $x = 4, 5, 6, \ldots, 13$ because the limit of f does not exist at these points.

62. f is discontinuous at $t = 20$, 40, and 60. When $t = 0$, the inventory stands at 750 reams. The level drops to about 200 reams by the twentieth day at which time a new order of 500 reams arrives to replenish the supply. A similar interpretation holds for the other values of t.

63. Having made steady progress up to $x = x_1$, Michael's progress comes to a standstill at that point. Then at $x = x_2$ a sudden breakthrough occurs and he then continues to solve the problem.

64. The total deposits of Franklin make a jump at each of these points as the deposits of the ailing institutions become a part of the total deposits of the parent company.

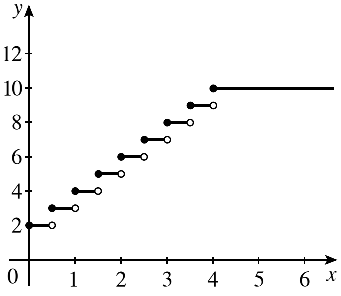
65. Conditions 2 and 3 are not satisfied at any of these points.

66. The function P is discontinuous at $t = 12$, 16, and 28. At $t = 12$, the prime interest rate jumped from $3\frac{1}{2}\%$ to 4%, at $t = 16$ it jumped to $4\frac{1}{2}\%$, and at $t = 28$ it jumped back down to 4%.

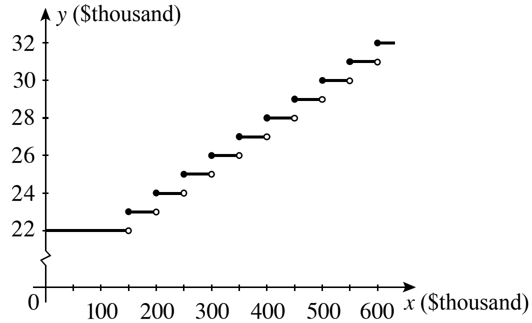
67.

$$f(x) = \begin{cases} 2 & \text{if } 0 < x \le \frac{1}{2} \\ 3 & \text{if } \frac{1}{2} < x \le 1 \\ \vdots & \quad \vdots \\ 10 & \text{if } 4\frac{1}{2} < x \le 5 \end{cases}$$

f is discontinuous at $x = \frac{1}{2}, 1, 1\frac{1}{2}, \ldots, 4$.

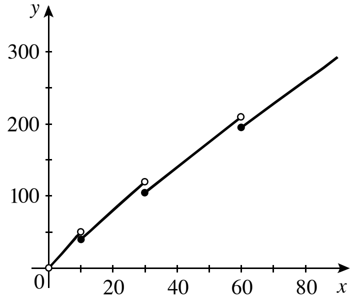


68.



f is discontinuous at $x = 150{,}000$, at $x = 200{,}000$, at $x = 250{,}000$, and so on.

69.



C is discontinuous at $x = 0$, 10, 30, and 60.

70. a. $\lim_{v\to u^+} \frac{aLv^3}{v-u} = \infty$. This reflects the fact that when the speed of the fish is very close to that of the current, the energy expended by the fish will be enormous.

b. $\lim_{v\to\infty} \frac{aLv^3}{v-u} = \infty$. This says that if the speed of the fish increases greatly, so does the amount of energy required to swim a distance of L ft.

71. a. $\lim_{t\to 0^+} S(t) = \lim_{t\to 0^+} \frac{a}{t} + b = \infty$. As the time taken to excite the tissue is made shorter and shorter, the electric current gets stronger and stronger.

b. $\lim_{t\to\infty} \frac{a}{t} + b = b$. As the time taken to excite the tissue is made longer and longer, the electric current gets weaker and weaker and approaches b.

72. a. $\lim_{D\to 0^+} L = \lim_{D\to 0^+} \frac{Y-(1-D)R}{D} = \infty$, so if the investor puts down next to nothing to secure the loan, the leverage approaches infinity.

b. $\lim_{D\to 1} L = \lim_{D\to 1} \frac{Y-(1-D)R}{D} = Y$, so if the investor puts down all of the money to secure the loan, the leverage is equal to the yield.

73. We require that $f(1) = 1 + 2 = 3 = \lim_{x\to 1^+} kx^2 = k$, so $k = 3$.

74. Because $\lim_{x\to -2} \frac{x^2-4}{x+2} = \lim_{x\to -2} \frac{(x-2)(x+2)}{x+2} = \lim_{x\to -2} (x-2) = -4$, we define $f(-2) = k = -4$, that is, take $k = -4$.

75. a. f is a polynomial of degree 2 and is therefore continuous everywhere, including the interval $[1, 3]$.

b. $f(1) = 3$ and $f(3) = -1$ and so f must have at least one zero in $(1, 3)$.

76. a. f is a polynomial of degree 3 and is thus continuous everywhere.

b. $f(0) = 14$ and $f(1) = -23$ and so f has at least one zero in $(0, 1)$.

77. a. f is a polynomial of degree 3 and is therefore continuous on $[-1, 1]$.

b. $f(-1) = (-1)^3 - 2(-1)^2 + 3(-1) + 2 = -1 - 2 - 3 + 2 = -4$ and $f(1) = 1 - 2 + 3 + 2 = 4$. Because $f(-1)$ and $f(1)$ have opposite signs, we see that f has at least one zero in $(-1, 1)$.

78. f is continuous on $[14, 16]$, $f(14) = 2(14)^{5/3} - 5(14)^{4/3} \approx -6.06$, and $f(16) = 2(16)^{5/3} - 5(16)^{4/3} \approx 1.60$. Thus, f has at least one zero in $(14, 16)$.

79. $f(0) = 6$, $f(3) = 3$, and f is continuous on $[0, 3]$. Thus, the Intermediate Value Theorem guarantees that there is at least one value of x for which $f(x) = 4$. Solving $f(x) = x^2 - 4x + 6 = 4$, we find $x^2 - 4x + 2 = 0$. Using the quadratic formula, we find that $x = 2 \pm \sqrt{2}$. Because $2 + \sqrt{2}$ does not lie in $[0, 3]$, we see that $x = 2 - \sqrt{2} \approx 0.59$.

80. Because $f(-1) = 3$, $f(4) = 13$, and f is continuous on $[-1, 4]$, the Intermediate Value Theorem guarantees that there is at least one value of x for which $f(x) = 7$ because $3 < 7 < 13$. Solving $f(x) = x^2 - x + 1 = 7$, we find $x^2 - x - 6 = (x-3)(x+2) = 0$, that is, $x = -2$ or 3. Because -2 does not lie in $[-1, 4]$, the required solution is 3.

81. $x^5 + 2x - 7 = 0$

Step	Interval in which a root lies
1	$(1, 2)$
2	$(1, 1.5)$
3	$(1.25, 1.5)$
4	$(1.25, 1.375)$
5	$(1.3125, 1.375)$
6	$(1.3125, 1.34375)$
7	$(1.328125, 1.34375)$
8	$(1.3359375, 1.34375)$

We see that a root is approximately 1.34.

82. $x^3 - x + 1 = 0$

Step	Interval in which a root lies
1	$(-2, -1)$
2	$(-1.5, -1)$
3	$(-1.5, -1.25)$
4	$(-1.375, -1.25)$
5	$(-1.375, -1.3125)$
6	$(-1.34375, -1.3125)$
7	$(-1.328125, -1.3125)$
8	$(-1.328125, -1.3203125)$
9	$(-1.32421875, -1.3203125)$

We see that a root is approximately -1.32.

83. a. $h(0) = 4 + 64(0) - 16(0) = 4$ and $h(2) = 4 + 64(2) - 16(4) = 68$.

b. The function h is continuous on $[0, 2]$. Furthermore, the number 32 lies between 4 and 68. Therefore, the Intermediate Value Theorem guarantees that there is at least one value of t in $(0, 2]$ such that $h(t) = 32$, that is, Joan must see the ball at least once during the time the ball is in the air.

c. We solve $h(t) = 4 + 64t - 16t^2 = 32$, obtaining $16t^2 - 64t + 28 = 0$, $4t^2 - 16t + 7 = 0$, and $(2t-1)(2t-7) = 0$. Thus, $t = \frac{1}{2}$ or $t = \frac{7}{2}$. Joan sees the ball on its way up half a second after it was thrown and again 3 seconds later when it is on its way down. Note that the ball hits the ground when $t \approx 4.06$, but Joan sees it approximately half a second before it hits the ground.

84. a. $f(0) = 100\left(\dfrac{0+0+100}{0+0+100}\right) = 100$ and $f(10) = 100\left(\dfrac{100+100+100}{100+200+100}\right) = \dfrac{30{,}000}{400} = 75$.

b. Because 80 lies between 75 and 100 and f is continuous on $[75, 100]$, we conclude that there exists some t in $[0, 10]$ such that $f(t) = 80$.

c. We solve $f(t) = 80$; that is, $100\left[\dfrac{t^2+10t+100}{t^2+20t+100}\right] = 80$, obtaining $5(t^2+10t+100) = 4(t^2+20t+100)$, and $t^2 - 30t + 100 = 0$. Thus, $t = \dfrac{30 \pm \sqrt{900-400}}{2} = \dfrac{30 \pm \sqrt{500}}{2} \approx 3.82$ or 26.18. Because 26.18 lies outside the interval of interest, we reject it. Thus, the oxygen content is 80% approximately 3.82 seconds after the organic waste has been dumped into the pond.

85. False. Take $f(x) = \begin{cases} -1 & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ 1 & \text{if } x > 2 \end{cases}$ Then $f(2) = 4$, but $\lim_{x\to 2} f(x)$ does not exist.

86. False. Take $f(x) = \begin{cases} x+3 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ Then $\lim_{x\to 0} f(x) = 3$, but $f(1)$ is not defined.

87. False. Consider $f(x) = \begin{cases} 0 & \text{if } x < 2 \\ 3 & \text{if } x \geq 2 \end{cases}$ Then $\lim_{x\to 2^+} f(x) = f(2) = 3$, but $\lim_{x\to 2^-} f(x) = 0$.

88. False. Consider $f(x) = \begin{cases} 2 & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ 4 & \text{if } x \geq 3 \end{cases}$ Then $\lim_{x\to 3^-} f(x) = 2$ and $\lim_{x\to 3^+} f(x) = 4$, so $\lim_{x\to 3} f(x)$ does not exist.

89. False. Consider $f(x) = \begin{cases} 2 & \text{if } x < 5 \\ 3 & \text{if } x > 5 \end{cases}$ Then $f(5)$ is not defined, but $\lim_{x\to 5^-} f(x) = 2$.

90. False. Consider the function $f(x) = x^2 - 1$ on the interval $[-2, 2]$. Here $f(-2) = f(2) = 3$, but f has zeros at $x = -1$ and $x = 1$.

91. False. Let $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ Then $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$, but $f(0) = 1$.

92. False. Let $f(x) = x$ and let $g(x) = \begin{cases} x & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$ Then $\lim_{x\to 1} f(x) = 1 = L$, $g(1) = 2 = M$, $\lim_{x\to 1} g(x) = 1$, and $\lim_{x\to 1} f(x) g(x) = \left[\lim_{x\to 1} f(x)\right]\left[\lim_{x\to 1} g(x)\right] = (1)(1) = 1 \neq 2 = LM$.

93. False. Let $f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ Then f is continuous for all $x \neq 0$ and $f(0) = 0$, but $\lim_{x\to 0} f(x)$ does not exist.

94. False. Consider $f(x) = \begin{cases} -1 & \text{if } -1 \leq x \leq 0 \\ 1 & \text{if } 0 < x \leq 1 \end{cases}$ and $g(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 0 \\ -1 & \text{if } 0 < x \leq 1 \end{cases}$

95. False. Consider $f(x) = \begin{cases} -1 & \text{if } -1 \leq x \leq 0 \\ 1 & \text{if } 0 < x \leq 1 \end{cases}$ and $g(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 0 \\ -1 & \text{if } 0 < x \leq 1 \end{cases}$

96. False. Let $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ and $g(x) = x^2$.

97. False. Consider $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ and $g(x) = \begin{cases} x+1 & \text{if } x < 0 \\ x-1 & \text{if } x \geq 0 \end{cases}$

98. The statement is false. The Intermediate Value Theorem says that there is at least one number c in $[a, b]$ such that $f(c) = M$ if M is a number between $f(a)$ and $f(b)$.

99. a. f is a rational function whose denominator is never zero, and so it is continuous for all values of x.

b. Because the numerator x^2 is nonnegative and the denominator is $x^2 + 1 \geq 1$ for all values of x, we see that $f(x)$ is nonnegative for all values of x.

c. $f(0) = \dfrac{0}{0+1} = \dfrac{0}{1} = 0$, and so f has a zero at $x = 0$. This does not contradict Theorem 5.

100. a. Both $g(x) = x$ and $h(x) = \sqrt{1 - x^2}$ are continuous on $[-1, 1]$ and so $f(x) = x - \sqrt{1 - x^2}$ is continuous on $[-1, 1]$.

b. $f(-1) = -1$ and $f(1) = 1$, and so f has at least one zero in $(-1, 1)$.

c. Solving $f(x) = 0$, we have $x = \sqrt{1 - x^2}$, $x^2 = 1 - x^2$, and $2x^2 = 1$, so $x = \frac{\pm\sqrt{2}}{2}$.

101. a. (i) Repeated use of Property 3 shows that $g(x) = x^n = \underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ times}}$ is a continuous function, since $f(x) = x$ is continuous by Property 1.

(ii) Properties 1 and 5 combine to show that $c \cdot x^n$ is continuous using the results of part (a)(i).

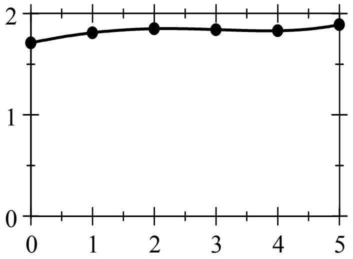
(iii) Each of the terms of $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is continuous and so Property 4 implies that p is continuous.

b. Property 6 now shows that $R(x) = \dfrac{p(x)}{q(x)}$ is continuous if $q(a) \neq 0$, since p and q are continuous at $x = a$.

102. Consider the function f defined by $f(x) = \begin{cases} -1 & \text{if } -1 \leq x < 0 \\ 1 & \text{if } 0 \leq x < 1 \end{cases}$ Then $f(-1) = -1$ and $f(1) = 1$, but if we take the number $\frac{1}{2}$, which lies between $y = -1$ and $y = 1$, there is no value of x such that $f(x) = \frac{1}{2}$.

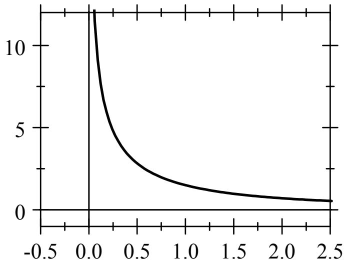
Using Technology page 136

1.



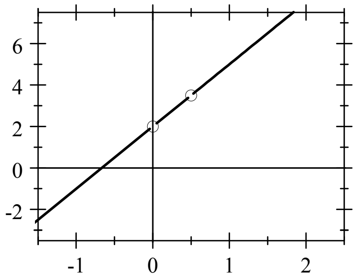
The function is discontinuous at $x = 0$ and $x = 1$.

2.



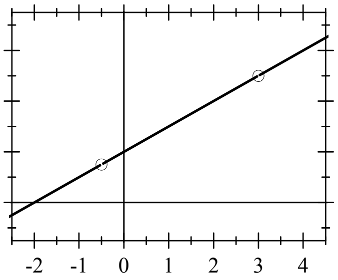
The function is undefined for $x \leq 0$.

3.



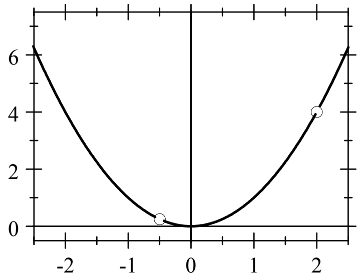
The function is discontinuous at $x = 0$ and $\frac{1}{2}$.

4.



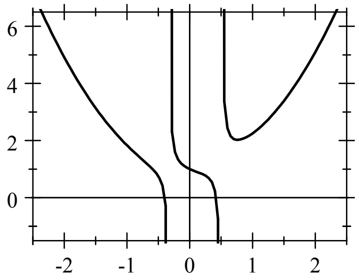
The function is discontinuous at $x = -\frac{1}{2}$ and 3.

5.



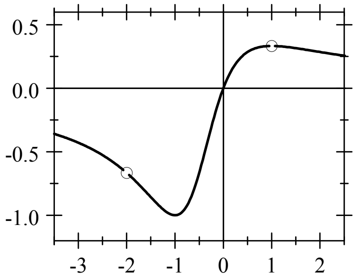
The function is discontinuous at $x = -\frac{1}{2}$ and 2.

6.



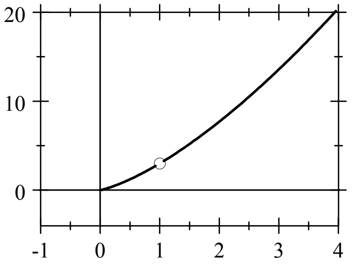
The function is discontinuous at $x = -\frac{1}{3}$ and $\frac{1}{2}$.

7.



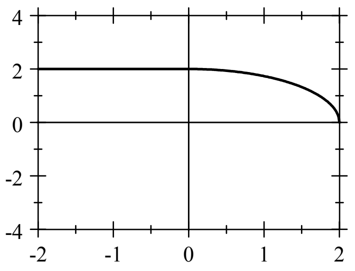
The function is discontinuous at $x = -2$ and 1.

8.



The function is discontinuous only at $x = -1$ and 1. Part of the graph is missing because some graphing calculators cannot evaluate cube roots of negative numbers.

9.



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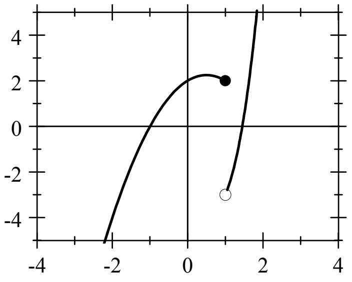

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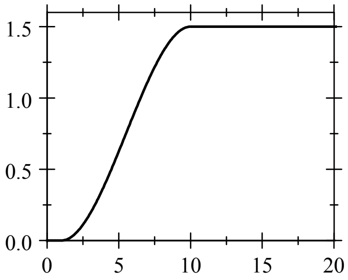

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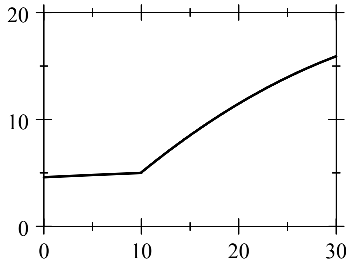


2.6 The Derivative

Concept Questions page 148

1. a. $m = \dfrac{f(2+h) - f(2)}{h}$

b. The slope of the tangent line is $\displaystyle\lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$.

2. a. The average rate of change is $\dfrac{f(2+h) - f(2)}{h}$.

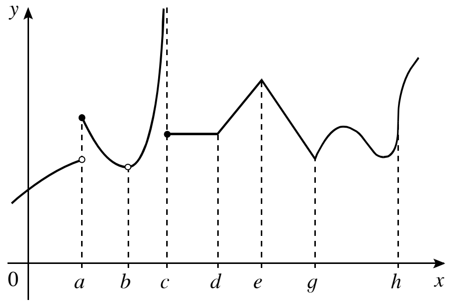
b. The instantaneous rate of change of f at 2 is $\displaystyle\lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$.

c. The expression for the slope of the secant line is the same as that for the average rate of change. The expression for the slope of the tangent line is the same as that for the instantaneous rate of change.

3. a. The expression $\dfrac{f(x+h) - f(x)}{h}$ gives (i) the slope of the secant line passing through the points $(x, f(x))$ and $(x+h, f(x+h))$, and (ii) the average rate of change of f over the interval $[x, x+h]$.

b. The expression $\displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ gives (i) the slope of the tangent line to the graph of f at the point $(x, f(x))$, and (ii) the instantaneous rate of change of f at x.

4. Loosely speaking, a function f does not have a derivative at a if the graph of f does not have a tangent line at a, or if the tangent line does exist, but is vertical. In the figure, the function fails to be differentiable at $x = a$, b, and c because it is discontinuous at each of these numbers. The derivative of the function does not exist at $x = d$, e, and g because it has a kink at each point on the graph corresponding to these numbers. Finally, the function is not differentiable at $x = h$ because the tangent line is vertical at $(h, f(h))$.



5. a. $C(500)$ gives the total cost incurred in producing 500 units of the product.

b. $C'(500)$ gives the rate of change of the total cost function when the production level is 500 units.

6. a. $P(5)$ gives the population of the city (in thousands) when $t = 5$.

b. $P'(5)$ gives the rate of change of the city's population (in thousands/year) when $t = 5$.

Exercises page 149

1. The rate of change of the average infant's weight when $t = 3$ is $\frac{7.5}{5}$, or 1.5 lb/month. The rate of change of the average infant's weight when $t = 18$ is $\frac{3.5}{6}$, or approximately 0.58 lb/month. The average rate of change over the infant's first year of life is $\frac{22.5-7.5}{12}$, or 1.25 lb/month.

2. The rate at which the wood grown is changing at the beginning of the 10th year is $\frac{4}{12}$, or $\frac{1}{3}$ cubic meter per hectare per year. At the beginning of the 30th year, it is $\frac{10}{8}$, or 1.25 cubic meters per hectare per year.

3. The rate of change of the percentage of households watching television at 4 p.m. is $\frac{12.3}{4}$, or approximately 3.1 percent per hour. The rate at 11 p.m. is $\frac{-42.3}{2} = -21.15$, that is, it is dropping off at the rate of 21.15 percent per hour.

4. The rate of change of the crop yield when the density is 200 aphids per bean stem is $\frac{-500}{300}$, a decrease of approximately 1.7 kg/4000 m^2 per aphid per bean stem. The rate of change when the density is 800 aphids per bean stem is $\frac{-150}{300}$, a decrease of approximately 0.5 kg/4000 m^2 per aphid per bean stem.

5. a. Car A is travelling faster than Car B at t_1 because the slope of the tangent line to the graph of f is greater than the slope of the tangent line to the graph of g at t_1.

b. Their speed is the same because the slope of the tangent lines are the same at t_2.

c. Car B is travelling faster than Car A.

d. They have both covered the same distance and are once again side by side at t_3.

6. a. At t_1, the velocity of Car A is greater than that of Car B because $f(t_1) > g(t_1)$. However, Car B has greater acceleration because the slope of the tangent line to the graph of g is increasing, whereas the slope of the tangent line to f is decreasing as you move across t_1.

b. Both cars have the same velocity at t_2, but the acceleration of Car B is greater than that of Car A because the slope of the tangent line to the graph of g is increasing, whereas the slope of the tangent line to the graph of f is decreasing as you move across t_2.

7. a. P_2 is decreasing faster at t_1 because the slope of the tangent line to the graph of g at t_1 is greater than the slope of the tangent line to the graph of f at t_1.

b. P_1 is decreasing faster than P_2 at t_2.

c. Bactericide B is more effective in the short run, but bactericide A is more effective in the long run.

8. a. The revenue of the established department store is decreasing at the slowest rate at $t = 0$.

b. The revenue of the established department store is decreasing at the fastest rate at t_3.

c. The revenue of the discount store first overtakes that of the established store at t_1.

d. The revenue of the discount store is increasing at the fastest rate at t_2 because the slope of the tangent line to the graph of f is greatest at the point $(t_2, f(t_2))$.

9. $f(x) = 13$.

Step 1 $f(x+h) = 13$.

Step 2 $f(x+h) - f(x) = 13 - 13 = 0$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{0}{h} = 0$.

Step 4 $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} 0 = 0$.

10. $f(x) = -6$.

Step 1 $f(x+h) = -6$.

Step 2 $f(x+h) - f(x) = -6 - (-6) = 0$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{0}{h} = 0$.

Step 4 $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} 0 = 0$.

11. $f(x) = 2x + 7$.

Step 1 $f(x+h) = 2(x+h) + 7$.

Step 2 $f(x+h) - f(x) = 2(x+h) + 7 - (2x+7) = 2h$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{2h}{h} = 2$.

Step 4 $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} 2 = 2$.

12. $f(x) = 8 - 4x$.

Step 1 $f(x+h) = 8 - 4(x+h) = 8 - 4x - 4h$.

Step 2 $f(x+h) - f(x) = (8 - 4x - 4h) - (8 - 4x) = -4h$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = -\dfrac{4h}{h} = -4$.

Step 4 $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} (-4) = -4$.

13. $f(x) = 3x^2$.

Step 1 $f(x+h) = 3(x+h)^2 = 3x^2 + 6xh + 3h^2$.

Step 2 $f(x+h) - f(x) = \left(3x^2 + 6xh + 3h^2\right) - 3x^2 = 6xh + 3h^2 = h(6x + 3h)$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{h(6x+3h)}{h} = 6x + 3h$.

Step 4 $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} (6x + 3h) = 6x$.

14. $f(x) = -\frac{1}{2}x^2$.

Step 1 $f(x+h) = -\frac{1}{2}(x+h)^2$.

Step 2 $f(x+h) - f(x) = -\frac{1}{2}x^2 - xh - \frac{1}{2}h^2 + \frac{1}{2}x^2 = -h\left(x + \frac{1}{2}h\right)$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{-h\left(x + \frac{1}{2}h\right)}{h} = -\left(x + \frac{1}{2}h\right)$.

Step 4 $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} -\left(x + \frac{1}{2}h\right) = -x$.

15. $f(x) = -x^2 + 3x$.

Step 1 $f(x+h) = -(x+h)^2 + 3(x+h) = -x^2 - 2xh - h^2 + 3x + 3h$.

Step 2 $f(x+h) - f(x) = \left(-x^2 - 2xh - h^2 + 3x + 3h\right) - \left(-x^2 + 3x\right) = -2xh - h^2 + 3h$

$= h(-2x - h + 3)$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{h(-2x - h + 3)}{h} = -2x - h + 3$.

Step 4 $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (-2x - h + 3) = -2x + 3$.

16. $f(x) = 2x^2 + 5x$.

Step 1 $f(x+h) = 2(x+h)^2 + 5(x+h) = 2x^2 + 4xh + 2h^2 + 5x + 5h$.

Step 2 $f(x+h) - f(x) = 2x^2 + 4xh + 2h^2 + 5x + 5h - 2x^2 - 5x = h(4x + 2h + 5)$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{h(4x + 2h + 5)}{h} = 4x + 2h + 5$.

Step 4 $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (4x + 2h + 5) = 4x + 5$.

17. $f(x) = 2x + 7$.

Step 1 $f(x+h) = 2(x+h) + 7 = 2x + 2h + 7$.

Step 2 $f(x+h) - f(x) = 2x + 2h + 7 - 2x - 7 = 2h$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{2h}{h} = 2$.

Step 4 $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 2 = 2$.

Therefore, $f'(x) = 2$. In particular, the slope at $x = 2$ is 2. Therefore, an equation of the tangent line is $y - 11 = 2(x - 2)$ or $y = 2x + 7$.

18. $f(x) = -3x + 4$. First, we find $f'(x) = -3$ using the four-step process. Thus, the slope of the tangent line is $f'(-1) = -3$ and an equation is $y - 7 = -3(x + 1)$ or $y = -3x + 4$.

19. $f(x) = 3x^2$.We first compute $f'(x) = 6x$ (see Exercise 13). Because the slope of the tangent line is $f'(1) = 6$, we use the point-slope form of the equation of a line and find that an equation is $y - 3 = 6(x - 1)$, or $y = 6x - 3$.

20. $f(x) = 3x - x^2$.

Step 1 $f(x+h) = 3(x+h) - (x+h)^2 = 3x + 3h - x^2 - 2xh - h^2$.

Step 2 $f(x+h) - f(x) = 3x + 3h - x^2 - 2xh - h^2 - 3x + x^2 = 3h - 2xh - h^2 = h(3 - 2x - h)$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{h(3 - 2x - h)}{h} = 3 - 2x - h$.

Step 4 $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (3 - 2x - h) = 3 - 2x$.

Therefore, $f'(x) = 3 - 2x$. In particular, $f'(-2) = 3 - 2(-2) = 7$. Using the point-slope form of an equation of a line, we find $y + 10 = 7(x + 2)$, or $y = 7x + 4$.

21. $f(x) = -1/x$. We first compute $f'(x)$ using the four-step process:

Step 1 $f(x+h) = -\dfrac{1}{x+h}$.

Step 2 $f(x+h) - f(x) = -\dfrac{1}{x+h} + \dfrac{1}{x} = \dfrac{-x + (x+h)}{x(x+h)} = \dfrac{h}{x(x+h)}$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{\dfrac{h}{x(x+h)}}{h} = \dfrac{1}{x(x+h)}$.

Step 4 $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{1}{x(x+h)} = \dfrac{1}{x^2}$.

The slope of the tangent line is $f'(3) = \frac{1}{9}$. Therefore, an equation is $y - \left(-\frac{1}{3}\right) = \frac{1}{9}(x-3)$, or $y = \frac{1}{9}x - \frac{2}{3}$.

22. $f(x) = \dfrac{3}{2x}$. First use the four-step process to find $f'(x) = -\dfrac{3}{2x^2}$. (This is similar to Exercise 21.) The slope of the tangent line is $f'(1) = -\frac{3}{2}$. Therefore, an equation is $y - \frac{3}{2} = -\frac{3}{2}(x-1)$ or $y = -\frac{3}{2}x + 3$.

23. **a.** $f(x) = 2x^2 + 1$.

Step 1 $f(x+h) = 2(x+h)^2 + 1 = 2x^2 + 4xh + 2h^2 + 1$.

Step 2 $f(x+h) - f(x) = \left(2x^2 + 4xh + 2h^2 + 1\right) - \left(2x^2 + 1\right)$

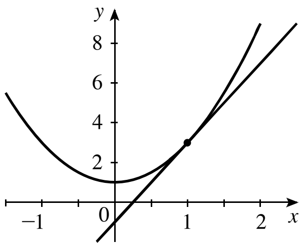
$$= 4xh + 2h^2 = h(4x + 2h).$$

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{h(4x+2h)}{h} = 4x + 2h$.

Step 4 $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} (4x + 2h) = 4x$.

b. The slope of the tangent line is $f'(1) = 4(1) = 4$. Therefore, an equation is $y - 3 = 4(x-1)$ or $y = 4x - 1$.

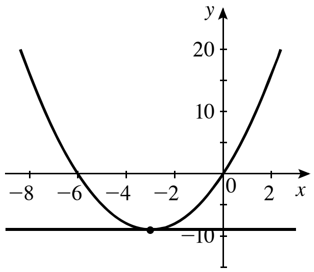
c.



24. **a.** $f(x) = x^2 + 6x$. Using the four-step process, we find that $f'(x) = 2x + 6$.

b. At a point on the graph of f where the tangent line to the curve is horizontal, $f'(x) = 0$. Then $2x + 6 = 0$, or $x = -3$. Therefore, $y = f(-3) = (-3)^2 + 6(-3) = -9$. The required point is $(-3, -9)$.

c.



25. **a.** $f(x) = x^2 - 2x + 1$. We use the four-step process:

Step 1 $f(x+h) = (x+h)^2 - 2(x+h) + 1 = x^2 + 2xh + h^2 - 2x - 2h + 1$.

Step 2 $f(x+h) - f(x) = \left(x^2 + 2xh + h^2 - 2x - 2h + 1\right) - \left(x^2 - 2x + 1\right) = 2xh + h^2 - 2h$

$$= h(2x + h - 2).$$

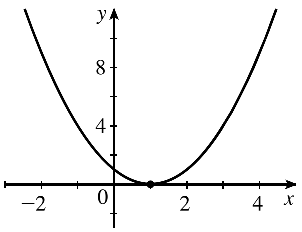
Step 3 $\dfrac{f(x+h)-f(x)}{h}=\dfrac{h(2x+h-2)}{h}=2x+h-2.$

Step 4 $f'(x)=\lim_{h\to0}\dfrac{f(x+h)-f(x)}{h}=\lim_{h\to0}(2x+h-2)$

$=2x-2.$

b. At a point on the graph of f where the tangent line to the curve is horizontal, $f'(x)=0$. Then $2x-2=0$, or $x=1$. Because $f(1)=1-2+1=0$, we see that the required point is $(1,0)$.

c.



d. It is changing at the rate of 0 units per unit change in x.

26. a. $f(x)=\dfrac{1}{x-1}.$

Step 1 $f(x+h)=\dfrac{1}{(x+h)-1}=\dfrac{1}{x+h-1}.$

Step 2 $f(x+h)-f(x)=\dfrac{1}{x+h-1}-\dfrac{1}{x-1}=\dfrac{x-1-(x+h-1)}{(x+h-1)(x-1)}=-\dfrac{h}{(x+h-1)(x-1)}.$

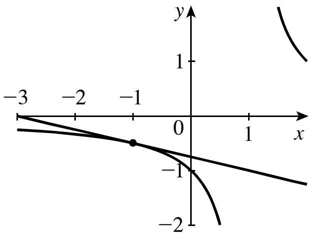
Step 3 $\dfrac{f(x+h)-f(x)}{h}=-\dfrac{1}{(x+h-1)(x-1)}.$

Step 4 $f'(x)=\lim_{h\to0}\dfrac{f(x+h)-f(x)}{h}$

$=\lim_{h\to0}-\dfrac{1}{(x+h-1)(x-1)}=-\dfrac{1}{(x-1)^2}.$

b. The slope is $f'(-1)=-\frac{1}{4}$, so, an equation is $y-\left(-\frac{1}{2}\right)=-\frac{1}{4}(x+1)$ or $y=-\frac{1}{4}x-\frac{3}{4}$.

c.



27. a. $f(x)=x^2+x$, so $\dfrac{f(3)-f(2)}{3-2}=\dfrac{(3^2+3)-(2^2+2)}{1}=6,$

$\dfrac{f(2.5)-f(2)}{2.5-2}=\dfrac{(2.5^2+2.5)-(2^2+2)}{0.5}=5.5$, and $\dfrac{f(2.1)-f(2)}{2.1-2}=\dfrac{(2.1^2+2.1)-(2^2+2)}{0.1}=5.1.$

b. We first compute $f'(x)$ using the four-step process.

Step 1 $f(x+h)=(x+h)^2+(x+h)=x^2+2xh+h^2+x+h.$

Step 2 $f(x+h)-f(x)=(x^2+2xh+h^2+x+h)-(x^2+x)=2xh+h^2+h=h(2x+h+1).$

Step 3 $\dfrac{f(x+h)-f(x)}{h}=\dfrac{h(2x+h+1)}{h}=2x+h+1.$

Step 4 $f'(x)=\lim_{h\to0}\dfrac{f(x+h)-f(x)}{h}=\lim_{h\to0}(2x+h+1)=2x+1.$

The instantaneous rate of change of y at $x=2$ is $f'(2)=2(2)+1$, or 5 units per unit change in x.

c. The results of part (a) suggest that the average rates of change of f at $x=2$ approach 5 as the interval $[2,2+h]$ gets smaller and smaller ($h=1$, 0.5, and 0.1). This number is the instantaneous rate of change of f at $x=2$ as computed in part (b).

28. a. $f(x)=x^2-4x$, so $\dfrac{f(4)-f(3)}{4-3}=\dfrac{(16-16)-(9-12)}{1}=3,$

$\dfrac{f(3.5)-f(3)}{3.5-3}=\dfrac{(12.25-14)-(9-12)}{0.5}=2.5$, and $\dfrac{f(3.1)-f(3)}{3.1-3}=\dfrac{(9.61-12.4)-(9-12)}{0.1}=2.1.$

b. We first compute $f'(x)$ using the four-step process:

Step 1 $f(x+h) = (x+h)^2 - 4(x+h) = x^2 + 2xh + h^2 - 4x - 4h$.

Step 2 $f(x+h) - f(x) = \left(x^2 + 2xh + h^2 - 4x - 4h\right) - \left(x^2 - 4x\right) = 2xh + h^2 - 4h = h(2x + h - 4)$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{h(2x+h-4)}{h} = 2x + h - 4$.

Step 4 $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} (2x + h - 4) = 2x - 4$.

The instantaneous rate of change of y at $x = 3$ is $f'(3) = 6 - 4 = 2$, or 2 units per unit change in x.

c. The results of part (a) suggest that the average rates of change of f over smaller and smaller intervals containing $x = 3$ approach the instantaneous rate of change of 2 units per unit change in x obtained in part (b).

29. a. $f(t) = 2t^2 + 48t$. The average velocity of the car over the time interval [20, 21] is $\dfrac{f(21) - f(20)}{21 - 20} = \dfrac{\left[2(21)^2 + 48(21)\right] - \left[2(20)^2 + 48(20)\right]}{1} = 130\ \dfrac{\text{ft}}{\text{s}}$. Its average velocity over [20, 20.1] is $\dfrac{f(20.1) - f(20)}{20.1 - 20} = \dfrac{\left[2(20.1)^2 + 48(20.1)\right] - \left[2(20)^2 + 48(20)\right]}{0.1} = 128.2\ \dfrac{\text{ft}}{\text{s}}$. Its average velocity over [20, 20.01] is $\dfrac{f(20.01) - f(20)}{20.01 - 20} = \dfrac{\left[2(20.01)^2 + 48(20.01)\right] - \left[2(20)^2 + 48(20)\right]}{0.01} = 128.02\ \dfrac{\text{ft}}{\text{s}}$.

b. We first compute $f'(t)$ using the four-step process.

Step 1 $f(t+h) = 2(t+h)^2 + 48(t+h) = 2t^2 + 4th + 2h^2 + 48t + 48h$.

Step 2 $f(t+h) - f(t) = \left(2t^2 + 4th + 2h^2 + 48t + 48h\right) - \left(2t^2 + 48t\right) = 4th + 2h^2 + 48h$

$= h(4t + 2h + 48)$.

Step 3 $\dfrac{f(t+h) - f(t)}{h} = \dfrac{h(4t + 2h + 48)}{h} = 4t + 2h + 48$.

Step 4 $f'(t) = \lim\limits_{t\to 0} \dfrac{f(t+h) - f(t)}{h} = \lim\limits_{t\to 0} (4t + 2h + 48) = 4t + 48$.

The instantaneous velocity of the car at $t = 20$ is $f'(20) = 4(20) + 48$, or 128 ft/s.

c. Our results show that the average velocities do approach the instantaneous velocity as the intervals over which they are computed decreases.

30. a. The average velocity of the ball over the time interval [2, 3] is $\dfrac{s(3) - s(2)}{3 - 2} = \dfrac{\left[128(3) - 16(3)^2\right] - \left[128(2) - 16(2)^2\right]}{1} = 48$, or 48 ft/s. Over the time interval [2, 2.5], it is $\dfrac{s(2.5) - s(2)}{2.5 - 2} = \dfrac{\left[128(2.5) - 16(2.5)^2\right] - \left[128(2) - 16(2)^2\right]}{0.5} = 56$, or 56 ft/s. Over the time interval [2, 2.1], it is $\dfrac{s(2.1) - s(2)}{2.1 - 2} = \dfrac{\left[128(2.1) - 16(2.1)^2\right] - \left[128(2) - 16(2)^2\right]}{0.1} = 62.4$, or 62.4 ft/s.

b. Using the four-step process, we find that the instantaneous velocity of the ball at any time t is given by $v(t) = 128 - 32t$. In particular, the velocity of the ball at $t = 2$ is $v(2) = 128 - 32(2) = 64$, or 64 ft/s.

c. At $t = 5$, $v(5) = 128 - 32(5) = -32$, so the speed of the ball at $t = 5$ is 32 ft/s and it is falling.

d. The ball hits the ground when $s(t) = 0$, that is, when $128t - 16t^2 = 0$, whence $t(128 - 16t) = 0$, so $t = 0$ or $t = 8$. Thus, it will hit the ground when $t = 8$.

31. a. We solve the equation $16t^2 = 400$ and find $t = 5$, which is the time it takes the screwdriver to reach the ground.

b. The average velocity over the time interval [0, 5] is $\dfrac{f(5)-f(0)}{5-0} = \dfrac{16(25)-0}{5} = 80$, or 80 ft/s.

c. The velocity of the screwdriver at time t is

$$v(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = \lim_{h\to 0} \frac{16(t+h)^2-16t^2}{h} = \lim_{h\to 0} \frac{16t^2+32th+16h^2-16t^2}{h}$$
$$= \lim_{h\to 0} \frac{(32t+16h)h}{h} = 32t.$$

In particular, the velocity of the screwdriver when it hits the ground (at $t = 5$) is $v(5) = 32(5) = 160$, or 160 ft/s.

32. a. We write $f(t) = \frac{1}{2}t^2 + \frac{1}{2}t$. The height after 40 seconds is $f(40) = \frac{1}{2}(40)^2 + \frac{1}{2}(40) = 820$.

b. Its average velocity over the time interval [0, 40] is $\dfrac{f(40)-f(0)}{40-0} = \dfrac{820-0}{40} = 20.5$, or 20.5 ft/s.

c. Its velocity at time t is

$$v(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = \lim_{h\to 0} \frac{\frac{1}{2}(t+h)^2 + \frac{1}{2}(t+h) - \left(\frac{1}{2}t^2+\frac{1}{2}t\right)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{1}{2}t^2 + th + \frac{1}{2}h^2 + \frac{1}{2}t + \frac{1}{2}h - \frac{1}{2}t^2 - \frac{1}{2}t}{h} = \lim_{h\to 0} \frac{th + \frac{1}{2}h^2 + \frac{1}{2}h}{h} = \lim_{h\to 0} \left(t + \frac{1}{2}h + \frac{1}{2}\right) = t + \frac{1}{2}.$$

In particular, the velocity at the end of 40 seconds is $v(40) = 40 + \frac{1}{2}$, or $40\frac{1}{2}$ ft/s.

33. a. We write $V = f(p) = \dfrac{1}{p}$. The average rate of change of V is $\dfrac{f(3)-f(2)}{3-2} = \dfrac{\frac{1}{3}-\frac{1}{2}}{1} = -\dfrac{1}{6}$, a decrease of $\frac{1}{6}$ liter/atmosphere.

b. $V'(t) = \lim_{h\to 0} \dfrac{\dfrac{f(p+h)-f(p)}{h}}{h} = \lim_{h\to 0} \dfrac{\dfrac{1}{p+h} - \dfrac{1}{p}}{h} = \lim_{h\to 0} \dfrac{p-(p+h)}{hp(p+h)} = \lim_{h\to 0} -\dfrac{1}{p(p+h)} = -\dfrac{1}{p^2}$. In particular, the rate of change of V when $p = 2$ is $V'(2) = -\dfrac{1}{2^2}$, a decrease of $\frac{1}{4}$ liter/atmosphere.

34. $C(x) = -10x^2 + 300x + 130$.

a. Using the four-step process, we find

$$C'(x) = \lim_{h\to 0} \frac{C(x+h)-C(x)}{h} = \lim_{h\to 0} \frac{h(-20x-10h+300)}{h} = -20x + 300.$$

b. The rate of change is $C'(10) = -20(10) + 300 = 100$, or \$100/surfboard.

35. a. $P(x) = -\frac{1}{3}x^2 + 7x + 30$. Using the four-step process, we find that

$$P'(x) = \lim_{h\to 0} \frac{P(x+h)-P(x)}{h} = \lim_{h\to 0} \frac{-\frac{1}{3}\left(x^2+2xh+h^2\right) + 7x + 7h + 30 - \left(-\frac{1}{3}x^2+7x+30\right)}{h}$$
$$= \lim_{h\to 0} \frac{-\frac{2}{3}xh - \frac{1}{3}h^2 + 7h}{h} = \lim_{h\to 0} \left(-\frac{2}{3}x - \frac{1}{3}h + 7\right) = -\frac{2}{3}x + 7.$$

b. $P'(10) = -\frac{2}{3}(10) + 7 \approx 0.333$, or approximately \$333 per \$1000 spent on advertising.

$P'(30) = -\frac{2}{3}(30) + 7 = -13$, a decrease of \$13,000 per \$1000 spent on advertising.

36. **a.** $f(x) = -0.1x^2 - x + 40$, so $\dfrac{f(5.05) - f(5)}{5.05 - 5} = \dfrac{\left[-0.1(5.05)^2 - 5.05 + 40\right] - \left[-0.1(5)^2 - 5 + 40\right]}{0.05} = -2.005$, or approximately $-\$2.01$ per 1000 tents. $\dfrac{f(5.01) - f(5)}{5.01 - 5} = \dfrac{\left[-0.1(5.01)^2 - 5.01 + 40\right] - \left[-0.1(5)^2 - 5 + 40\right]}{0.01} = -2.001$, or approximately $-\$2.00$ per 1000 tents.

b. We compute $f'(x)$ using the four-step process, obtaining $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \dfrac{h(-0.2x - 0.1h - 1)}{h} = \lim\limits_{h \to 0} (-0.2x - 0.1h - 1) = -0.2x - 1$. The rate of change of the unit price if $x = 5000$ is $f'(5) = -0.2(5) - 1 = -2$, a decrease of \$2 per 1000 tents.

37. $N(t) = t^2 + 2t + 50$. We first compute $N'(t)$ using the four-step process.

Step 1 $N(t+h) = (t+h)^2 + 2(t+h) + 50 = t^2 + 2th + h^2 + 2t + 2h + 50$.

Step 2 $N(t+h) - N(t) = \left(t^2 + 2th + h^2 + 2t + 2h + 50\right) - \left(t^2 + 2t + 50\right) = 2th + h^2 + 2h = h(2t + h + 2)$.

Step 3 $\dfrac{N(t+h) - N(t)}{h} = 2t + h + 2$.

Step 4 $N'(t) = \lim\limits_{h \to 0} (2t + h + 2) = 2t + 2$.

The rate of change of the country's GNP two years from now is $N'(2) = 2(2) + 2 = 6$, or \$6 billion/yr. The rate of change four years from now is $N'(4) = 2(4) + 2 = 10$, or \$10 billion/yr.

38. $f(t) = 3t^2 + 2t + 1$. Using the four-step process, we obtain $f'(t) = \lim\limits_{t \to 0} \dfrac{f(t+h) - f(t)}{h} = \lim\limits_{t \to 0} \dfrac{h(6t + 3h + 2)}{h} = \lim\limits_{t \to 0} (6t + 3h + 2) = 6t + 2$. Next, $f'(10) = 6(10) + 2 = 62$, and we conclude that the rate of bacterial growth at $t = 10$ is 62 bacteria per minute.

39. **a.** $f'(h)$ gives the instantaneous rate of change of the temperature with respect to height at a given height h, in °F per foot.

b. Because the temperature decreases as the altitude increases, the sign of $f'(h)$ is negative.

c. Because $f'(1000) = -0.05$, the change in the air temperature as the altitude changes from 1000 ft to 1001 ft is approximately $-0.05°$ F.

40. **a.** $\dfrac{f(b) - f(a)}{b - a}$ measures the average rate of change in revenue as the advertising expenditure changes from a thousand dollars to b thousand dollars. The units of measurement are thousands of dollars per thousands of dollars.

b. $f'(x)$ gives the instantaneous rate of change in the revenue when x thousand dollars is spent on advertising. It is measured in thousands of dollars per thousands of dollars.

c. Because $f'(20) \cdot (21 - 20) = 3 \cdot 1 = 3$, the approximate change in revenue is \$3000.

41. $\dfrac{f(a+h) - f(a)}{h}$ gives the average rate of change of the seal population over the time interval $[a, a+h]$.

$\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ gives the instantaneous rate of change of the seal population at $x = a$.

42. $\dfrac{f(a+h) - f(a)}{h}$ gives the average rate of change of the prime interest rate over the time interval $[a, a+h]$.

$\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ gives the instantaneous rate of change of the prime interest rate at $x = a$.

43. $\dfrac{f(a+h)-f(a)}{h}$ gives the average rate of change of the country's industrial production over the time interval $[a,a+h]$. $\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ gives the instantaneous rate of change of the country's industrial production at $x=a$.

44. $\dfrac{f(a+h)-f(a)}{h}$ gives the average rate of change of the cost incurred in producing the commodity over the production level $[a,a+h]$. $\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ gives the instantaneous rate of change of the cost of producing the commodity at $x=a$.

45. $\dfrac{f(a+h)-f(a)}{h}$ gives the average rate of change of the atmospheric pressure over the altitudes $[a,a+h]$. $\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ gives the instantaneous rate of change of the atmospheric pressure with respect to altitude at $x=a$.

46. $\dfrac{f(a+h)-f(a)}{h}$ gives the average rate of change of the fuel economy of a car over the speeds $[a,a+h]$. $\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ gives the instantaneous rate of change of the fuel economy at $x=a$.

47. **a.** f has a limit at $x=a$.
b. f is not continuous at $x=a$ because $f(a)$ is not defined.
c. f is not differentiable at $x=a$ because it is not continuous there.

48. **a.** f has a limit at $x=a$.
b. f is continuous at $x=a$.
c. f is differentiable at $x=a$.

49. **a.** f has a limit at $x=a$.
b. f is continuous at $x=a$.
c. f is not differentiable at $x=a$ because f has a kink at the point $x=a$.

50. **a.** f does not have a limit at $x=a$ because the left-hand and right-hand limits are not equal.
b. f is not continuous at $x=a$ because the limit does not exist there.
c. f is not differentiable at $x=a$ because it is not continuous there.

51. **a.** f does not have a limit at $x=a$ because it is unbounded in the neighborhood of a.
b. f is not continuous at $x=a$.
c. f is not differentiable at $x=a$ because it is not continuous there.

52. **a.** f does not have a limit at $x=a$ because the left-hand and right-hand limits are not equal.
b. f is not continuous at $x=a$ because the limit does not exist there.
c. f is not differentiable at $x=a$ because it is not continuous there.

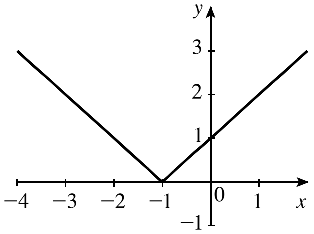
53. $s(t)=-0.1t^3+2t^2+24t$. Our computations yield the following results: 32.1, 30.939, 30.814, 30.8014, 30.8001, and 30.8000. The motorcycle's instantaneous velocity at $t=2$ is approximately 30.8 ft/s.

54. $C(x) = 0.000002x^3 + 5x + 400$. Our computations yield the following results: 5.060602, 5.06006002, 5.060006, 5.0600006, and 5.0600001. The rate of change of the total cost function when the level of production is 100 cases a day is approximately \$5.06.

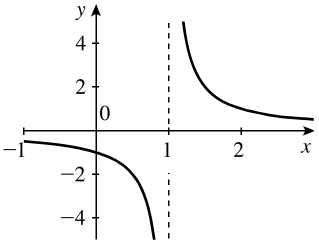
55. False. Let $f(x) = |x|$. Then f is continuous at $x = 0$, but is not differentiable there.

56. True. If g is differentiable at $x = a$, then it is continuous there. Therefore, the product fg is continuous, and so $\lim_{x\to a} f(x)g(x) = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] = f(a)g(a)$.

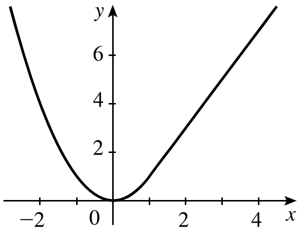
57. Observe that the graph of f has a kink at $x = -1$. We have $\dfrac{f(-1+h) - f(-1)}{h} = 1$ if $h > 0$, and -1 if $h < 0$, so that $\lim_{h\to 0} \dfrac{f(-1+h) - f(-1)}{h}$ does not exist.



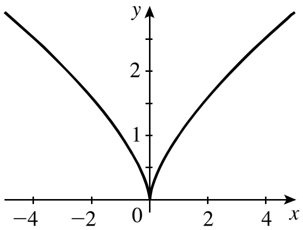
58. f does not have a derivative at $x = 1$ because it is not continuous there.



59. For continuity, we require that $f(1) = 1 = \lim_{x\to 1^+} (ax + b) = a + b$, or $a + b = 1$. Next, using the four-step process, we have $f'(x) = \begin{cases} 2x & \text{if } x < 1 \\ a & \text{if } x > 1 \end{cases}$ In order that the derivative exist at $x = 1$, we require that $\lim_{x\to 1^-} 2x = \lim_{x\to 1^+} a$, or $2 = a$. Therefore, $b = -1$ and so $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$



60. f is continuous at $x = 0$, but $f'(0)$ does not exist because the graph of f has a vertical tangent line at $x = 0$.



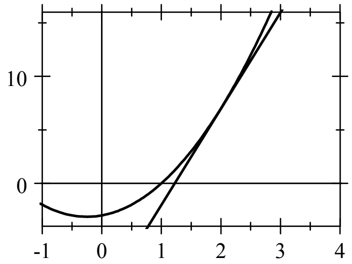
61. We have $f(x) = x$ if $x > 0$ and $f(x) = -x$ if $x < 0$. Therefore, when $x > 0$, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = 1$, and when $x < 0$, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{-x-h-(-x)}{h} = \lim_{h\to 0} \frac{-h}{h} = -1$. Because the right-hand limit does not equal the left-hand limit, we conclude that $\lim_{h\to 0} f(x)$ does not exist.

62. From $f(x) - f(a) = \left[\frac{f(x) - f(a)}{x-a}\right](x-a)$, we see that $\lim_{x\to a} \left[f(x) - f(a)\right] = \lim_{x\to a} \left[\frac{f(x) - f(a)}{x-a}\right] \lim_{x\to a} (x-a) = f'(a) \cdot 0 = 0$, and so $\lim_{x\to a} f(x) = f(a)$. This shows that f is continuous at $x = a$.

Using Technology page 155

1. a. 9

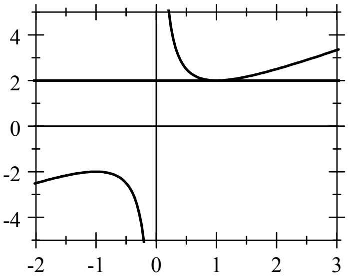
b.



c. $y = 9x - 11$

2. a. 0

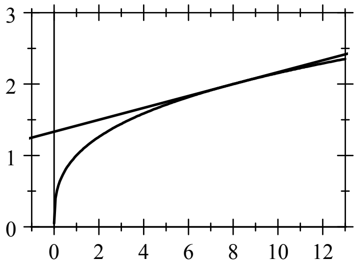
b.



c. $y = 2$

3. a. 0.083

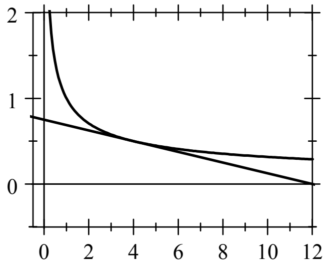
b.



c. $y = \frac{1}{12}x + \frac{4}{3}$

4. a. -0.0625

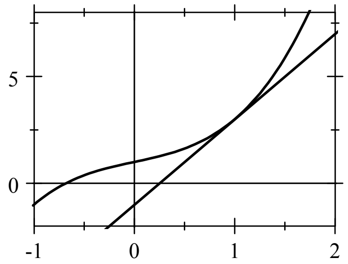
b.



c. $y = -\frac{1}{16}x + \frac{3}{4}$

5. **a.** 4

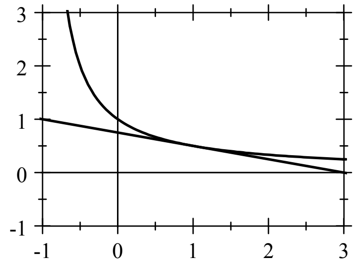
b.



c. $y = 4x - 1$

6. **a.** -0.25

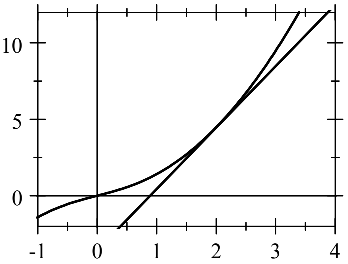
b.



c. $y = -\frac{1}{4}x + \frac{3}{4}$

7. **a.** 4.02

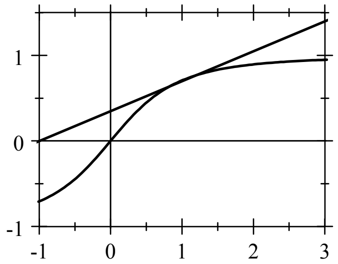
b.



c. $y = 4.02x - 3.57$

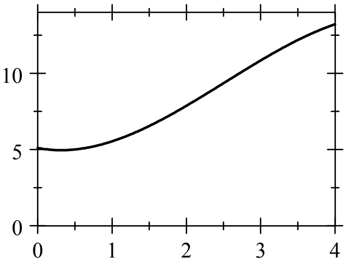
8. **a.** 0.35

b.



c. $y = 0.35x + 0.35$

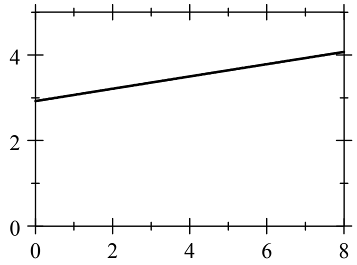
9. **a.**



b. $f'(3) = 2.8826$ (million per decade)

10. **a.** $S(t) = -0.000114719t^2 + 0.144618t + 2.92202$

b.



c. \$3.786 billion

d. \$143 million/yr

CHAPTER 2 Concept Review Questions page 156

1. domain, range, B

2. domain, $f(x)$, vertical, point

3. $f(x) \pm g(x)$, $f(x)\,g(x)$, $\dfrac{f(x)}{g(x)}$, $A \cap B$, $A \cap B$, 0

4. $g(f(x))$, f, $f(x)$, g

5. a. $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n \neq 0$ and n is a positive integer

b. linear, quadratic, cubic **c.** quotient, polynomials **d.** x^r, where r is a real number

6. $f(x), L, a$

7. a. L^r **b.** $L \pm M$ **c.** LM **d.** $\frac{L}{M}, M \neq 0$

8. a. L, x **b.** M, negative, absolute

9. a. right **b.** left **c.** L, L

10. a. continuous **b.** discontinuous **c.** every

11. a. $a, a, g(a)$ **b.** everywhere **c.** Q

12. a. $[a, b]$, $f(c) = M$ **b.** $f(x) = 0$, (a, b)

13. a. $f'(a)$ **b.** $y - f(a) = m(x - a)$

14. a. $\frac{f(a+h) - f(a)}{h}$ **b.** $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

CHAPTER 2 Review Exercises page 157

1. a. $9 - x \geq 0$ gives $x \leq 9$, and the domain is $(-\infty, 9]$.

b. $2x^2 - x - 3 = (2x - 3)(x + 1)$, and $x = \frac{3}{2}$ or -1. Because the denominator of the given expression is zero at these points, we see that the domain of f cannot include these points and so the domain of f is $(-\infty, -1) \cup \left(-1, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$.

2. a. We must have $2 - x \geq 0$ and $x + 3 \neq 0$. This implies $x \leq 2$ and $x \neq -3$, so the domain of f is $(-\infty, -3) \cup (-3, 2]$.

b. The domain is $(-\infty, \infty)$.

3. a. $f(-2) = 3(-2)^2 + 5(-2) - 2 = 0$.

b. $f(a+2) = 3(a+2)^2 + 5(a+2) - 2 = 3a^2 + 12a + 12 + 5a + 10 - 2 = 3a^2 + 17a + 20$.

c. $f(2a) = 3(2a)^2 + 5(2a) - 2 = 12a^2 + 10a - 2$.

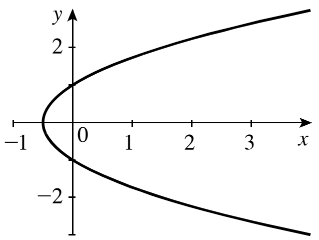
d. $f(a+h) = 3(a+h)^2 + 5(a+h) - 2 = 3a^2 + 6ah + 3h^2 + 5a + 5h - 2$.

4. a.

$$\begin{aligned} f(x-1) + f(x+1) &= \left[2(x-1)^2 - (x-1) + 1\right] + \left[2(x+1)^2 - (x+1) + 1\right] \\ &= \left(2x^2 - 4x + 2 - x + 1 + 1\right) + \left(2x^2 + 4x + 2 - x - 1 + 1\right) = 4x^2 - 2x + 6. \end{aligned}$$

b. $f(x + 2h) = 2(x + 2h)^2 - (x + 2h) + 1 = 2x^2 + 8xh + 8h^2 - x - 2h + 1$.

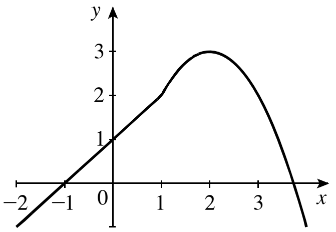
5. a.



b. For each value of $x > 0$, there are two values of y. We conclude that y is not a function of x. (We could also note that the function fails the vertical line test.)

c. Yes. For each value of y, there is only one value of x.

6.



7. a. $f(x)\,g(x) = \dfrac{2x+3}{x}$.

b. $\dfrac{f(x)}{g(x)} = \dfrac{1}{x(2x+3)}$.

c. $f(g(x)) = \dfrac{1}{2x+3}$.

d. $g(f(x)) = 2\left(\dfrac{1}{x}\right) + 3 = \dfrac{2}{x} + 3$.

8. a. $(f \circ g)(x) = f(g(x)) = 2g(x) - 1 = 2\left(x^2+4\right) - 1 = 2x^2 + 7$ and

$(g \circ f)(x) = g(f(x)) = \left[f(x)\right]^2 + 4 = (2x-1)^2 + 4 = 4x^2 - 4x + 5$.

b. $(f \circ g)(x) = f(g(x)) = 1 - g(x) = 1 - \dfrac{1}{3x+4} = \dfrac{3x+3}{3x+4} = \dfrac{3(x+1)}{3x+4}$ and

$(g \circ f)(x) = g(f(x)) = \dfrac{1}{3f(x)+4} = \dfrac{1}{3(1-x)+4} = \dfrac{1}{7-3x}$.

c. $(f \circ g)(x) = f(g(x)) = g(x) - 3 = \dfrac{1}{\sqrt{x+1}} - 3$ and

$(g \circ f)(x) = g(f(x)) = \dfrac{1}{\sqrt{f(x)+1}} = \dfrac{1}{\sqrt{(x-3)+1}} = \dfrac{1}{\sqrt{x-2}}$.

9. a. Take $f(x) = 2x^2 + x + 1$ and $g(x) = \dfrac{1}{x^3}$.

b. Take $f(x) = x^2 + x + 4$ and $g(x) = \sqrt{x}$.

10. We have $c(4)^2 + 3(4) - 4 = 2$, so $16c + 12 - 4 = 2$, or $c = -\dfrac{6}{16} = -\dfrac{3}{8}$.

11. $\lim\limits_{x\to 0}(5x-3) = 5(0) - 3 = -3$.

12. $\lim\limits_{x\to 1}\left(x^2+1\right) = (1)^2 + 1 = 1 + 1 = 2$.

13. $\lim\limits_{x\to -1}\left(3x^2+4\right)(2x-1) = \left[3(-1)^2+4\right]\left[2(-1)-1\right] = -21$.

14. $\lim\limits_{x\to 3}\dfrac{x-3}{x+4} = \dfrac{3-3}{3+4} = 0$

15. $\lim\limits_{x\to 2}\dfrac{x+3}{x^2-9} = \dfrac{2+3}{4-9} = -1$.

16. $\lim\limits_{x\to -2}\dfrac{x^2-2x-3}{x^2+5x+6}$ does not exist. (The denominator is 0 at $x = -2$.)

17. $\lim_{x\to 3}\sqrt{2x^3-5}=\sqrt{2(27)-5}=7.$

18. $\lim_{x\to 3}\dfrac{4x-3}{\sqrt{x+1}}=\dfrac{12-3}{\sqrt{4}}=\dfrac{9}{2}.$

19. $\lim_{x\to 1^+}\dfrac{x-1}{x(x-1)}=\lim_{x\to 1^+}\dfrac{1}{x}=1.$

20. $\lim_{x\to 1^-}\dfrac{\sqrt{x}-1}{x-1}=\lim_{x\to 1^-}\dfrac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)}=\lim_{x\to 1^-}\dfrac{x-1}{(x-1)(\sqrt{x}+1)}=\lim_{x\to 1^-}\dfrac{1}{\sqrt{x}+1}=\dfrac{1}{2}.$

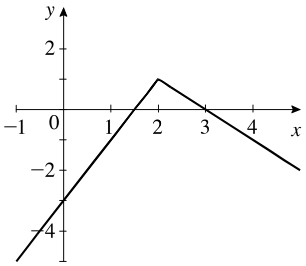
21. $\lim_{x\to\infty}\dfrac{x^2}{x^2-1}=\lim_{x\to\infty}\dfrac{1}{1-\dfrac{1}{x^2}}=1.$

22. $\lim_{x\to-\infty}\dfrac{x+1}{x}=\lim_{x\to-\infty}\left(1+\dfrac{1}{x}\right)=1.$

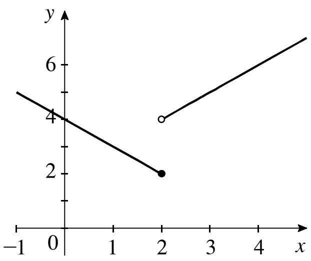
23. $\lim_{x\to\infty}\dfrac{3x^2+2x+4}{2x^2-3x+1}=\lim_{x\to\infty}\dfrac{3+\dfrac{2}{x}+\dfrac{4}{x^2}}{2-\dfrac{3}{x}+\dfrac{1}{x^2}}=\dfrac{3}{2}.$

24. $\lim_{x\to-\infty}\dfrac{x^2}{x+1}=\lim_{x\to-\infty}\left(x\cdot\dfrac{1}{1+\dfrac{1}{x}}\right)=-\infty$, so the limit does not exist.

25. $\lim_{x\to 2^+}f(x)=\lim_{x\to 2^+}(-x+3)=-2+3=1$ and
$\lim_{x\to 2^-}f(x)=\lim_{x\to 2^-}(2x-3)=2(2)-3=4-3=1.$
Therefore, $\lim_{x\to 2}f(x)=1.$



26. $\lim_{x\to 2^+}f(x)=\lim_{x\to 2^+}(x+2)=4$ and
$\lim_{x\to 2^-}f(x)=\lim_{x\to 2^-}(4-x)=2.$ Therefore, $\lim_{x\to 2}f(x)$ does not exist.



27. The function is discontinuous at $x=2$.

28. Because the denominator $4x^2-2x-2=2\left(2x^2-x-1\right)=2(2x+1)(x-1)=0$ if $x=-\frac{1}{2}$ or 1, we see that f is discontinuous at these points.

29. Because $\lim_{x\to-1} f(x) = \lim_{x\to-1} \dfrac{1}{(x+1)^2} = \infty$ (does not exist), we see that f is discontinuous at $x = -1$.

30. The function is discontinuous at $x = 0$.

31. **a.** Let $f(x) = x^2 + 2$. Then the average rate of change of y over $[1, 2]$ is $\dfrac{f(2) - f(1)}{2 - 1} = \dfrac{(4+2) - (1+2)}{1} = 3$. Over $[1, 1.5]$, it is $\dfrac{f(1.5) - f(1)}{1.5 - 1} = \dfrac{(2.25+2) - (1+2)}{0.5} = 2.5$. Over $[1, 1.1]$, it is $\dfrac{f(1.1) - f(1)}{1.1 - 1} = \dfrac{(1.21+2) - (1+2)}{0.1} = 2.1$.

b. Computing $f'(x)$ using the four-step process., we obtain $f'(x) = \lim_{h\to0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to0} \dfrac{h(2x+h)}{h} = \lim_{h\to0} (2x+h) = 2$. Therefore, the instantaneous rate of change of f at $x = 1$ is $f'(1) = 2$, or 2 units per unit change in x.

32. $f(x) = 4x + 5$. We use the four-step process:

Step 1 $f(x+h) = 4(x+h) + 5 = 4x + 4h + 5$.

Step 2 $f(x+h) - f(x) = 4x + 4h + 5 - 4x - 5 = 4h$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{4h}{h} = 4$.

Step 4 $f'(x) = \lim_{h\to0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to0} (4) = 4$.

33. $f(x) = \frac{3}{2}x + 5$. We use the four-step process:

Step 1 $f(x+h) = \frac{3}{2}(x+h) + 5 = \frac{3}{2}x + \frac{3}{2}h + 5$.

Step 2 $f(x+h) - f(x) = \frac{3}{2}x + \frac{3}{2}h + 5 - \frac{3}{2}x - 5 = \frac{3}{2}h$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{3}{2}$.

Step 4 $f'(x) = \lim_{h\to0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to0} \dfrac{3}{2} = \dfrac{3}{2}$.

Therefore, the slope of the tangent line to the graph of the function f at the point $(-2, 2)$ is $\frac{3}{2}$. To find the equation of the tangent line to the curve at the point $(-2, 2)$, we use the point-slope form of the equation of a line, obtaining $y - 2 = \frac{3}{2}[x - (-2)]$ or $y = \frac{3}{2}x + 5$.

34. $f(x) = -x^2$. We use the four-step process:

Step 1 $f(x+h) = -(x+h)^2 = -x^2 - 2xh - h^2$.

Step 2 $f(x+h) - f(x) = \left(-x^2 - 2xh - h^2\right) - \left(-x^2\right) = -2xh - h^2 = h(-2x - h)$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = -2x - h$.

Step 4 $f'(x) = \lim_{h\to0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to0} (-2x - h) = -2x$.

The slope of the tangent line is $f'(2) = -2(2) = -4$. An equation of the tangent line is $y - (-4) = -4(x - 2)$, or $y = -4x + 4$.

35. $f(x) = -\frac{1}{x}$. We use the four-step process:

Step 1 $f(x+h) = -\frac{1}{x+h}$.

Step 2 $f(x+h) - f(x) = -\frac{1}{x+h} - \left(-\frac{1}{x}\right) = -\frac{1}{x+h} + \frac{1}{x} = \frac{h}{x(x+h)}$.

Step 3 $\frac{f(x+h) - f(x)}{h} = \frac{1}{x(x+h)}$.

Step 4 $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{x(x+h)} = \frac{1}{x^2}$.

36. **a.** f is continuous at $x = a$ because the three conditions for continuity are satisfied at $x = a$; that is, 1. $f(x)$ is defined. 2. $\lim_{x \to a} f(x)$ exists. 3. $\lim_{x \to a} f(x) = f(a)$.

b. f is not differentiable at $x = a$ because the graph of f has a kink at $x = a$.

37. $S(4) = 6000(4) + 30{,}000 = 54{,}000$.

38. **a.** The line passes through (0, 2.4) and (5, 7.4) and has slope $m = \frac{7.4 - 2.4}{5 - 0} = 1$. Letting y denote the sales, we see that an equation of the line is $y - 2.4 = 1(t - 0)$, or $y = t + 2.4$. We can also write this in the form $S(t) = t + 2.4$.

b. The sales in 2011 are $S(3) = 3 + 2.4 = 5.4$, or \$5.4 million.

39. **a.** $C(x) = 6x + 30{,}000$.

b. $R(x) = 10x$.

c. $P(x) = R(x) - C(x) = 10x - (6x + 30{,}000) = 4x - 30{,}000$.

d. $P(6000) = 4(6000) - 30{,}000 = -6000$, or a loss of \$6000. $P(8000) = 4(8000) - 30{,}000 = 2000$, or a profit of \$2000. $P(12{,}000) = 4(12{,}000) - 30{,}000 = 18{,}000$, or a profit of \$18,000.

40. Substituting the first equation into the second yields $3x - 2\left(\frac{3}{4}x + 6\right) + 3 = 0$, so $\frac{3}{2}x - 12 + 3 = 0$ and $x = 6$. Substituting this value of x into the first equation then gives $y = \frac{21}{2}$, so the point of intersection is $\left(6, \frac{21}{2}\right)$.

41. The profit function is given by $P(x) = R(x) - C(x) = 20x - (12x + 20{,}000) = 8x - 20{,}000$.

42. We solve the system $\begin{cases} 3x + p - 40 = 0 \\ 2x - p + 10 = 0 \end{cases}$ Adding these two equations, we obtain $5x - 30 = 0$, or $x = 6$. Thus, $p = 2x + 10 = 12 + 10 = 22$. Therefore, the equilibrium quantity is 6000 and the equilibrium price is \$22.

43. The child should receive $D(35) = \frac{500(35)}{150} \approx 117$, or approximately 117 mg.

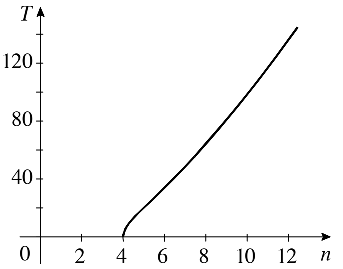
44. When 1000 units are produced, $R(1000) = -0.1(1000)^2 + 500(1000) = 400{,}000$, or \$400,000.

45. $R(30) = -\frac{1}{2}(30)^2 + 30(30) = 450$, or \$45,000.

46. $N(0) = 200(4 + 0)^{1/2} = 400$, and so there are 400 members initially. $N(12) = 200(4 + 12)^{1/2} = 800$, and so there are 800 members after one year.

47. The population will increase by $P(9) - P(0) = \left[50{,}000 + 30(9)^{3/2} + 20(9)\right] - 50{,}000$, or 990, during the next 9 months. The population will increase by $P(16) - P(0) = \left[50{,}000 + 30(16)^{3/2} + 20(16)\right] - 50{,}000$, or 2240 during the next 16 months.

48. $T = f(n) = 4n\sqrt{n-4}$. $f(4) = 0$, $f(5) = 20\sqrt{1} = 20$, $f(6) = 24\sqrt{2} \approx 33.9$, $f(7) = 28\sqrt{3} \approx 48.5$, $f(8) = 32\sqrt{4} = 64$, $f(9) = 36\sqrt{5} \approx 80.5$, $f(10) = 40\sqrt{6} \approx 98$, $f(11) = 44\sqrt{7} \approx 116$, and $f(12) = 48\sqrt{8} \approx 135.8$.



49. We need to find the point of intersection of the two straight lines representing the given linear functions. We solve the equation $2.3 + 0.4t = 1.2 + 0.6t$, obtaining $1.1 = 0.2t$ and thus $t = 5.5$. This tells us that the annual sales of the Cambridge Drug Store first surpasses that of the Crimson Drug store $5\frac{1}{2}$ years from now.

50. We solve $-1.1x^2 + 1.5x + 40 = 0.1x^2 + 0.5x + 15$, obtaining $1.2x^2 - x - 25 = 0$, $12x^2 - 10x - 250 = 0$, $6x^2 - 5x - 125 = 0$, and $(x-5)(6x+25) = 0$. Therefore, $x = 5$. Substituting this value of x into the second supply equation, we have $p = 0.1(5)^2 + 0.5(5) + 15 = 20$. So the equilibrium quantity is 5000 and the equilibrium price is \$20.

51. The life expectancy of a female whose current age is 65 is $C(65) \approx 16.80$ (years). The life expectancy of a female whose current age is 75 is $C(75) \approx 10.18$ (years).

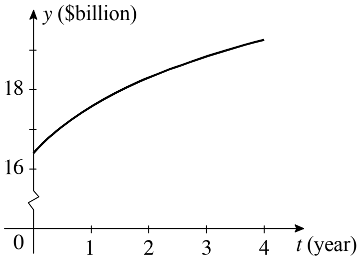
52. **a.** The amount of Medicare benefits paid out in 2010 is $B(0) = 0.25$, or \$250 billion.

b. The amount of Medicare benefits projected to be paid out in 2040 is $B(3) = 0.09(3)^2 + (0.102)(3) + 0.25 = 1.366$, or \$1.366 trillion.

53. $N(0) = 648$, or 648,000, $N(1) = -35.8 + 202 + 87.7 + 648 \approx 902$ or 902,000, $N(2) = -35.8(2)^3 + 202(2)^2 + 87.8(2) + 648 = 1345.2$ or 1,345,200, and $N(3) = -35.8(3)^3 + 202(3)^2 + 87.8(3) + 648 = 1762.8$ or 1,762,800.

54. **a.** $A(0) = 16.4$, or \$16.4 billion; $A(1) = 16.4(1+1)^{0.1} \approx 17.58$, or \$17.58 billion; $A(2) = 16.4(2+1)^{0.1} \approx 18.30$, or \$18.3 billion; $A(3) = 16.4(3+1)^{0.1} \approx 18.84$, or \$18.84 billion; and $A(4) = 16.4(4+1)^{0.1} \approx 19.26$, or \$19.26 billion. The nutritional market grew over the years 1999 to 2003.

b.



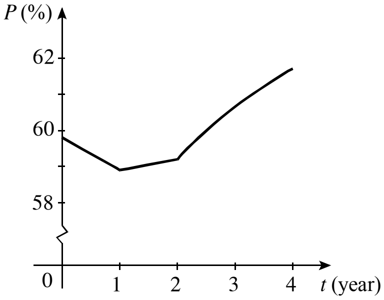
55. **a.** $f(t) = 267$; $g(t) = 2t^2 + 46t + 733$.

b. $h(t) = (f+g)(t) = f(t) + g(t) = 267 + \left(2t^2 + 46t + 733\right) = 2t^2 + 46t + 1000$.

c. $h(13) = 2(13)^2 + 46(13) + 1000 = 1936$, or 1936 tons.

56. a. $P(0) = 59.8$, $P(1) = 0.3(1) + 58.6 = 58.9$,
$P(2) = 56.79(2)^{0.06} \approx 59.2$, $P(3) = 56.79(3)^{0.06} \approx 60.7$, and
$P(4) = 56.79(4)^{0.06} \approx 61.7$.

b.



c. $P(3) \approx 60.7$, or 60.7%.

57. a. $f(r) = \pi r^2$.

b. $g(t) = 2t$.

c. $h(t) = (f \circ g)(t) = f(g(t)) = \pi [g(t)]^2 = 4\pi t^2$.

d. $h(30) = 4\pi(30^2) = 3600\pi$, or 3600π ft^2.

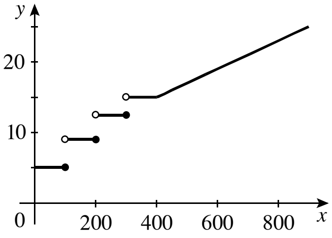
58. Measured in inches, the sides of the resulting box have length $20 - 2x$ and the height is x, so its volume is $V = x(20 - 2x)^2$ in^3.

59. Let h denote the height of the box. Then its volume is $V = (x)(2x)h = 30$, so that $h = \dfrac{15}{x^2}$. Thus, the cost is

$$\begin{aligned} C(x) &= 30(x)(2x) + 15[2xh + 2(2x)h] + 20(x)(2x) \\ &= 60x^2 + 15(6xh) + 40x^2 = 100x^2 + (15)(6)x\left(\frac{15}{x^2}\right) \\ &= 100x^2 + \frac{1350}{x}. \end{aligned}$$

60.

$$C(x) = \begin{cases} 5 & \text{if } 1 \le x \le 100 \\ 9 & \text{if } 100 < x \le 200 \\ 12.50 & \text{if } 200 < x \le 300 \\ 15.00 & \text{if } 300 < x \le 400 \\ 7 + 0.02x & \text{if } x > 400 \end{cases}$$



61. $\lim_{x\to\infty} \overline{C}(x) = \lim_{x\to\infty}\left(20 + \frac{400}{x}\right) = 20$. As the level of production increases without bound, the average cost of producing the commodity steadily decreases and approaches \$20 per unit.

62. a. $C'(x)$ gives the instantaneous rate of change of the total manufacturing cost c in dollars when x units of a certain product are produced.

b. Positive

c. Approximately \$20.

63. True. If $x < 0$, then $\sqrt{x}$ is not defined, and if $x > 0$, then $\sqrt{-x}$ is not defined. Therefore $f(x)$ is defined nowhere, and is not a function.

64. False. Let $f(x) = x^{1/3} + 1$. Then $f'(x) = \frac{1}{3}x^{-2/3}$, so $f'(1) = \frac{1}{3}$ and an equation of the tangent line to the graph of f at the point $(1, 2)$ is $y - 2 = \frac{1}{3}(x - 1)$ or $y = \frac{1}{3}x + \frac{5}{3}$. This tangent line intersects the graph of f at the point $(-8, -1)$, as can be easily verified.

CHAPTER 2 Before Moving On... page 160

1. a. $f(-1) = -2(-1) + 1 = 3$.

b. $f(0) = 2$.

c. $f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 + 2 = \frac{17}{4}$.

2. a. $(f + g)(x) = f(x) + g(x) = \dfrac{1}{x+1} + x^2 + 1$.

b. $(fg)(x) = f(x)g(x) = \dfrac{x^2+1}{x+1}$.

c. $(f \circ g)(x) = f(g(x)) = \dfrac{1}{g(x)+1} = \dfrac{1}{x^2+2}$.

d. $(g \circ f)(x) = g(f(x)) = \left[f(x)\right]^2 + 1 = \dfrac{1}{(x+1)^2} + 1$.

3. $4x + h = 108$, so $h = 108 - 4x$. The volume is $V = x^2h = x^2(108 - 4x) = 108x^2 - 4x^3$.

4. $\displaystyle\lim_{x\to-1} \frac{x^2+4x+3}{x^2+3x+2} = \lim_{x\to-1} \frac{(x+3)(x+1)}{(x+2)(x+1)} = 2.$

5. a. $\displaystyle\lim_{x\to1^-} f(x) = \lim_{x\to1^-} \left(x^2 - 1\right) = 0.$

b. $\displaystyle\lim_{x\to1^+} f(x) = \lim_{x\to1^+} x^3 = 1.$

Because $\displaystyle\lim_{x\to1^-} f(x) \neq \lim_{x\to1^+} f(x)$, f is not continuous at 1.

6. The slope of the tangent line at any point is

$$\begin{aligned}\lim_{h\to0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to0} \frac{(x+h)^2 - 3(x+h) + 1 - \left(x^2 - 3x + 1\right)}{h} \\ &= \lim_{h\to0} \frac{x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x - 1}{h} \\ &= \lim_{h\to0} \frac{h(2x+h-3)}{h} = \lim_{h\to0} (2x + h - 3) = 2x - 3.\end{aligned}$$

Therefore, the slope at 1 is $2(1) - 3 = -1$. An equation of the tangent line is $y - (-1) = -1(x - 1)$, or $y + 1 = -x + 1$, or $y = -x$.

CHAPTER 2 Explore & Discuss

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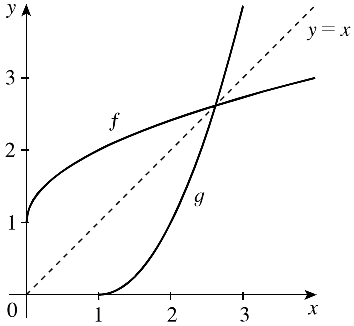
1. $(g \circ f)(x) = g(f(x)) = [f(x) - 1]^2 = [(\sqrt{x} + 1) - 1]^2 = (\sqrt{x})^2 = x$ and $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} + 1 = \sqrt{(x-1)^2} + 1 = (x-1) + 1 = x$.

2. From the figure, we see that the graph of one is the mirror reflection of the other if we place a mirror along the line $y = x$.



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1. As x approaches 0 from either direction, $h(x)$ oscillates more and more rapidly between -1 and 1 and therefore cannot approach a specific number. But this says $\lim_{x \to 0} h(x)$ does not exist.

2. The function f fails to have a limit at $x = 0$ because $f(x)$ approaches 1 from the right but -1 from the left. The function g fails to have a limit at $x = 0$ because $g(x)$ is unbounded on either side of $x = 0$. The function h here does not approach any number from either the right or the left and has no limit at 0, as explained earlier.

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1. $\lim_{x \to \infty} f(x)$ does not exist because no matter how large x is, $f(x)$ takes on values between -1 and 1. In other words, $f(x)$ does not approach a definite number as x approaches infinity. Similarly, $\lim_{x \to -\infty} f(x)$ fails to exist.

2. The function of Example 10 fails to have a limit at infinity (negative infinity) because $f(x)$ increases (decreases) without bound as x approaches infinity (negative infinity). On the other hand, the function whose graph is depicted here, though bounded (its values lie between -1 and 1), does not approach any specific number as x increases (decreases) without bound and this is the reason it fails to have a limit at infinity or negative infinity.

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The average rate of change of a function f is measured over an interval. Thus, the average rate of change of f over the interval $[a, b]$ is the number $\dfrac{f(b) - f(a)}{b - a}$. On the other hand, the instantaneous rate of change of a function measures the rate of change of the function at a point. As we have seen, this quantity can be found by taking the limit of an appropriate difference quotient. Specifically, the instantaneous rate of change of f at $x = a$ is $\lim_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$.

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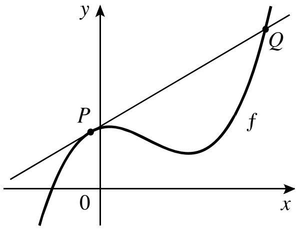
Yes. Here the line tangent to the graph of f at P also intersects the graph at the point Q lying on the graph of f.



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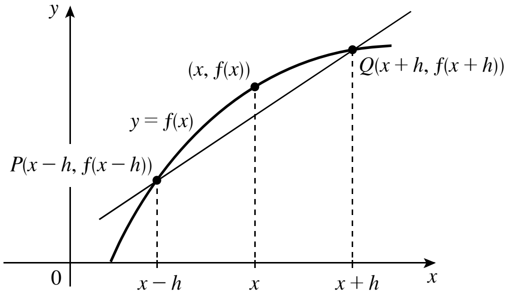
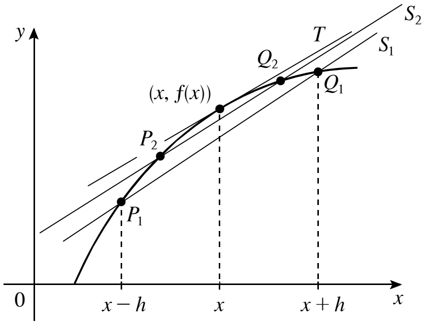
1. The quotient gives the slope of the secant line passing through $P\,(x - h, f\,(x - h))$ and $Q\,(x + h, f\,(x + h))$. It also gives the average rate of change of f over the interval $[x - h, x + h]$.



2. The limit gives the slope of the tangent line to the graph of f at the point $(x, f\,(x))$. It also gives the instantaneous rate of change of f at the point $(x, f\,(x))$. As h gets smaller and smaller, the secant lines approach the tangent line T.



3. The observation in part (b) suggests that this definition makes sense. We can also justify this observation as follows: From the definition of $f'\,(x)$, we have $f'\,(x) = \lim\limits_{h\to 0} \dfrac{f\,(x+h) - f\,(x)}{h}$. Replacing h by $-h$ gives $f'\,(x) = \lim\limits_{h\to 0} \dfrac{f\,(x-h) - f\,(x)}{-h} = \lim\limits_{h\to 0} \dfrac{f\,(x) - f\,(x-h)}{h}$. Thus, $2f'\,(x) = \lim\limits_{h\to 0} \left[\dfrac{f\,(x+h) - f\,(x)}{h} + \dfrac{f\,(x) - f\,(x-h)}{h}\right]$, and so $f'\,(x) = \lim\limits_{h\to 0} \dfrac{f\,(x+h) - f\,(x-h)}{2h}$, in agreement with the result of Example 3.

4. **Step 1** Compute $f\,(x + h)$ and $f\,(x - h)$.
   Step 2 Form the difference $f\,(x + h) - f\,(x - h)$.
   Step 3 Form the quotient $\dfrac{f\,(x+h) - f\,(x-h)}{2h}$.
   Step 4 Compute $f'\,(x) = \lim\limits_{h\to 0} \dfrac{f\,(x+h) - f\,(x-h)}{2h}$.

   For the function $f\,(x) = x^2$, we have the following:

   Step 1 $f\,(x + h) = (x + h)^2 = x^2 + 2xh + h^2$ and $f\,(x - h) = (x - h)^2 = x^2 - 2xh + h^2$.
   Step 2 $f\,(x + h) - f\,(x - h) = \left(x^2 + 2xh + h^2\right) - \left(x^2 - 2xh + h^2\right) = 4xh$.
   Step 3 $\dfrac{f\,(x+h) - f\,(x)}{2h} = \dfrac{4xh}{2h} = 2x$.
   Step 4 $f'\,(x) = \lim\limits_{h\to 0} \dfrac{f\,(x+h) - f\,(x)}{2h} = \lim\limits_{h\to 0} 2x = 2x$, in agreement with the result of Example 3.

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No. The slope of the tangent line to the graph of f at $(a, f(a))$ is defined by $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, and because the limit must be unique (see the definition of a limit), there is only one number $f'(a)$ giving the slope of the tangent line. Furthermore, since there can only be one straight line with a given slope, $f'(a)$, passing through a given point, $(a, f(a))$, our conclusion follows.

CHAPTER 2 Exploring with Technology

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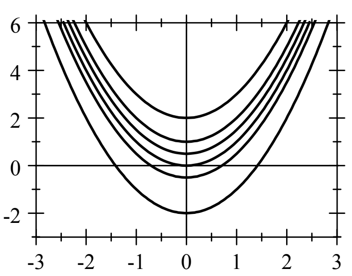
1.



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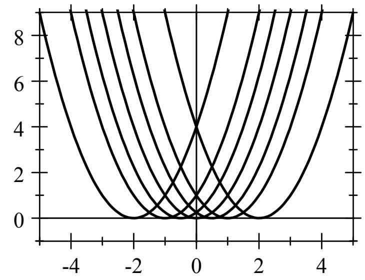

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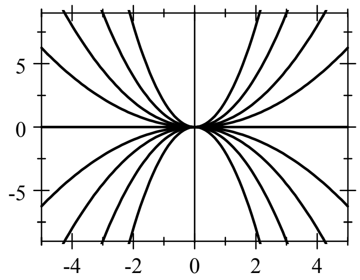

4. The graph of $f(x) + c$ is obtained by translating the graph of f along the y-axis by c units. The graph of $f(x + c)$ is obtained by translating the graph of f along the x-axis by c units. Finally, the graph of cf is obtained from that of f by "expanding"(if $c > 1$) or "contracting"(if $0 < c < 1$) that of f. If $c < 0$, the graph of cf is obtained from that of f by reflecting it with respect to the x-axis as well as expanding or contracting it.

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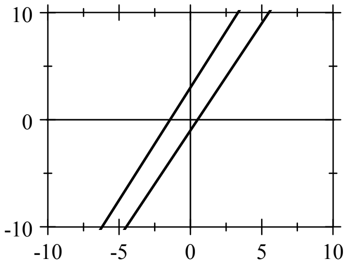
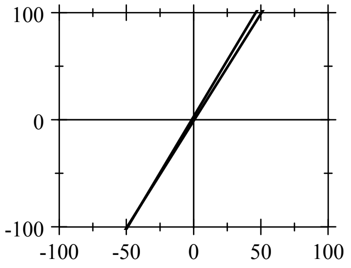
1. a.



The lines seem to be parallel to each other and do not appear to intersect.

b.

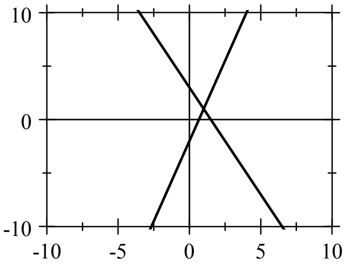


They appear to intersect. But finding the point of intersection using TRACE and ZOOM with any degree of accuracy seems to be an impossible task. Using the intersection feature of the graphing utility yields the point of intersection $(-40, -81)$ immediately.

c. Substituting the first equation into the second gives $2x - 1 = 2.1x + 3$, $-4 = 0.1x$, and thus $x = -40$. The corresponding y-value is -81.

d. Using TRACE and ZOOM is not effective. The intersection feature gives the desired result immediately. The algebraic method also yields the answer with little effort and without the use of a graphing utility.

2. a.



Plotting the straight lines L_1 and L_2 and using TRACE and ZOOM repeatedly, you will see that the iterations approach the answer (1, 1). Using the intersection feature of the graphing utility gives the result $x = 1$ and $y = 1$, immediately.

b. Substituting the first equation into the second yields $3x - 2 = -2x + 3$, so $5x = 5$ and $x = 1$. Substituting this value of x into either equation gives $y = 1$.

c. The iterations obtained using TRACE and ZOOM converge to the solution (1, 1). The use of the intersection feature is clearly superior to the first method. The algebraic method also yields the desired result easily.

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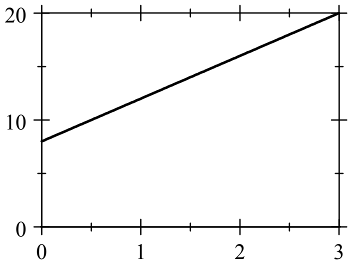
1.



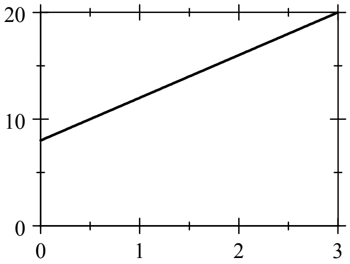
2. Using TRACE and ZOOM repeatedly, we find that $g(x)$ approaches 16 as x approaches 2.

3. If we try to use the evaluation function of the graphing utility to find $g(2)$ it will fail. This is because $x = 2$ is not in the domain of g.

4. The results obtained here confirm those obtained in the preceding example.

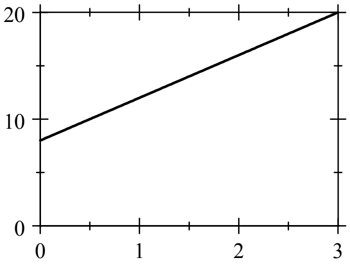
Page 109 (First Box)

1.



Using TRACE, we find $\lim_{x \to 2} \frac{4(x^2 - 4)}{x - 2} = 16$.

2.



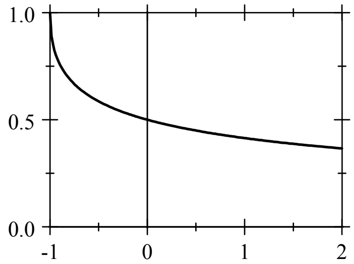
Using TRACE, we find $\lim_{x \to 2} 4(x + 2) = 16$. When $x = 2$, $y = 16$. The function $f(x) = 4(x + 2)$ is defined at $x = 2$ and so $f(2) = 16$ is defined.

3. No.

4. As we saw in Example 5, the function f is not defined at $x = 2$, but g is defined there.

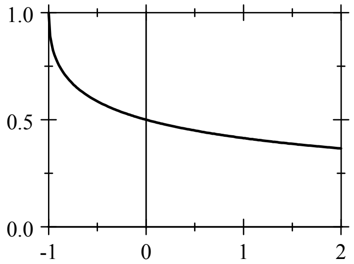
Page 109 (Second Box)

1.



Using TRACE and ZOOM, we see that $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = 0.5$.

2.



The graph of f is the same as that of g except that the domain of f includes $x = 0$. (This is not evident from simply looking at the graphs!) Using the evaluation function to find the value of y, we obtain $y = 0.5$ when $x = 0$. This is to be expected since $x = 0$ lies in the domain of g.

3. As mentioned in part 2, the graphs are indistinguishable even though $x = 0$ is in the domain of g but not in the domain of f.

4. The functions f and g are the same everywhere except at $x = 0$ and so $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{2}$, as seen in Example 6.

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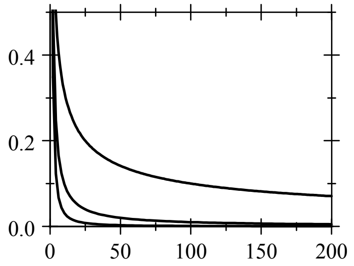
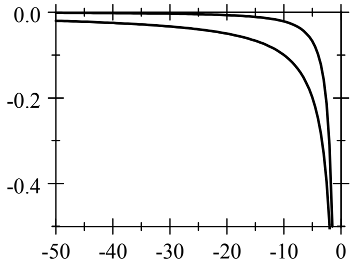
1.



The results suggest that $\frac{1}{x^n}$ goes to zero (as x increases) with increasing rapidity as n gets larger, as predicted by Theorem 2.

2.



The results suggest that $\frac{1}{x^n}$ goes to zero (as negative x increases in absolute value) with increasing rapidity as n gets larger, as predicted by Theorem 2.

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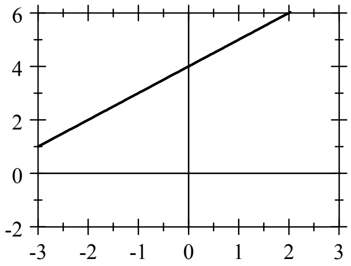
1.



2. Using ZOOM repeatedly, we find $\lim_{x\to 0} g(x) = 4$.

3. The fact that the limit found in part 2 is $f'(2)$ is an illustration of the definition of a derivative.

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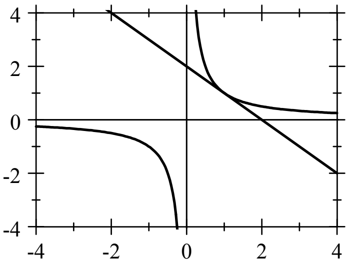
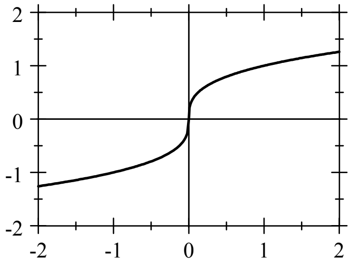


Page 147

1.



2. The graphing utility will indicate an error when you try to draw the tangent line to the graph of f at $(0, 0)$. This happens because the slope of the tangent line to the graph of $f(x)$ is not defined at $x = 0$.

3 DIFFERENTIATION

3.1 Basic Rules of Differentiation

Concept Questions page 169

1. a. The derivative of a constant is zero.

b. The derivative of $f(x) = x^n$ is n times x raised to the $(n-1)$th power.

c. The derivative of a constant times a function is the constant times the derivative of the function.

d. The derivative of the sum is the sum of the derivatives.

2. a. $h'(x) = 2f'(x)$, so $h'(2) = 2f'(2) = 2(3) = 6$.

b. $F'(x) = 3f'(x) - 4g'(x)$, so $F'(2) = 3f'(2) - 4g'(2) = 3(3) - 4(-2) = 17$.

3. a. $F'(x) = \dfrac{d}{dx}\left[af(x) + bg(x)\right] = \dfrac{d}{dx}\left[af(x)\right] + \dfrac{d}{dx}\left[bg(x)\right] = af'(x) + bg'(x)$.

b. $F'(x) = \dfrac{d}{dx}\left[\dfrac{f(x)}{a}\right] = \dfrac{1}{a}\dfrac{d}{dx}\left[f(x)\right] = \dfrac{f'(x)}{a}$.

4. No. The expression on the left is the derivative at a of the function f, whereas the expression on the right is the derivative of the constant obtained by evaluating f at a. For example, if $f(x) = x^2$ and $a = 1$, then $\left[f'(x)\right](a) = 2x|_{x=1} = 2$, but $\dfrac{d}{dx}\left[f(a)\right] = \dfrac{d}{dx}\left(1^2\right) = 0$.

Exercises page 169

1. $f'(x) = \frac{d}{dx}(-3) = 0$.

2. $f'(x) = \frac{d}{dx}(365) = 0$.

3. $f'(x) = \frac{d}{dx}\left(x^5\right) = 5x^4$.

4. $f'(x) = \frac{d}{dx}\left(x^7\right) = 7x^6$.

5. $f'(x) = \frac{d}{dx}\left(x^{3.1}\right) = 3.1x^{2.1}$.

6. $f'(x) = \frac{d}{dx}\left(x^{0.8}\right) = 0.8x^{-0.2}$.

7. $f'(x) = \frac{d}{dx}\left(3x^2\right) = 6x$.

8. $f'(x) = \frac{d}{dx}\left(-2x^3\right) = -6x^2$.

9. $f'(r) = \frac{d}{dr}\left(\pi r^2\right) = 2\pi r$.

10. $f'(r) = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$.

11. $f'(x) = \frac{d}{dx}\left(9x^{1/3}\right) = \frac{1}{3}(9)x^{(1/3-1)} = 3x^{-2/3}$.

12. $f'(x) = \frac{d}{dx}\left(\frac{5}{4}x^{4/5}\right) = \left(\frac{4}{5}\right)\left(\frac{5}{4}\right)x^{-1/5} = x^{-1/5}$.

13. $f'(x) = \frac{d}{dx}\left(3\sqrt{x}\right) = \frac{d}{dx}\left(3x^{1/2}\right) = \frac{1}{2}(3)x^{-1/2} = \frac{3}{2}x^{-1/2} = \frac{3}{2\sqrt{x}}$.

14. $f'(u) = \frac{d}{du}\left(\frac{2}{\sqrt{u}}\right) = \frac{d}{du}\left(2u^{-1/2}\right) = -\frac{1}{2}(2)u^{-3/2} = -u^{-3/2}$.

15. $f'(x) = \frac{d}{dx}\left(7x^{-12}\right) = (-12)(7)x^{-12-1} = -84x^{-13}.$

16. $f'(x) = \frac{d}{dx}\left(0.3x^{-1.2}\right) = (0.3)(-1.2)x^{-2.2} = -0.36x^{-2.2}.$

17. $f'(x) = \frac{d}{dx}\left(5x^2 - 3x + 7\right) = 10x - 3.$

18. $f'(x) = \frac{d}{dx}\left(x^3 - 3x^2 + 1\right) = 3x^2 - 6x.$

19. $f'(x) = \frac{d}{dx}\left(-x^3 + 2x^2 - 6\right) = -3x^2 + 4x.$

20. $f'(x) = \frac{d}{dx}\left[\left(1 + 2x^2\right)^2 + 2x^3\right] = \frac{d}{dx}\left(1 + 4x^2 + 4x^4 + 2x^3\right) = 8x + 16x^3 + 6x^2 = 2x\left(8x^2 + 3x + 4\right).$

21. $f'(x) = \frac{d}{dx}\left(0.03x^2 - 0.4x + 10\right) = 0.06x - 0.4.$

22. $f'(x) = \frac{d}{dx}\left(0.002x^3 - 0.05x^2 + 0.1x - 20\right) = 0.006x^2 - 0.1x + 0.1.$

23. $f(x) = \dfrac{2x^3 - 4x^2 + 3}{x} = 2x^2 - 4x + \dfrac{3}{x}$, so $f'(x) = \dfrac{d}{dx}\left(2x^2 - 4x + 3x^{-1}\right) = 4x - 4 - \dfrac{3}{x^2}.$

24. $f(x) = \dfrac{x^3 + 2x^2 + x - 1}{x} = x^2 + 2x + 1 - x^{-1}$, so $f'(x) = \frac{d}{dx}\left(x^2 + 2x + 1 - x^{-1}\right) = 2x + 2 + x^{-2}.$

25. $f'(x) = \frac{d}{dx}\left(4x^4 - 3x^{5/2} + 2\right) = 16x^3 - \frac{15}{2}x^{3/2}.$

26. $f'(x) = \frac{d}{dx}\left(5x^{4/3} - \frac{2}{3}x^{3/2} + x^2 - 3x + 1\right) = \frac{20}{3}x^{1/3} - x^{1/2} + 2x - 3.$

27. $f'(x) = \frac{d}{dx}\left(5x^{-1} + 4x^{-2}\right) = -5x^{-2} - 8x^{-3} = \dfrac{-5}{x^2} - \dfrac{8}{x^3}.$

28. $f'(x) = \frac{d}{dx}\left[-\frac{1}{3}\left(x^{-3} - x^6\right)\right] = -\frac{1}{3}\left(-3x^{-4} - 6x^5\right) = x^{-4} + 2x^5.$

29. $f'(t) = \frac{d}{dt}\left(4t^{-4} - 3t^{-3} + 2t^{-1}\right) = -16t^{-5} + 9t^{-4} - 2t^{-2} = -\dfrac{16}{t^5} + \dfrac{9}{t^4} - \dfrac{2}{t^2}.$

30. $f'(x) = \frac{d}{dx}\left(5x^{-3} - 2x^{-2} - x^{-1} + 200\right) = -15x^{-4} + 4x^{-3} + x^{-2} = -\dfrac{15}{x^4} + \dfrac{4}{x^3} + \dfrac{1}{x^2}.$

31. $f'(x) = \frac{d}{dx}\left(3x - 5x^{1/2}\right) = 3 - \frac{5}{2}x^{-1/2} = 3 - \dfrac{5}{2\sqrt{x}}.$

32. $f'(t) = \frac{d}{dt}\left(2t^2 + t^{3/2}\right) = 4t + \frac{3}{2}t^{1/2}.$

33. $f'(x) = \dfrac{d}{dx}\left(2x^{-2} - 3x^{-1/3}\right) = -4x^{-3} + x^{-4/3} = -\dfrac{4}{x^3} + \dfrac{1}{x^{4/3}}.$

34. $f'(x) = \dfrac{d}{dx}\left(\dfrac{3}{x^3} + \dfrac{4}{\sqrt{x}} + 1\right) = \dfrac{d}{dx}\left(3x^{-3} + 4x^{-1/2} + 1\right) = -9x^{-4} - 2x^{-3/2} = -\dfrac{9}{x^4} - \dfrac{2}{x^{3/2}}.$

35. $f'(x) = \frac{d}{dx}\left(2x^3 - 4x\right) = 6x^2 - 4.$

a. $f'(-2) = 6(-2)^2 - 4 = 20.$ **b.** $f'(0) = 6(0) - 4 = -4.$ **c.** $f'(2) = 6(2)^2 - 4 = 20.$

36. $f'(x) = \frac{d}{dx}\left(4x^{5/4} + 2x^{3/2} + x\right) = 5x^{1/4} + 3x^{1/2} + 1.$

a. $f'(4) = 5(4)^{1/4} + 3(4)^{1/2} + 1 = 5(4)^{1/4} + 6 + 1 = 5(4)^{1/4} + 7 = 5\sqrt{2} + 7.$

b. $f'(16) = 5(16)^{1/4} + 3(16)^{1/2} + 1 = 10 + 12 + 1 = 23.$

37. The given limit is $f'(1)$, where $f(x) = x^3$. Because $f'(x) = 3x^2$, we have $\lim_{h\to 0} \frac{(1+h)^3 - 1}{h} = f'(1) = 3.$

38. Letting $h = x - 1$ or $x = h + 1$ and observing that $h \to 0$ as $x \to 1$, we find $\lim_{x\to 1} \frac{x^5 - 1}{x - 1} = \lim_{h\to 0} \frac{(h+1)^5 - 1}{h} = f'(1)$, where $f(x) = x^5$. Because $f'(x) = 5x^4$, we have $f'(1) = 5$, the value of the limit; that is, $\lim_{x\to 1} \frac{x^5 - 1}{x - 1} = 5.$

39. Let $f(x) = 3x^2 - x$. Then $\lim_{h\to 0} \frac{3(2+h)^2 - (2+h) - 10}{h} = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$ because $f(2+h) - f(2) = 3(2+h)^2 - (2+h) - [3(4) - 2] = 3(2+h)^2 - (2+h) - 10$. But the last limit is simply $f'(2)$. Because $f'(x) = 6x - 1$, we have $f'(2) = 11$. Therefore, $\lim_{h\to 0} \frac{3(2+h)^2 - (2+h) - 10}{h} = 11.$

40. Write $\lim_{t\to 0} \frac{1 - (1+t)^2}{t(1+t)^2} = \lim_{t\to 0} \frac{1}{(1+t)^2} \cdot \lim_{t\to 0} \frac{1 - (1+t)^2}{t}$. Now let $f(t) = -t^2$. Then $\lim_{t\to 0} \frac{1 - (1+t)^2}{t} = \lim_{t\to 0} \frac{f(1+t) - f(1)}{t} = f'(1)$. Because $f'(t) = -2t$, we find $f'(1) = -2$. Therefore, $\lim_{t\to 0} \frac{1 - (1+t)^2}{t(1+t)^2} = \lim_{t\to 0} \frac{1}{(1+t)^2} \cdot f'(1) = 1 \cdot (-2) = -2.$

41. $f(x) = 2x^2 - 3x + 4$. The slope of the tangent line at any point $(x, f(x))$ on the graph of f is $f'(x) = 4x - 3$. In particular, the slope of the tangent line at the point $(2, 6)$ is $f'(2) = 4(2) - 3 = 5$. An equation of the required tangent line is $y - 6 = 5(x - 2)$ or $y = 5x - 4$.

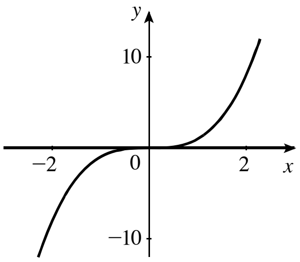
42. $f(x) = -\frac{5}{3}x^2 + 2x + 2$, so $f'(x) = -\frac{10}{3}x + 2$. The slope is $f'(-1) = \frac{10}{3} + 2 = \frac{16}{3}$. An equation of the tangent line is $y + \frac{5}{3} = \frac{16}{3}(x + 1)$ or $y = \frac{16}{3}x + \frac{11}{3}$.

43. $f(x) = x^4 - 3x^3 + 2x^2 - x + 1$, so $f'(x) = 4x^3 - 9x^2 + 4x - 1$. The slope is $f'(2) = 4(2)^3 - 9(2)^2 + 4(2) - 1 = 3$. An equation of the tangent line is $y - (-1) = 3(x - 2)$ or $y = 3x - 7$.

44. $f(x) = \sqrt{x} + 1/\sqrt{x}$. The slope of the tangent line at any point $(x, f(x))$ on the graph of f is $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$. In particular, the slope of the tangent line at the point $\left(4, \frac{5}{2}\right)$ is $f'(4) = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$. An equation of the required tangent line is $y - \frac{5}{2} = \frac{3}{16}(x - 4)$ or $y = \frac{3}{16}x + \frac{7}{4}$.

45. **a.** $f'(x) = 3x^2$. At a point where the tangent line is horizontal, $f'(x) = 0$, or $3x^2 = 0$, and so $x = 0$. Therefore, the point is $(0, 0)$.

b.

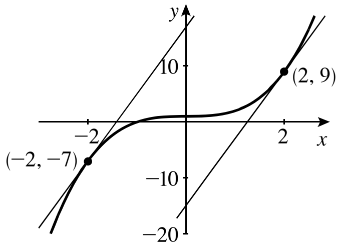


46. $f(x) = x^3 - 4x^2$, so $f'(x) = 3x^2 - 8x = x(3x - 8)$. Thus, $f'(x) = 0$ if $x = 0$ or $x = \frac{8}{3}$. Therefore, the points are $(0, 0)$ and $\left(\frac{8}{3}, -\frac{256}{27}\right)$.

47. **a.** $f(x) = x^3 + 1$. The slope of the tangent line at any point $(x, f(x))$ on the graph of f is $f'(x) = 3x^2$. At the point(s) where the slope is 12, we have $3x^2 = 12$, so $x = \pm 2$. The required points are $(-2, -7)$ and $(2, 9)$.

b. The tangent line at $(-2, -7)$ has equation $y - (-7) = 12[x - (-2)]$, or $y = 12x + 17$, and the tangent line at $(2, 9)$ has equation $y - 9 = 12(x - 2)$, or $y = 12x - 15$.

c.



48. $f(x) = \frac{2}{3}x^3 + x^2 - 12x + 6$, so $f'(x) = 2x^2 + 2x - 12$.

a. $f'(x) = -12$ gives $2x^2 + 2x - 12 = -12$, $2x^2 + 2x = 0$, $2x(x + 1) = 0$; that is, $x = 0$ or $x = -1$.

b. $f'(x) = 0$ gives $2x^2 + 2x - 12 = 0$, $2\left(x^2 + x - 6\right) = 2(x + 3)(x - 2) = 0$, and so $x = -3$ or $x = 2$.

c. $f'(x) = 12$ gives $2x^2 + 2x - 12 = 12$, $2\left(x^2 + x - 12\right) = 2(x + 4)(x - 3) = 0$, and so $x = -4$ or $x = 3$.

49. $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2$, so $f'(x) = x^3 - x^2 - 2x$.

a. $f'(x) = x^3 - x^2 - 2x = -2x$ implies $x^3 - x^2 = 0$, so $x^2(x - 1) = 0$. Thus, $x = 0$ or $x = 1$. $f(1) = \frac{1}{4}(1)^4 - \frac{1}{3}(1)^3 - (1)^2 = -\frac{13}{12}$ and $f(0) = \frac{1}{4}(0)^4 - \frac{1}{3}(0)^3 - (0)^2 = 0$. We conclude that the corresponding points on the graph are $\left(1, -\frac{13}{12}\right)$ and $(0, 0)$.

b. $f'(x) = x^3 - x^2 - 2x = 0$ implies $x\left(x^2 - x - 2\right) = 0$, $x(x - 2)(x + 1) = 0$, and so $x = 0, 2$, or -1. $f(0) = 0$, $f(2) = \frac{1}{4}(2)^4 - \frac{1}{3}(2)^3 - (2)^2 = 4 - \frac{8}{3} - 4 = -\frac{8}{3}$, and $f(-1) = \frac{1}{4}(-1)^4 - \frac{1}{3}(-1)^3 - (-1)^2 = \frac{1}{4} + \frac{1}{3} - 1 = -\frac{5}{12}$. We conclude that the corresponding points are $(0, 0)$, $\left(2, -\frac{8}{3}\right)$, and $\left(-1, -\frac{5}{12}\right)$.

c. $f'(x) = x^3 - x^2 - 2x = 10x$ implies $x^3 - x^2 - 12x = 0$, $x\left(x^2 - x - 12\right) = 0$, $x(x - 4)(x + 3) = 0$, so $x = 0, 4$, or -3. $f(0) = 0$, $f(4) = \frac{1}{4}(4)^4 - \frac{1}{3}(4)^3 - (4)^2 = 48 - \frac{64}{3} = \frac{80}{3}$, and $f(-3) = \frac{1}{4}(-3)^4 - \frac{1}{3}(-3)^3 - (-3)^2 = \frac{81}{4} + 9 - 9 = \frac{81}{4}$. We conclude that the corresponding points are $(0, 0)$, $\left(4, \frac{80}{3}\right)$, and $\left(-3, \frac{81}{4}\right)$.

50. $y = x^3 - 3x + 1$, so $\frac{dy}{dx} = 3x^2 - 3$. The slope of the tangent line to the given graph is $\left.\frac{dy}{dx}\right|_{x=2} = 3(4) - 3 = 9$. Therefore, an equation of the tangent line at $(2, 3)$ is $y - 3 = 9(x - 2)$, or $y = 9x - 15$. The slope of the normal line through the point $(2, 3)$ is $-\frac{1}{9}$. Therefore, an equation of the required normal line is $y - 3 = -\frac{1}{9}(x - 2)$ or $y = -\frac{1}{9}x + \frac{29}{9}$.

51. $V(r) = \frac{4}{3}\pi r^3$, so $V'(r) = 4\pi r^2$.

a. $V'\left(\frac{2}{3}\right) = 4\pi\left(\frac{4}{9}\right) = \frac{16}{9}\pi\ \text{cm}^3/\text{cm}$.

b. $V'\left(\frac{5}{4}\right) = 4\pi\left(\frac{25}{16}\right) = \frac{25}{4}\pi\ \text{cm}^3/\text{cm}$.

52. $v(r) = k\left(R^2 - r^2\right) = 1000\left(0.04 - r^2\right)$, so $v(0.1) = 1000\left(0.04 - 0.1^2\right) = 1000(0.03) = 30$. This says that the velocity of blood 0.1 cm from the central axis is 30 cm/sec. Next, $v'(r) = -2000r$, and so $v'(0.1) = -200$. This says that at a point 0.1 cm from the central axis, the velocity of blood is decreasing at the rate of 200 cm/sec per cm along a line perpendicular to the central axis.

53. a. The number of tablets and smartphones in use in 2011 was $f(2) = 128.1(2)^{1.94} \approx 491.5269$, or approximately 491.5 million.

b. $f'(t) = 128.1 \cdot 1.94t^{0.94} = 248.514t^{0.94}$, so the number of tablets and smartphones in 2011 was changing at the rate of $f'(2) = 248.514(2)^{0.94} \approx 476.7811$, or approximately 476.8 million/year.

54. a. The percentage of the UK population that is expected to watch video content on mobile phones in 2015 is $P(4) = 13.86(4)^{0.535} \approx 29.098$, or approximately 29.10%.

b. $P'(t) = 13.86 \cdot 0.535t^{-0.465}$, so the percentage of mobile phone video viewers in 2015 is projected to be changing at the rate of $P'(4) = 13.86 \cdot 0.535(4)^{-0.465} \approx 3.8919$, or approximately 3.89%/year.

55. a.

1970 ($t = 1$)	1990 ($t = 3$)	2010 ($t = 5$)
49.6%	36.9%	32.1%

b. $P'(t) = (49.6)\left(-0.27t^{-1.27}\right) = -\dfrac{13.392}{t^{1.27}}$. In 1990, $P'(3) \approx -3.3$, or decreasing at 3.3%/decade. In 2000, $P'(4) \approx -2.3$, or decreasing at 2.3%/decade.

56. $\dfrac{dA}{dx} = 26.5\dfrac{d}{dx}\left(x^{-0.45}\right) = 26.5(-0.45)x^{-1.45} = -\dfrac{11.925}{x^{1.45}}$. Therefore, $\left.\dfrac{dA}{dx}\right|_{x=0.25} = -\dfrac{11.925}{(0.25)^{1.45}} \approx -89.01$ and $\left.\dfrac{dA}{dx}\right|_{x=2} = -\dfrac{11.925}{(2)^{1.45}} \approx -4.36$. Our computations reveal that if you make 0.25 stops per mile, your average speed decreases at the rate of approximately 89.01 mph per stop per mile. If you make 2 stops per mile, your average speed decreases at the rate of approximately 4.36 mph per stop per mile.

57. a. The percentage of households with annual incomes within 50 percent of the median in 2010 was $P(4) = 50.3(4)^{-0.09} \approx 44.40$ (percent).

b. $P'(t) = 50.3\left(-0.09t^{-1.09}\right) \approx -4.527t^{-1.09}$, so the percentage of households with annual incomes within 50 percent of the median in 2010 was changing at the rate of $P'(4) = -4.527(4)^{-1.09} \approx -1.00$; that is, decreasing at the rate of approximately 1%/decade.

58. **a.** $f(x) = -0.1x^2 - 0.4x + 35$, so $f'(x) = -0.2x - 0.4$.

b. $f'(10) = -0.2(10) - 0.4 = -2.4$; that is, it is decreasing at the rate of \$2.40 per 1000 lamps. The unit price at this level of demand is $f(10) = -0.1\left(10^2\right) - 0.4(10) + 35 = 21$, or \$21.

59. **a.** $f(t) = 120t - 15t^2$, so $v = f'(t) = 120 - 30t$.

b. $v(0) = 120$ ft/sec

c. Setting $v = 0$ gives $120 - 30t = 0$, or $t = 4$. Therefore, the stopping distance is $f(4) = 120(4) - 15(16)$ or 240 ft.

60. **a.** The total percentage was $P(2) = 0.257\left(2^2\right) + 0.57(2) + 3.9 = 6.068$, or approximately 6.1%.

b. In 2008, the percentage was changing at the rate of $P'(2) = (0.514t + 0.57)\big|_{t=2} = 0.514(2) + 0.57 = 1.598$, or approximately 1.6%/year.

61. **a.** The approximate average medical cost for a family of four in 2010 was $C(10) = 22.9883(10)^2 + 830.358(10) + 7513 = 18{,}115.41$, or \$18,115.41.

b. $C'(t) = 45.9766t + 830.358$, so the rate at which the average medical cost for a family of four was increasing in 2010 was approximately $C'(10) = 45.9766(10) + 830.358 = 1290.124$, or \$1290.12/year.

62. **a.** $P(t) = 0.27t^2 + 1.4t + 2.2$, so $P'(t) = 0.54t + 1.4$. In 2010, $P'(1) = 0.54(1) + 1.4 = 1.94$ or 1.94%/decade. In 2020, $P'(2) = 0.54(2) + 1.4 = 2.48$, or 2.48%/decade.

b. In 2010, $P(1) = 0.27\left(1^2\right) + 1.4(1) + 2.2 = 3.87$, or 3.87%. In 2020, $P(2) = 0.27\left(2^2\right) + 1.4(2) + 2.2 = 6.08$, or 6.08%.

63. $I(t) = -0.2t^3 + 3t^2 + 100$, so $I'(t) = -0.6t^2 + 6t$.

a. In 2008, it was changing at a rate of $I'(5) = -0.6(25) + 6(5)$, or 15 points/yr. In 2010, it is $I'(7) = -0.6(49) + 6(7)$, or 12.6 points/yr. In 2013, it is $I'(10) = -0.6(100) + 6(10)$, or 0 points/yr.

b. The average rate of increase of the CPI over the period from 2008 to 2013 is

$$\frac{I(10) - I(5)}{5} = \frac{[-0.2(1000) + 3(100) + 100] - [-0.2(125) + 3(25) + 100]}{5} = \frac{200 - 150}{5} = 10, \text{ or}$$

10 points/yr.

64. **a.** $N(t) = -t^3 + 6t^2 + 15t$. The rate is given by $N'(t) = -3t^2 + 12t + 15$.

b. The rate at which the average worker is assembling walkie-talkies at 10 a.m. is $N'(2) = -3(2)^2 + 12(2) + 15 = 27$, or 27 walkie-talkies/hour. At 11 a.m., we have $N'(3) = -3(3)^2 + 12(3) + 15 = 24$, or 24 walkie-talkies/hour.

c. The number will be $N(3) - N(2) = (-27 + 54 + 45) - (-8 + 24 + 30) = 26$, or 26 walkie-talkies.

65. $P(t) = -\frac{1}{3}t^3 + 64t + 3000$, so $P'(t) = -t^2 + 64$. The rates of change at the end of years one, two, three and four are $P'(1) = -1 + 64 = 63$, or 63,000 people/yr; $P'(2) = -4 + 64 = 60$, or 60,000 people/yr; $P'(3) = -9 + 64 = 55$, or 55,000 people/yr; and $P'(4) = -16 + 64 = 48$, or 48,000 people/yr. It appears that the plan is working.

66. $N(t) = 2t^3 + 3t^2 - 4t + 1000$, so $N'(t) = 6t^2 + 6t - 4$. $N'(2) = 6(4) + 6(2) - 4 = 32$, or 32 turtles/yr; and $N'(8) = 6(64) + 6(8) - 4 = 428$, or 428 turtles/yr. The population ten years after implementation of the conservation measures will be $N(10) = 2\left(10^3\right) + 3\left(10^2\right) - 4(10) + 1000$, or 3260 turtles.

67. a. $f(t) = -2t^3 + 12t^2 + 5$, so $v = f'(t) = -6t^2 + 24t$.

b. $f'(0) = 0$, or 0 ft/sec; $f'(2) = -6(4) + 24(2) = 24$, or 24 ft/sec; $f'(4) = -6(16) + 24(4) = 0$, or 0 ft/sec; and $f'(6) = -6(36) + 24(6) = -72$, or -72 ft/sec. The rocket starts out at an initial velocity of 0 ft/sec. It climbs upward until a maximum altitude is attained 4 seconds into flight. It then descends until it hits the ground.

c. At the highest point, $v = 0$. But this occurs when $t = 4$ (see part (b)). The maximum altitude is $f(4) = -2(4)^3 + 12(4)^2 + 5 = 69$, or 69 feet.

68. a. $f'(x) = \frac{d}{dx}\left[0.0001x^{5/4} + 10\right] = \frac{5}{4}\left(0.0001x^{1/4}\right) = 0.000125x^{1/4}$.

b. $f'(10{,}000) = 0.000125(10{,}000)^{1/4} = 0.00125$, or \$0.00125/radio.

69. $P(t) = 50{,}000 + 30t^{3/2} + 20t$. The rate at which the population is increasing at any time t is $P'(t) = 45t^{1/2} + 20$. Nine months from now the population will be increasing at the rate of $P'(9) = 45(9)^{1/2} + 20$, or 155 people/month. Sixteen months from now the population will be increasing at the rate of $P'(16) = 45(16)^{1/2} + 20$, or 200 people/month.

70. a. $f(t) = 20t - 40\sqrt{t} + 50$, so $f'(t) = 20 - 40\left(\frac{1}{2}\right)t^{-1/2} = 20\left(1 - \frac{1}{\sqrt{t}}\right)$.

b. $f(0) = 20(0) - 40\sqrt{0} + 50 = 50$, so $f(1) = 20(1) - 40\sqrt{1} + 50 = 30$ and $f(2) = 20(2) - 40\sqrt{2} + 50 \approx 33.43$. The average velocities at 6, 7, and 8 a.m. are 50, 30, and 33.43 mph, respectively.

c. $f'\left(\frac{1}{2}\right) = 20 - 20\left(\frac{1}{2}\right)^{-1/2} \approx -8.28$, $f'(1) = 20 - 20(1)^{-1/2} \approx 0$, and $f'(2) = 20 - 20(2)^{-1/2} \approx 5.86$. At 6:30 a.m. the average velocity is decreasing at the rate of 8.28 mph/hr, at 7 a.m. it is not changing, and at 8 a.m. it is increasing at the rate of 5.86 mph.

71. $S(x) = -0.002x^3 + 0.6x^2 + x + 500$, so $S'(x) = -0.006x^2 + 1.2x + 1$.

a. When $x = 100$, $S'(100) = -0.006(100)^2 + 1.2(100) + 1 = 61$, or \$61,000 per thousand dollars.

b. When $x = 150$, $S'(150) = -0.006(150)^2 + 1.2(150) + 1 = 46$, or \$46,000 per thousand dollars. We conclude that the company's total sales increase at a faster rate with option (a); that is, when \$100,000 is spent on advertising.

72. a. The per capita health spending in 2010 was $C(10) = -1.1708(10)^3 + 7.029(10)^2 + 389.69(10) + 4780 = 8209$, or \$8209.

b. $C'(t) = -3.5124t^2 + 14.058t + 389.69$, so the per capita health spending in 2010 was changing at the rate of $C'(10) = -3.5124(10)^2 + 14.058(10) + 389.69 = 179.03$; that is, it was increasing at \$179.03/year.

73. a. $P(t) = 0.0004t^3 + 0.0036t^2 + 0.8t + 12$. At the beginning of 1991, $P(0) = 12\%$. At the beginning of 2010, $P(19) = 0.0004(19)^3 + 0.0036(19)^2 + 0.8(19) + 12 \approx 31.2$, or approximately 31.2%.

b. $P'(t) = 0.0012t^2 + 0.0072t + 0.8$. At the beginning of 1991, $P'(0) = 0.8$, or 0.8%/yr. At the beginning of 2010, $P'(19) = 0.0012(19)^2 + 0.0072(19) + 0.8 \approx 1.4$, or approximately 1.4%/yr.

74. a. At any time t, the function $D = g + f$ at t, $D(t) = (g + f)(t) = g(t) + f(t)$, gives the total population aged 65 and over of the developed and the underdeveloped/emerging countries.

b. $D(t) = g(t) + f(t) = \left(0.46t^2 + 0.16t + 287.8\right) + (3.567t + 175.2) = 0.46t^2 + 3.727t + 463$, so $D'(t) = 0.92t + 3.727$. Therefore, $D'(10) = 0.92(10) + 3.727 = 12.927$, which says that the combined population is growing at the rate of approximately 13 million people per year in 2010.

75. a. $$G(t) = J(t) - N(t) = \begin{cases} -0.0002t^2 + 0.032t + 0.1 & \text{if } 0 \le t < 5 \\ 0.0002t^2 - 0.006t + 0.28 & \text{if } 5 \le t < 10 \\ -0.0012t^2 + 0.082t - 0.46 & \text{if } 10 \le t < 15 \end{cases}$$

b. In 2008, where $t = 8$, the gap is changing at a rate of $G'(8) = \left[\frac{d}{dt}\left(0.0002t^2 - 0.006t + 0.28\right)\right]_{t=8} = (0.0004t - 0.006)|_{t=8} = -0.0028$; that is, the gap is narrowing at a rate of 2800 jobs/yr. In 2012, where $t = 12$, the gap is changing at a rate of $G'(12) = \left[\frac{d}{dt}\left(-0.0012t^2 + 0.082t - 0.46\right)\right]_{t=12} = (-0.0024t + 0.082)|_{t=12} = 0.0532$; that is, the gap is increasing at a rate of 53,200 jobs/yr.

76. True. $\frac{d}{dx}\left[2f(x) - 5g(x)\right] = \frac{d}{dx}\left[2f(x)\right] - \frac{d}{dx}\left[5g(x)\right] = 2f'(x) - 5g'(x)$.

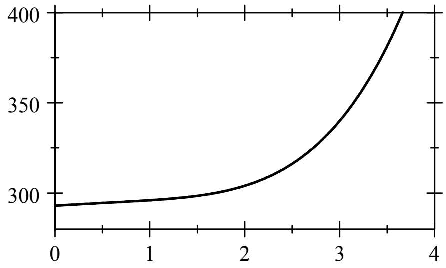
77. False. f is not a power function.

78. $$\frac{d}{dx}\left(x^3\right) = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h\to 0}\frac{h\left(3x^2 + 3xh + h^2\right)}{h}$$
$$= \lim_{h\to 0}\left(3x^2 + 3xh + h^2\right) = 3x^2.$$

Using Technology page 175

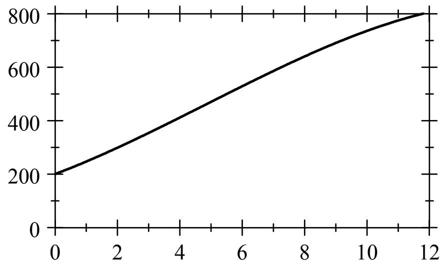
1. 1	**2.** 3.0720	**3.** 0.4226	**4.** 0.0732	**5.** 0.1613	**6.** 3.9730

7. a.



b. 3.4295 parts/million per 40 years;
164.239 parts/million per 40 years

8. a.



b. 42,272 cases/year

3.2 The Product and Quotient Rules

Concept Questions page 181

1. **a.** The derivative of the product of two functions is equal to the first function times the derivative of the second function plus the second function times the derivative of the first function.

   b. The derivative of the quotient of two functions is equal to the quotient whose numerator is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator and whose denominator is the square of the denominator of the quotient.

2. **a.** $h'(x) = f(x)g'(x) + f'(x)g(x)$, so $h'(1) = f(1)g'(1) + f'(1)g(1) = (3)(4) + (-1)(2) = 10$.

   b. $F'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$, so $F'(1) = \dfrac{g(1)f'(1) - f(1)g'(1)}{[g(1)]^2} = \dfrac{2(-1) - 3(4)}{2^2} = -\dfrac{7}{2}$.

Exercises page 181

1. $f(x) = 2x(x^2+1)$, so $f'(x) = 2x\frac{d}{dx}(x^2+1) + (x^2+1)\frac{d}{dx}(2x) = 2x(2x) + (x^2+1)(2) = 6x^2+2$.

2. $f(x) = 3x^2(x-1)$, so $f'(x) = 3x^2\frac{d}{dx}(x-1) + (x-1)\frac{d}{dx}(3x^2) = 3x^2 + (x-1)(6x) = 9x^2 - 6x$.

3. $f(t) = (t-1)(2t+1)$, so
   $f'(t) = (t-1)\frac{d}{dt}(2t+1) + (2t+1)\frac{d}{dt}(t-1) = (t-1)(2) + (2t+1)(1) = 4t-1$.

4. $f(x) = (2x+3)(3x-4)$, so
   $f'(x) = (2x+3)\frac{d}{dx}(3x-4) + (3x-4)\frac{d}{dx}(2x+3) = (2x+3)(3) + (3x-4)(2) = 12x+1$.

5. $f(x) = (3x+1)(x^2-2)$, so
   $f'(x) = (3x+1)\frac{d}{dx}(x^2-2) + (x^2-2)\frac{d}{dx}(3x+1) = (3x+1)(2x) + (x^2-2)(3) = 9x^2+2x-6$.

6. $f(x) = (x+1)(2x^2-3x+1)$, so
   $$f'(x) = (x+1)\tfrac{d}{dx}(2x^2-3x+1) + (2x^2-3x+1)\tfrac{d}{dx}(x+1) = (x+1)(4x-3) + (2x^2-3x+1)(1)$$
   $$= 4x^2 - 3x + 4x - 3 + 2x^2 - 3x + 1 = 6x^2 - 2x - 2 = 2(3x^2 - x - 1).$$

7. $f(x) = (x^3-1)(x+1)$, so
   $f'(x) = (x^3-1)\frac{d}{dx}(x+1) + (x+1)\frac{d}{dx}(x^3-1) = (x^3-1)(1) + (x+1)(3x^2) = 4x^3+3x^2-1$.

8. $f(x) = (x^3-12x)(3x^2+2x)$, so
   $$f'(x) = (x^3-12x)\tfrac{d}{dx}(3x^2+2x) + (3x^2+2x)\tfrac{d}{dx}(x^3-12x)$$
   $$= (x^3-12x)(6x+2) + (3x^2+2x)(3x^2-12)$$
   $$= 6x^4 + 2x^3 - 72x^2 - 24x + 9x^4 + 6x^3 - 36x^2 - 24x = 15x^4 + 8x^3 - 108x^2 - 48x.$$

9. $f(w) = \left(w^3 - w^2 + w - 1\right)\left(w^2 + 2\right)$, so

$$\begin{aligned} f'(w) &= \left(w^3 - w^2 + w - 1\right)\frac{d}{dw}\left(w^2 + 2\right) + \left(w^2 + 2\right)\frac{d}{dw}\left(w^3 - w^2 + w - 1\right) \\ &= \left(w^3 - w^2 + w - 1\right)(2w) + \left(w^2 + 2\right)\left(3w^2 - 2w + 1\right) \\ &= 2w^4 - 2w^3 + 2w^2 - 2w + 3w^4 - 2w^3 + w^2 + 6w^2 - 4w + 2 = 5w^4 - 4w^3 + 9w^2 - 6w + 2. \end{aligned}$$

10. $f(x) = \frac{1}{5}x^5 + \left(x^2 + 1\right)\left(x^2 - x - 1\right) + 28$, so

$$\begin{aligned} f'(x) &= x^4 + \left(x^2 + 1\right)(2x - 1) + 2x\left(x^2 - x - 1\right) = x^4 + 2x^3 - x^2 + 2x - 1 + 2x^3 - 2x^2 - 2x \\ &= x^4 + 4x^3 - 3x^2 - 1. \end{aligned}$$

11. $f(x) = \left(5x^2 + 1\right)\left(2\sqrt{x} - 1\right)$, so

$$\begin{aligned} f'(x) &= \left(5x^2 + 1\right)\tfrac{d}{dx}\left(2x^{1/2} - 1\right) + \left(2x^{1/2} - 1\right)\tfrac{d}{dx}\left(5x^2 + 1\right) = \left(5x^2 + 1\right)\left(x^{-1/2}\right) + \left(2x^{1/2} - 1\right)(10x) \\ &= 5x^{3/2} + x^{-1/2} + 20x^{3/2} - 10x = \frac{25x^2 - 10x\sqrt{x} + 1}{\sqrt{x}}. \end{aligned}$$

12. $f(t) = \left(1 + \sqrt{t}\right)\left(2t^2 - 3\right)$, so

$$\begin{aligned} f'(t) &= \left(1 + t^{1/2}\right)(4t) + \left(2t^2 - 3\right)\left(\tfrac{1}{2}t^{-1/2}\right) = 4t + 4t^{3/2} + t^{3/2} - \tfrac{3}{2}t^{-1/2} = 5t^{3/2} + 4t - \tfrac{3}{2}t^{-1/2} \\ &= \frac{10t^2 + 8t\sqrt{t} - 3}{2\sqrt{t}}. \end{aligned}$$

13. $f(x) = \left(x^2 - 5x + 2\right)\left(x - \dfrac{2}{x}\right)$, so

$$\begin{aligned} f'(x) &= \left(x^2 - 5x + 2\right)\frac{d}{dx}\left(x - \frac{2}{x}\right) + \left(x - \frac{2}{x}\right)\frac{d}{dx}\left(x^2 - 5x + 2\right) \\ &= \frac{\left(x^2 - 5x + 2\right)\left(x^2 + 2\right)}{x^2} + \frac{\left(x^2 - 2\right)(2x - 5)}{x} = \frac{\left(x^2 - 5x + 2\right)\left(x^2 + 2\right) + x\left(x^2 - 2\right)(2x - 5)}{x^2} \\ &= \frac{x^4 + 2x^2 - 5x^3 - 10x + 2x^2 + 4 + 2x^4 - 5x^3 - 4x^2 + 10x}{x^2} = \frac{3x^4 - 10x^3 + 4}{x^2}. \end{aligned}$$

14. $f(x) = \left(x^3 + 2x + 1\right)\left(2 + \dfrac{1}{x^2}\right) = 2x^3 + 4x + 2 + x + \dfrac{2}{x} + \dfrac{1}{x^2}$, so

$$f'(x) = \frac{d}{dx}\left(2x^3 + 5x + 2 + \frac{2}{x} + \frac{1}{x^2}\right) = 6x^2 + 5 - \frac{2}{x^2} - \frac{2}{x^3} = \frac{6x^5 + 5x^3 - 2x - 2}{x^3}.$$

15. $f(x) = \dfrac{1}{x - 2}$, so $f'(x) = \dfrac{(x - 2)\frac{d}{dx}(1) - (1)\frac{d}{dx}(x - 2)}{(x - 2)^2} = \dfrac{0 - 1(1)}{(x - 2)^2} = -\dfrac{1}{(x - 2)^2}$.

16. $g(x) = \dfrac{3}{2x + 4} + 2x^2$, so $g'(x) = \dfrac{d}{dx}\left(\dfrac{3}{2x + 4}\right) + \dfrac{d}{dx}\left(2x^2\right) = \dfrac{(2x + 4)(0) - 3(2)}{(2x + 4)^2} + 4x = -\dfrac{6}{(2x + 4)^2} + 4x$.

17. $f(x) = \dfrac{2x - 1}{2x + 1}$, so

$$f'(x) = \frac{(2x + 1)\frac{d}{dx}(2x - 1) - (2x - 1)\frac{d}{dx}(2x + 1)}{(2x + 1)^2} = \frac{2(2x + 1) - (2x - 1)(2)}{(2x + 1)^2} = \frac{4}{(2x + 1)^2}.$$

18. $f(t) = \dfrac{1-2t}{1+3t}$, so $f'(t) = \dfrac{(1+3t)(-2)-(1-2t)(3)}{(1+3t)^2} = \dfrac{-5}{(1+3t)^2}$.

19. $f(x) = \dfrac{1}{x^2+x+2}$, so $f'(x) = \dfrac{\left(x^2+x+2\right)(0)-(1)(2x+1)}{\left(x^2+x+2\right)^2} = -\dfrac{2x+1}{\left(x^2+x+2\right)^2}$.

20. $f(u) = \dfrac{u}{u^2+1}$, so $f'(u) = \dfrac{\left(u^2+1\right)\dfrac{d}{du}(u) - u\dfrac{d}{du}\left(u^2+1\right)}{\left(u^2+1\right)^2} = \dfrac{\left(u^2+1\right)(1)-u(2u)}{\left(u^2+1\right)^2} = \dfrac{1-u^2}{\left(u^2+1\right)^2}$.

21. $f(s) = \dfrac{s^2-4}{s+1}$, so

$$f'(s) = \frac{(s+1)\dfrac{d}{ds}\left(s^2-4\right)-\left(s^2-4\right)\dfrac{d}{ds}(s+1)}{(s+1)^2} = \frac{(s+1)(2s)-\left(s^2-4\right)(1)}{(s+1)^2} = \frac{s^2+2s+4}{(s+1)^2}.$$

22. $f(x) = \dfrac{x^3-2}{x^2+1}$, so

$$f'(x) = \frac{\left(x^2+1\right)\frac{d}{dx}\left(x^3-2\right)-\left(x^3-2\right)\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2} = \frac{\left(x^2+1\right)\left(3x^2\right)-\left(x^3-2\right)(2x)}{\left(x^2+1\right)^2} = \frac{x\left(x^3+3x+4\right)}{\left(x^2+1\right)^2}.$$

23. $f(x) = \dfrac{\sqrt{x}+1}{x^2+1}$, so

$$f'(x) = \frac{\left(x^2+1\right)\frac{d}{dx}\left(x^{1/2}\right)-\left(x^{1/2}+1\right)\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2} = \frac{\left(x^2+1\right)\left(\frac{1}{2}x^{-1/2}\right)-\left(x^{1/2}+1\right)(2x)}{\left(x^2+1\right)^2}$$

$$= \frac{\left(\frac{1}{2}x^{-1/2}\right)\left[\left(x^2+1\right)-\left(x^{1/2}+1\right)4x^{3/2}\right]}{\left(x^2+1\right)^2} = \frac{1-3x^2-4x^{3/2}}{2\sqrt{x}\left(x^2+1\right)^2}.$$

24. $f(x) = \dfrac{x}{\sqrt{x}+2} = \dfrac{x}{x^{1/2}+2}$, so

$$f'(x) = \frac{\left(x^{1/2}+2\right)(1)-x\left(\frac{1}{2}x^{-1/2}\right)}{\left(x^{1/2}+2\right)^2} = \frac{x^{1/2}+2-\frac{1}{2}x^{1/2}}{\left(x^{1/2}+2\right)^2} = \frac{\frac{1}{2}x^{1/2}+2}{\left(x^{1/2}+2\right)^2} = \frac{\frac{1}{2}\left(x^{1/2}+4\right)}{\left(x^{1/2}+2\right)^2} = \frac{\sqrt{x}+4}{2\left(\sqrt{x}+2\right)^2}.$$

25. $f(x) = \dfrac{x^2+2}{x^2+x+1}$, so

$$f'(x) = \frac{\left(x^2+x+1\right)\frac{d}{dx}\left(x^2+2\right)-\left(x^2+2\right)\frac{d}{dx}\left(x^2+x+1\right)}{\left(x^2+x+1\right)^2}$$

$$= \frac{\left(x^2+x+1\right)(2x)-\left(x^2+2\right)(2x+1)}{\left(x^2+x+1\right)^2} = \frac{2x^3+2x^2+2x-2x^3-x^2-4x-2}{\left(x^2+x+1\right)^2} = \frac{x^2-2x-2}{\left(x^2+x+1\right)^2}.$$

26. $f(x) = \dfrac{x+1}{2x^2+2x+3}$, so

$$f'(x) = \frac{\left(2x^2+2x+3\right)(1)-(x+1)(4x+2)}{\left(2x^2+2x+3\right)^2} = \frac{2x^2+2x+3-4x^2-2x-4x-2}{\left(2x^2+2x+3\right)^2} = \frac{-2x^2-4x+1}{\left(2x^2+2x+3\right)^2}.$$

27. $f(x) = \dfrac{(x+1)\left(x^2+1\right)}{x-2} = \dfrac{\left(x^3+x^2+x+1\right)}{x-2}$, so

$$\begin{aligned} f'(x) &= \frac{(x-2)\,\frac{d}{dx}\left(x^3+x^2+x+1\right) - \left(x^3+x^2+x+1\right)\frac{d}{dx}(x-2)}{(x-2)^2} \\ &= \frac{(x-2)\left(3x^2+2x+1\right) - \left(x^3+x^2+x+1\right)}{(x-2)^2} \\ &= \frac{3x^3+2x^2+x-6x^2-4x-2-x^3-x^2-x-1}{(x-2)^2} = \frac{2x^3-5x^2-4x-3}{(x-2)^2}. \end{aligned}$$

28. $f(x) = \left(3x^2-1\right)\left(x^2-\dfrac{1}{x}\right)$, so

$$f'(x) = 6x\left(x^2-\frac{1}{x}\right) + \left(3x^2-1\right)\left(2x+\frac{1}{x^2}\right) = 6x^3-6+6x^3+3-2x-\frac{1}{x^2} = 12x^3-2x-3-\frac{1}{x^2}.$$

29. $f(x) = \dfrac{x}{x^2-4} - \dfrac{x-1}{x^2+4} = \dfrac{x\left(x^2+4\right)-(x-1)\left(x^2-4\right)}{\left(x^2-4\right)\left(x^2+4\right)} = \dfrac{x^2+8x-4}{\left(x^2-4\right)\left(x^2+4\right)}$, so

$$\begin{aligned} f'(x) &= \frac{\left(x^2-4\right)\left(x^2+4\right)\frac{d}{dx}\left(x^2+8x-4\right) - \left(x^2+8x-4\right)\frac{d}{dx}\left(x^4-16\right)}{\left(x^2-4\right)^2\left(x^2+4\right)^2} \\ &= \frac{\left(x^2-4\right)\left(x^2+4\right)(2x+8) - \left(x^2+8x-4\right)\left(4x^3\right)}{\left(x^2-4\right)^2\left(x^2+4\right)^2} \\ &= \frac{2x^5+8x^4-32x-128-4x^5-32x^4+16x^3}{\left(x^2-4\right)^2\left(x^2+4\right)^2} = \frac{-2x^5-24x^4+16x^3-32x-128}{\left(x^2-4\right)^2\left(x^2+4\right)^2}. \end{aligned}$$

30. $f(x) = \dfrac{x+\sqrt{3x}}{3x-1}$, so

$$\begin{aligned} f'(x) &= \frac{(3x-1)\left(1+\frac{1}{2}\sqrt{3}x^{-1/2}\right) - \left(x+\sqrt{3}x^{1/2}\right)(3)}{(3x-1)^2} \\ &= \frac{3x+\frac{3}{2}\sqrt{3}x^{1/2}-1-\frac{1}{2}\sqrt{3}x^{-1/2}-3x-3\sqrt{3}x^{1/2}}{(3x-1)^2} = -\frac{3\sqrt{3}x+2\sqrt{x}+\sqrt{3}}{2\sqrt{x}\,(3x-1)^2}. \end{aligned}$$

31. $h(x) = f(x)\,g(x)$, so $h'(x) = f(x)\,g'(x) + f'(x)\,g(x)$ by the Product Rule. Therefore, $h'(1) = f(1)\,g'(1) + f'(1)\,g(1) = (2)(3) + (-1)(-2) = 8$.

32. $h(x) = \left(x^2+1\right)g(x)$, so $h'(x) = \left(x^2+1\right)g'(x) + \frac{d}{dx}\left(x^2+1\right)\cdot g(x) = \left(x^2+1\right)g'(x) + 2xg(x)$. Therefore, $h'(1) = 2g'(1) + 2g(1) = (2)(3) + 2(-2) = 2$.

33. $h(x) = \dfrac{xf(x)}{x+g(x)}$. Using the Quotient Rule followed by the Product Rule, we obtain

$$h'(x) = \frac{[x+g(x)]\,\frac{d}{dx}[xf(x)] - xf(x)\,\frac{d}{dx}[x+g(x)]}{[x+g(x)]^2} = \frac{[x+g(x)]\,[xf'(x)+f(x)] - xf(x)\,[1+g'(x)]}{[x+g(x)]^2}.$$

Therefore,

$$h'(1) = \frac{[1+g(1)]\,[f'(1)+f(1)] - f(1)\,[1+g'(1)]}{[1+g(1)]^2} = \frac{(1-2)(-1+2)-2(1+3)}{(1-2)^2} = \frac{-1-8}{1} = -9.$$

34. $h(x) = \dfrac{f(x)\,g(x)}{f(x) - g(x)}$. Using the Quotient Rule followed by the Product Rule and the Sum Rule, we obtain

$$\begin{aligned} h'(x) &= \frac{\left[f(x) - g(x)\right] \frac{d}{dx}\left[f(x)\,g(x)\right] - f(x)\,g(x)\,\frac{d}{dx}\left[f(x) - g(x)\right]}{\left[f(x) - g(x)\right]^2} \\ &= \frac{\left[f(x) - g(x)\right]\left[f(x)\,g'(x) + f'(x)\,g(x)\right] - f(x)\,g(x)\left[f'(x) - g'(x)\right]}{\left[f(x) - g(x)\right]^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} h'(1) &= \frac{\left[f(1) - g(1)\right]\left[f(1)\,g'(1) + f'(1)\,g(1)\right] - f(1)\,g(1)\left[f'(1) - g'(1)\right]}{\left[f(1) - g(1)\right]^2} \\ &= \frac{[2 - (-2)]\,[(2)\,(3) + (-1)\,(-2)] - (2)\,(-2)\,[(-1) - 3]}{[2 - (-2)]^2} = \frac{(4)\,(8) - (-4)\,(-4)}{4^2} = \frac{16}{16} = 1. \end{aligned}$$

35. $f(x) = (2x - 1)\left(x^2 + 3\right)$, so

$$\begin{aligned} f'(x) &= (2x - 1)\,\tfrac{d}{dx}\left(x^2 + 3\right) + \left(x^2 + 3\right)\tfrac{d}{dx}\,(2x - 1) = (2x - 1)\,(2x) + \left(x^2 + 3\right)(2) \\ &= 6x^2 - 2x + 6 = 2\left(3x^2 - x + 3\right). \end{aligned}$$

At $x = 1$, $f'(1) = 2\left[3\,(1)^2 - (1) + 3\right] = 2\,(5) = 10$.

36. $f(x) = \dfrac{2x + 1}{2x - 1}$, so

$$\begin{aligned} f'(x) &= \frac{(2x - 1)\,\frac{d}{dx}\,(2x + 1) - (2x + 1)\,\frac{d}{dx}\,(2x - 1)}{(2x - 1)^2} = \frac{(2x - 1)\,(2) - (2x + 1)\,(2)}{(2x - 1)^2} \\ &= \frac{4x - 2 - 4x - 2}{(2x - 1)^2} = -\frac{4}{(2x - 1)^2}. \end{aligned}$$

At $x = 2$, $f'(2) = \dfrac{-4}{[2\,(2) - 1]^2} = -\dfrac{4}{9}$.

37. $f(x) = \dfrac{x}{x^4 - 2x^2 - 1}$, so

$$\begin{aligned} f'(x) &= \frac{\left(x^4 - 2x^2 - 1\right)\frac{d}{dx}\,(x) - x\,\frac{d}{dx}\left(x^4 - 2x^2 - 1\right)}{\left(x^4 - 2x^2 - 1\right)^2} = \frac{\left(x^4 - 2x^2 - 1\right)(1) - x\left(4x^3 - 4x\right)}{\left(x^4 - 2x^2 - 1\right)^2} \\ &= \frac{-3x^4 + 2x^2 - 1}{\left(x^4 - 2x^2 - 1\right)^2}. \end{aligned}$$

Therefore, $f'(-1) = \dfrac{-3 + 2 - 1}{(1 - 2 - 1)^2} = -\dfrac{2}{4} = -\dfrac{1}{2}$.

38. $f(x) = \left(x^{1/2} + 2x\right)\left(x^{3/2} - x\right) = x^2 - x^{3/2} + 2x^{5/2} - 2x^2 = 2x^{5/2} - x^2 - x^{3/2}$, so $f'(x) = 5x^{3/2} - 2x - \frac{3}{2}x^{1/2}$. At $x = 4$, $f'(4) = 5\,(4)^{3/2} - 2\,(4) - \frac{3}{2}\,(4)^{1/2} = 5\,(8) - 2\,(4) - \frac{3}{2}\,(2) = 29$.

39. $f(x) = \left(x^3 + 1\right)\left(x^2 - 2\right)$, so

$f'(x) = \left(x^3 + 1\right)\frac{d}{dx}\left(x^2 - 2\right) + \left(x^2 - 2\right)\frac{d}{dx}\left(x^3 + 1\right) = \left(x^3 + 1\right)(2x) + \left(x^2 - 2\right)\left(3x^2\right)$. The slope of the tangent line at $(2, 18)$ is $f'(2) = (8 + 1)\,(4) + (4 - 2)\,(12) = 60$. An equation of the tangent line is $y - 18 = 60\,(x - 2)$, or $y = 60x - 102$.

40. $f(x) = \dfrac{x^2}{x+1}$, so $f'(x) = \dfrac{(x+1)(2x) - x^2(1)}{(x+1)^2} = \dfrac{x^2+2x}{(x+1)^2}$. The slope of the tangent line at $x = 2$ is $f'(2) = \frac{8}{9}$. An equation of the line is $y - \frac{4}{3} = \frac{8}{9}(x-2)$, or $y = \frac{8}{9}x - \frac{4}{9}$.

41. $f(x) = \dfrac{x+1}{x^2+1}$, so

$$f'(x) = \frac{\left(x^2+1\right)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2} = \frac{\left(x^2+1\right)(1) - (x+1)(2x)}{\left(x^2+1\right)^2} = \frac{-x^2-2x+1}{\left(x^2+1\right)^2}.$$

At $x = 1$, $f'(1) = \dfrac{-1-2+1}{4} = -\dfrac{1}{2}$. Therefore, the slope of the tangent line at $x = 1$ is $-\frac{1}{2}$ and an equation is $y - 1 = -\frac{1}{2}(x-1)$ or $y = -\frac{1}{2}x + \frac{3}{2}$.

42. $f(x) = \dfrac{1+2x^{1/2}}{1+x^{3/2}}$, so $f'(x) = \dfrac{\left(1+x^{3/2}\right)\left(x^{-1/2}\right) - \left(1+2x^{1/2}\right)\left(\frac{3}{2}x^{1/2}\right)}{\left(1+x^{3/2}\right)^2}$. The slope of the tangent line at $x = 4$ is $f'(4) = \dfrac{(1+8)\left(\frac{1}{2}\right) - (1+4)(3)}{9^2} = -\dfrac{7}{54}$ and an equation is $y - \frac{5}{9} = -\frac{7}{54}(x-4)$, or $y = -\frac{7}{54}x + \frac{29}{27}$.

43. Using the Product Rule, we find
$g'(x) = \frac{d}{dx}\left[x^2 f(x)\right] = x^2\frac{d}{dx}\left[f(x)\right] + f(x)\frac{d}{dx}\left(x^2\right) = x^2 f'(x) + 2xf(x)$. Therefore,
$g'(2) = 2^2 f'(2) + 2(2)f(2) = (4)(-1) + 4(3) = 8$.

44. Using the Product Rule, we find
$g'(x) = \frac{d}{dx}\left[\left(x^2+1\right)f(x)\right] = \left(x^2+1\right)\frac{d}{dx}\left[f(x)\right] + f(x)\frac{d}{dx}\left(x^2+1\right) = \left(x^2+1\right)f'(x) + 2xf(x)$. Therefore,
$g'(2) = \left(2^2+1\right)f'(2) + 2(2)f(2) = (5)(-1) + 4(3) = 7$.

45. $f(x) = \left(x^3+1\right)\left(3x^2-4x+2\right)$, so

$$\begin{aligned} f'(x) &= \left(x^3+1\right)\frac{d}{dx}\left(3x^2-4x+2\right) + \left(3x^2-4x+2\right)\frac{d}{dx}\left(x^3+1\right) \\ &= \left(x^3+1\right)(6x-4) + \left(3x^2-4x+2\right)\left(3x^2\right) \\ &= 6x^4 + 6x - 4x^3 - 4 + 9x^4 - 12x^3 + 6x^2 = 15x^4 - 16x^3 + 6x^2 + 6x - 4. \end{aligned}$$

At $x = 1$, $f'(1) = 15(1)^4 - 16(1)^3 + 6(1) + 6(1) - 4 = 7$. Thus, the slope of the tangent line at the point $x = 1$ is 7 and an equation is $y - 2 = 7(x-1)$, or $y = 7x - 5$.

46. $f(x) = \dfrac{3x}{x^2-2}$. The slope of the tangent line at any point $(x, f(x))$ lying on the graph of f is

$$f'(x) = \frac{\left(x^2-2\right)\frac{d}{dx}(3x) - (3x)\frac{d}{dx}\left(x^2-2\right)}{\left(x^2-2\right)^2} = \frac{\left(x^2-2\right)(3) - 3x(2x)}{\left(x^2-2\right)^2} = \frac{-3x^2-6}{\left(x^2-2\right)^2} = \frac{-3\left(x^2+2\right)}{\left(x^2-2\right)^2}.$$

In particular, the slope of the tangent line at $(2, 3)$ is $f'(2) = \dfrac{-3(4+2)}{4} = -\dfrac{9}{2}$. Therefore, an equation of the tangent line is $y - 3 = -\frac{9}{2}(x-2)$ or $y = -\frac{9}{2}x + 12$.

47. $f(x) = \left(x^2+1\right)(2-x)$, so
$f'(x) = \left(x^2+1\right)\frac{d}{dx}(2-x) + (2-x)\frac{d}{dx}\left(x^2+1\right) = \left(x^2+1\right)(-1) + (2-x)(2x) = -3x^2+4x-1$. At a point where the tangent line is horizontal, we have $f'(x) = -3x^2+4x-1 = 0$ or $3x^2-4x+1 = (3x-1)(x-1) = 0$, giving $x = \frac{1}{3}$ or $x = 1$. Because $f\left(\frac{1}{3}\right) = \left(\frac{1}{9}+1\right)\left(2-\frac{1}{3}\right) = \frac{50}{27}$ and $f(1) = 2(2-1) = 2$, we see that the required points are $\left(\frac{1}{3}, \frac{50}{27}\right)$ and $(1, 2)$.

48. $f(x) = \dfrac{x}{x^2+1}$, so $f'(x) = \dfrac{\left(x^2+1\right)(1) - x(2x)}{\left(x^2+1\right)^2} = \dfrac{1-x^2}{\left(x^2+1\right)^2}$. At a point where the tangent line is horizontal, we have $f'(x) = 0$ or $1-x^2 = 0$, giving $x = \pm 1$. Therefore, the required points are $\left(-1, -\frac{1}{2}\right)$ and $\left(1, \frac{1}{2}\right)$.

49. $f(x) = \left(x^2+6\right)(x-5)$, so

$$f'(x) = \left(x^2+6\right)\frac{d}{dx}(x-5) + (x-5)\frac{d}{dx}\left(x^2+6\right) = \left(x^2+6\right)(1) + (x-5)(2x)$$
$$= x^2+6+2x^2-10x = 3x^2-10x+6.$$

At a point where the slope of the tangent line is -2, we have $f'(x) = 3x^2-10x+6 = -2$. This gives $3x^2-10x+8 = (3x-4)(x-2) = 0$, so $x = \frac{4}{3}$ or $x = 2$. Because $f\left(\frac{4}{3}\right) = \left(\frac{16}{9}+6\right)\left(\frac{4}{3}-5\right) = -\frac{770}{27}$ and $f(2) = (4+6)(2-5) = -30$, the required points are $\left(\frac{4}{3}, -\frac{770}{27}\right)$ and $(2, -30)$.

50. $f(x) = \dfrac{x+1}{x-1}$. The slope of the tangent line at any point $(x, f(x))$ on the graph of f is $f'(x) = \dfrac{(x-1)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2} = \dfrac{(x-1)(1)-(x+1)(1)}{(x-1)^2} = -\dfrac{2}{(x-1)^2}$. At the point(s) where the slope is equal to $-\frac{1}{2}$, we have $-\dfrac{2}{(x-1)^2} = -\dfrac{1}{2}$, so $(x-1)^2 = 4$ and $x = 1 \pm 2 = -1$ or 3. Therefore, the required points are $(-1, 0)$ and $(3, 2)$.

51. $y = \dfrac{1}{1+x^2}$, so $y' = \dfrac{\left(1+x^2\right)\frac{d}{dx}(1) - (1)\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2} = \dfrac{-2x}{\left(1+x^2\right)^2}$. Thus, the slope of the tangent line at $\left(1, \frac{1}{2}\right)$ is $y'\big|_{x=1} = \left.\dfrac{-2x}{\left(1+x^2\right)^2}\right|_{x=1} = \dfrac{-2}{4} = -\dfrac{1}{2}$ and an equation of the tangent line is $y - \frac{1}{2} = -\frac{1}{2}(x-1)$, or $y = -\frac{1}{2}x + 1$. Next, the slope of the required normal line is 2 and its equation is $y - \frac{1}{2} = 2(x-1)$, or $y = 2x - \frac{3}{2}$.

52. $C(t) = \dfrac{0.2t}{t^2+1}$.

a. $C'(t) = \dfrac{\left(t^2+1\right)\dfrac{d}{dt}(0.2t) - (0.2t)\dfrac{d}{dt}\left(t^2+1\right)}{\left(t^2+1\right)^2} = \dfrac{\left(t^2+1\right)(0.2) - (0.2t)(2t)}{\left(t^2+1\right)^2} = \dfrac{-0.2t^2+0.2}{\left(t^2+1\right)^2}$
$= \dfrac{0.2\left(1-t^2\right)}{\left(t^2+1\right)^2}$.

b. The rate of change of the concentration of the drug one-half hour after injection is $C'\left(\frac{1}{2}\right)=\frac{0.2\left(1-\frac{1}{4}\right)}{\left(\frac{1}{4}+1\right)^2}=\frac{0.2\,(0.75)}{(1.25)^2}=0.096$, or 0.096%/hr. The rate of change of the concentration of the drug one hour after injection is $C'(1)=\frac{0.2\,(1-1)}{(1+1)^2}=0$, or 0%/hr. The rate of change of the concentration of the drug 2 hours after injection is $C'(2)=\frac{0.2\left(1-2^2\right)}{\left(2^2+1\right)^2}=\frac{0.2\,(-3)}{25}=-0.024$, or -0.024%/hr.

53. $C(x)=\frac{0.5x}{100-x}$, so $C'(x)=\frac{(100-x)(0.5)-0.5x(-1)}{(100-x)^2}=\frac{50}{(100-x)^2}$. $C'(80)=\frac{50}{20^2}=0.125$, $C'(90)=\frac{50}{10^2}=0.5$, $C'(95)=\frac{50}{5^2}=2$, and $C'(99)=\frac{50}{1}=50$. The rates of change of the cost of removing 80%, 90%, 95%, and 99% of the toxic waste are 0.125, 0.5, 2, and 50 million dollars per 1% increase in waste removed. It is too costly to remove all of the pollutant.

54. $D(t)=\frac{500t}{t+12}$, so $D'(t)=\frac{(t+12)\,500-500t}{(t+12)^2}=\frac{6000}{(t+12)^2}$. The rates of change for a six-year-old child and a ten-year-old child are $D'(6)=\frac{6000}{(18)^2}\approx 18.5$ mg/yr and $D'(10)=\frac{6000}{(22)^2}\approx 12.4$ mg/yr.

55. $N(t)=\frac{10{,}000}{1+t^2}+2000$, so $N'(t)=\frac{d}{dt}\left[10{,}000\left(1+t^2\right)^{-1}+2000\right]=-\frac{10{,}000}{\left(1+t^2\right)^2}(2t)=-\frac{20{,}000t}{\left(1+t^2\right)^2}$.The rates of change after 1 minute and 2 minutes are $N'(1)=-\frac{20{,}000}{\left(1+1^2\right)^2}=-5000$ and $N'(2)=-\frac{20{,}000\,(2)}{\left(1+2^2\right)^2}=-1600$. The population of bacteria after one minute is $N(1)=\frac{10{,}000}{1+1}+2000=7000$, and the population after two minutes is $N(2)=\frac{10{,}000}{1+4}+2000=4000$.

56. a. $d(x)=\frac{50}{0.01x^2+1}$, so $d'(x)=\frac{\left(0.01x^2+1\right)(0)-50\,(0.02x)}{\left(0.01x^2+1\right)^2}=-\frac{x}{\left(0.01x^2+1\right)^2}$.

b. $d'(5)=-\frac{5}{(0.25+1)^2}=-3.2$, $d'(10)=-\frac{10}{(2)^2}=-2.5$, and $d'(15)=-\frac{15}{(3.25)^2}\approx-1.4$, so the rates of change of the price when the demand is 5,000, 10,000, and 15,000 units are decreasing at the rates of \$3200, \$2500, and \$1400 per 1000 watches, respectively.

57. a. $R(x)=xd(x)=\frac{50x}{0.01x^2+1}$.

b. $R'(x)=\frac{d}{dx}\left(\frac{50x}{0.01x^2+1}\right)=50\frac{d}{dx}\left(\frac{x}{0.01x^2+1}\right)=50\cdot\frac{\left(0.01x^2+1\right)(1)-x\,(0.02x)}{\left(0.01x^2+1\right)^2}=\frac{50\left(1-0.01x^2\right)}{\left(0.01x^2+1\right)^2}$.

c. $R'(8)\approx 6.69$, $R'(10)=0$, and $R'(12)\approx-3.70$, so the revenue is increasing at the rate of approximately \$6700 per thousand watches at a sales level of 8000 watches per week, the revenue is stable at a sales level of 10,000 watches per week, and the revenue is decreasing by approximately \$3700 per thousand watches at a sales level of 12,000 watches per week.

58. a. $P(x) = \dfrac{50x}{0.01x^2+1} - 0.025x^3 + 0.35x^2 - 10x - 30$. $P(0) = -30$, indicating that the company loses \$30,000 per week if no watches are sold.

b. Using the result of Exercise 57, we find $P'(x) = \dfrac{50\left(1-0.01x^2\right)}{\left(0.01x^2+1\right)^2} - 0.075x^2 + 0.7x - 10$, so $P'(5) = 15.625$ and $P'(10) = -10.5$. Thus, the profit increases by approximately \$15,625 per thousand watches at a sales level of 5000 watches per week, and the profit decreases by approximately 10,500 per thousand watches at a sales level of 10,000 watches per week.

59. a. The average 30-year fixed mortgage rate in the first week of May in 2010 was $M(1) = \dfrac{55.9}{1-0.31+11.2} \approx 4.701$, or approximately 4.7% per year.

b. $M'(t) = \dfrac{\left(t^2-0.31t+11.2\right)(0) - 55.9(2t-0.31)}{\left(t^2-0.31t+11.2\right)^2} = \dfrac{-55.9(2t-0.31)}{\left(t^2-0.31t+11.2\right)^2}$. Thus, the 30-year fixed mortgage rate was changing at the rate of $M'(1) = \dfrac{-55.9(2-0.31)}{(1-0.31+11.2)^2} \approx -0.668$ in the first week of May in 2010. That is, it was decreasing at approximately 0.67% per year.

60. $T(x) = \dfrac{120x^2}{x^2+4}$, so

$$T'(x) = \frac{\left(x^2+4\right)\frac{d}{dx}\left(120x^2\right) - \left(120x^2\right)\frac{d}{dx}\left(x^2+4\right)}{\left(x^2+4\right)^2} = \frac{\left(x^2+4\right)(240x) - \left(120x^2\right)(2x)}{\left(x^2+4\right)^2} = \frac{960x}{\left(x^2+4\right)^2}.$$

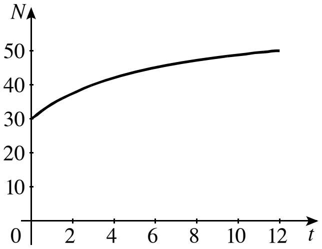
$T'(1) = \dfrac{960}{(1+4)^2} = \dfrac{960}{25} = 38.4$, or \$38.4 million per year; $T'(3) = \dfrac{960(3)}{(9+4)^2} \approx 17.04$, or approximately \$17.04 million per year; and $T'(5) = \dfrac{960(5)}{(25+4)^2} \approx 5.71$ or approximately \$5.71 million per year.

61. a. $N(t) = \dfrac{60t+180}{t+6}$, so

$$N'(t) = \frac{(t+6)\dfrac{d}{dt}(60t+180) - (60t+180)\dfrac{d}{dt}(t+6)}{(t+6)^2} = \frac{(t+6)(60) - (60t+180)(1)}{(t+6)^2} = \frac{180}{(t+6)^2}.$$

b. $N'(1) = \dfrac{180}{(1+6)^2} \approx 3.7$, $N'(3) = \dfrac{180}{(3+6)^2} \approx 2.2$, $N'(4) = \dfrac{180}{(4+6)^2} = 1.8$, and $N'(7) = \dfrac{180}{(7+6)^2} \approx 1.1$. We conclude that the rates at which the average student is increasing his or her speed one week, three weeks, four weeks, and seven weeks into the course are approximately 3.7, 2.2, 1.8, and 1.1 words per minute, respectively.

c. Yes.



d. $N(12) = \dfrac{60(12)+180}{12+6} = 50$, or 50 words/minute.

62. $f(t) = \dfrac{5t+300}{t^2+25}$, so

$$f'(t) = \frac{(t^2+25)(5)-(5t+300)(2t)}{(t^2+25)^2} = \frac{5t^2+125-10t^2-600t}{(t^2+25)^2} = \frac{-5(t^2+120t-25)}{(t^2+25)^2} \text{ and}$$

$f'(3) = \dfrac{-5[3^2+120(3)-25]}{(3^2+25)^2} \approx -1.488$, or approximately -1.49% per year.

63. $f(t) = \dfrac{0.055t+0.26}{t+2}$, so $f'(t) = \dfrac{(t+2)(0.055)-(0.055t+0.26)(1)}{(t+2)^2} = -\dfrac{0.15}{(t+2)^2}$. At the beginning, the formaldehyde level is changing at the rate of $f'(0) = -\dfrac{0.15}{4} = -0.0375$; that is, it is decreasing at the rate of 0.0375 parts per million per year. Next, $f'(3) = -\dfrac{0.15}{5^2} = -0.006$, and so the level is decreasing at the rate of 0.006 parts per million per year at the beginning of the fourth year (when $t = 3$).

64. a. $P(t) = \dfrac{25t^2+125t+200}{t^2+5t+40}$. The rate at which Glen Cove's population is changing with respect to time is

$$\begin{aligned} P'(t) &= \frac{(t^2+5t+40)\frac{d}{dt}(25t^2+125t+200)-(25t^2+125t+200)\frac{d}{dt}(t^2+5t+40)}{(t^2+5t+40)^2} \\ &= \frac{(t^2+5t+40)(50t+125)-(25t^2+125t+200)(2t+5)}{(t^2+5t+40)^2} \\ &= \frac{25(2t+5)(t^2+5t+40-t^2-5t-8)}{(t^2+5t+40)^2} = \frac{(25)(32)(2t+5)}{(t^2+5t+40)^2} = \frac{800(2t+5)}{(t^2+5t+40)^2}. \end{aligned}$$

b. After ten years the population will be $P(10) = \dfrac{25(10)^2+125(10)+200}{(10)^2+5(10)+40} = \dfrac{3950}{190} = 20.789$, or 20,789. After ten years the population will be increasing at the rate of $P'(10) = \dfrac{800[2(10)+5]}{[(10)^2+5(10)+40]^2} = \dfrac{20{,}000}{190^2} \approx 0.554$, or 554 people per year.

65. a. $R'(x) = \frac{d}{dx}[xD(x)] = xD'(x)+(1)D(x) = xD'(x)+D(x)$.

b. Here $p = D(x) = a-bx$, so $D'(x) = -b$. Therefore, $R'(x) = x(-b)+(a-bx) = a-2bx$.

c. $R(x) = xD(x) = x(a-bx) = ax-bx^2$, so $R'(x) = a-2bx$, as obtained in part (a).

66. The per capita income of the country at time t is $P(t) = \dfrac{g(t)}{f(t)} = \dfrac{60+4t}{3+0.06t}$. Thus, $P'(t) = \dfrac{(3+0.06t)(4)-(60+4t)(0.06)}{(3+0.06t)^2} = \dfrac{8.4}{(3+0.06t)^2}$. The per capita income in two years' time is projected to be changing at $P'(2) = \dfrac{8.4}{(3+0.06\cdot 2)^2} \approx 0.8629$, or approximately \$0.86 billion/year.

67. a. If there is a substrate present, then the relative growth rate is $R(0) = 0$.

b. $\lim\limits_{s\to\infty} R(s) = \lim\limits_{s\to\infty}\dfrac{cs}{k+s} = \lim\limits_{s\to\infty}\dfrac{c}{\frac{k}{s}+1} = c$. Thus, the relative growth rate approaches c when the substrate is present in great excess.

c. $R'(s) = \dfrac{(k+s)(c)-cs(1)}{(k+s)^2} = \dfrac{kc}{(k+s)^2}$.

68. **a.** $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$ with $q = 2.5$ and $p = 50$. Thus, $\frac{1}{f} = \frac{1}{2.5} + \frac{1}{50} = 0.42$, and so $f = \frac{1}{0.42} \approx 2.38$, or approximately 2.38 cm.

b. $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$, so $f = \frac{1}{\frac{1}{p} + \frac{1}{q}} = \frac{pq}{p+q}$. Differentiating with respect to p, we have $\frac{df}{dp} = \frac{(p+q)\,q - pq\,(1)}{(p+q)^2} = \left(\frac{q}{p+q}\right)^2$. When $p = 50$, we have $\frac{df}{dp} = \left(\frac{2.5}{50+2.5}\right)^2 \approx 0.00227$, or 0.00227 cm/cm.

69. False. Take $f(x) = x$ and $g(x) = x$. Then $f(x)\,g(x) = x^2$, so $\frac{d}{dx}\left[f(x)\,g(x)\right] = \frac{d}{dx}\left(x^2\right) = 2x \neq f'(x)\,g'(x) = 1$.

70. True. Using the Product Rule, $\frac{d}{dx}\left[xf(x)\right] = f(x)\,\frac{d}{dx}(x) + x\frac{d}{dx}\left[f(x)\right] = f(x)\,(1) + xf'(x)$.

71. False. Let $f(x) = x^3$. Then $\frac{d}{dx}\left[\frac{f(x)}{x^2}\right] = \frac{d}{dx}\left(\frac{x^3}{x^2}\right) = \frac{d}{dx}(x) = 1 \neq \frac{f'(x)}{2x} = \frac{3x^2}{2x} = \frac{3}{2}x$.

72. True. Using the Quotient Rule followed by the Product Rule,

$$\frac{d}{dx}\left[\frac{f(x)\,g(x)}{h(x)}\right] = \frac{h(x)\,\frac{d}{dx}\left[f(x)\,g(x)\right] - f(x)\,g(x)\,\frac{d}{dx}\left[h(x)\right]}{\left[h(x)\right]^2}$$
$$= \frac{h(x)\left[f'(x)\,g(x) + f(x)\,g'(x)\right] - f(x)\,g(x)\,h'(x)}{\left[h(x)\right]^2}.$$

73. Let $f(x) = u(x)\,v(x)$ and $g(x) = w(x)$. Then $h(x) = f(x)\,g(x)$. Therefore, $h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$. But $f'(x) = u(x)\,v'(x) + u'(x)\,v(x)$, so

$$h'(x) = \left[u(x)\,v'(x) + u'(x)\,v(x)\right]g(x) + u(x)\,v(x)\,w'(x)$$
$$= u(x)\,v(x)\,w'(x) + u(x)\,v'(x)\,w(x) + u'(x)\,v(x)\,w(x).$$

74. Let $k(x) = \frac{f(x)}{g(x)}$.

a. $\frac{k(x+h) - k(x)}{h} = \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \frac{f(x+h)\,g(x) - f(x)\,g(x+h)}{hg(x+h)\,g(x)}$.

b. By adding $-f(x)\,g(x) + f(x)\,g(x)$ (which is equal to zero) to the numerator and simplifying, we have

$$\frac{k(x+h) - k(x)}{h} = \frac{1}{g(x+h)\,g(x)}\left\{\left[\frac{f(x+h) - f(x)}{h}\right]g(x) - \left[\frac{g(x+h) - g(x)}{h}\right]f(x)\right\}.$$

c. Taking the limit and using the definition of the derivative, we find

$$k'(x) = \lim_{h\to 0}\frac{k(x+h) - k(x)}{h} = \frac{1}{\left[g(x)\right]^2}\left[f'(x)\,g(x) - g'(x)\,f(x)\right] = \frac{f'(x)\,g(x) - g'(x)\,f(x)}{\left[g(x)\right]^2}.$$

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1. 0.8750 **2.** 16.7980 **3.** 0.0774 **4.** −0.1314 **5.** −0.5000 **6.** 2.875

7. 31,312 per year **8.** **a.** 20,790 **b.** 554/year

3.3 The Chain Rule

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1. The derivative of $h(x) = g(f(x))$ is equal to the derivative of g evaluated at $f(x)$ times the derivative of f.

2. $h'(x) = \frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$.

3. $(g \circ f)'(t) = [(g \circ f)(t)]' = g'(f(t)) f'(t)$ describes the rate of change of the revenue as a function of time.

4. $(f \circ g)'(t)$ gives the rate of change of the air temperature.

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1. $f(x) = (2x-1)^3$, so $f'(x) = 3(2x-1)^2 \frac{d}{dx}(2x-1) = 3(2x-1)^2(2) = 6(2x-1)^2$.

2. $f(x) = (1-x)^4$, so $f'(x) = 4(1-x)^3(-1) = -4(1-x)^3$.

3. $f(x) = (x^2+2)^5$, so $f'(x) = 5(x^2+2)^4(2x) = 10x(x^2+2)^4$.

4. $f(t) = 2(t^3-1)^5$, so $f'(t) = (2)(5)(t^3-1)^4(3t^2) = 30t^2(t^3-1)^4$.

5. $f(x) = (2x-x^2)^3$, so $f'(x) = 3(2x-x^2)^2 \frac{d}{dx}(2x-x^2) = 3(2x-x^2)^2(2-2x) = 6x^2(1-x)(2-x)^2$.

6. $f(x) = 3(x^3-x)^4$, so $f'(x) = (3)(4)(x^3-x)^3(3x^2-1) = 12(3x^2-1)(x^3-x)^3$.

7. $f(x) = (2x+1)^{-2}$, so $f'(x) = -2(2x+1)^{-3} \frac{d}{dx}(2x+1) = -2(2x+1)^{-3}(2) = -4(2x+1)^{-3}$.

8. $f(t) = \frac{1}{2}(2t^2+t)^{-3}$, so $f'(t) = \frac{1}{2}(-3)(2t^2+t)^{-4}(4t+1) = -\frac{3(1+4t)}{2(2t^2+t)^4}$.

9. $f(x) = (x^2-4)^{5/2}$, so $f'(x) = \frac{5}{2}(x^2-4)^{3/2} \frac{d}{dx}(x^2-4) = \frac{5}{2}(x^2-4)^{3/2}(2x) = 5x(x^2-4)^{3/2}$.

10. $f(t) = (3t^2-2t+1)^{3/2}$, so $f'(t) = \frac{3}{2}(3t^2-2t+1)^{1/2}(6t-2) = 3(3t-1)(3t^2-2t+1)^{1/2}$.

11. $f(x) = \sqrt{3x-2} = (3x-2)^{1/2}$, so $f'(x) = \frac{1}{2}(3x-2)^{-1/2}(3) = \frac{3}{2}(3x-2)^{-1/2} = \frac{3}{2\sqrt{3x-2}}$.

12. $f(t) = \sqrt{3t^2-t} = (3t^2-t)^{1/2}$, so $f'(t) = \frac{1}{2}(3t^2-t)^{-1/2}(6t-1) = \frac{6t-1}{2\sqrt{3t^2-t}}$.

13. $f(x) = \sqrt[3]{1-x^2}$, so

$$f'(x) = \frac{d}{dx}(1-x^2)^{1/3} = \frac{1}{3}(1-x^2)^{-2/3} \frac{d}{dx}(1-x^2) = \frac{1}{3}(1-x^2)^{-2/3}(-2x) = -\frac{2}{3}x(1-x^2)^{-2/3}$$
$$= \frac{-2x}{3(1-x^2)^{2/3}}.$$

14. $f(x) = \sqrt{2x^2-2x+3}$, so $f'(x) = \frac{1}{2}(2x^2-2x+3)^{-1/2}(4x-2) = (2x-1)(2x^2-2x+3)^{-1/2}$.

15. $f(x)=\dfrac{1}{(2x+3)^3}=(2x+3)^{-3}$, so $f'(x)=-3(2x+3)^{-4}(2)=-6(2x+3)^{-4}=-\dfrac{6}{(2x+3)^4}$.

16. $f(x)=\dfrac{2}{\left(x^2-1\right)^4}$, so $f'(x)=2\frac{d}{dx}\left(x^2-1\right)^{-4}=2(-4)\left(x^2-1\right)^{-5}(2x)=-16x\left(x^2-1\right)^{-5}$.

17. $f(t)=\dfrac{1}{\sqrt{2t-4}}$, so $f'(t)=\frac{d}{dt}(2t-4)^{-1/2}=-\frac{1}{2}(2t-4)^{-3/2}(2)=-(2t-4)^{-3/2}=-\dfrac{1}{(2t-4)^{3/2}}$.

18. $f(x)=\dfrac{1}{\sqrt{2x^2-1}}=\left(2x^2-1\right)^{-1/2}$, so $f'(x)=-\frac{1}{2}\left(2x^2-1\right)^{-3/2}(4x)=-\dfrac{2x}{\sqrt{\left(2x^2-1\right)^3}}$.

19. $y=\dfrac{1}{\left(4x^4+x\right)^{3/2}}$, so $\frac{dy}{dx}=\frac{d}{dx}\left(4x^4+x\right)^{-3/2}=-\frac{3}{2}\left(4x^4+x\right)^{-5/2}\left(16x^3+1\right)=-\frac{3}{2}\left(16x^3+1\right)\left(4x^4+x\right)^{-5/2}$.

20. $f(t)=\dfrac{4}{\sqrt[3]{2t^2+t}}$, so $f'(t)=4\frac{d}{dt}\left(2t^2+t\right)^{-1/3}=-\frac{4}{3}\left(2t^2+t\right)^{-4/3}(4t+1)=-\frac{4}{3}(4t+1)\left(2t^2+t\right)^{-4/3}$.

21. $f(x)=\left(3x^2+2x+1\right)^{-2}$, so

$$\begin{aligned}f'(x)&=-2\left(3x^2+2x+1\right)^{-3}\frac{d}{dx}\left(3x^2+2x+1\right)=-2\left(3x^2+2x+1\right)^{-3}(6x+2)\\&=-4(3x+1)\left(3x^2+2x+1\right)^{-3}.\end{aligned}$$

22. $f(t)=\left(5t^3+2t^2-t+4\right)^{-3}$, so $f'(t)=-3\left(5t^3+2t^2-t+4\right)^{-4}\left(15t^2+4t-1\right)$.

23. $f(x)=\left(x^2+1\right)^3-\left(x^3+1\right)^2$, so

$$\begin{aligned}f'(x)&=3\left(x^2+1\right)^2\frac{d}{dx}\left(x^2+1\right)-2\left(x^3+1\right)\frac{d}{dx}\left(x^3+1\right)=3\left(x^2+1\right)^2(2x)-2\left(x^3+1\right)\left(3x^2\right)\\&=6x\left[\left(x^2+1\right)^2-x\left(x^3+1\right)\right]=6x\left(2x^2-x+1\right).\end{aligned}$$

24. $f(t)=(2t-1)^4+(2t+1)^4$, so $f'(t)=4(2t-1)^3(2)+4(2t+1)^3(2)=8\left[(2t-1)^3+(2t+1)^3\right]$.

25. $f(t)=\left(t^{-1}-t^{-2}\right)^3$, so $f'(t)=3\left(t^{-1}-t^{-2}\right)^2\frac{d}{dt}\left(t^{-1}-t^{-2}\right)=3\left(t^{-1}-t^{-2}\right)^2\left(-t^{-2}+2t^{-3}\right)$.

26. $f(v)=\left(v^{-3}+4v^{-2}\right)^3$, so $f'(v)=3\left(v^{-3}+4v^{-2}\right)^2\left(-3v^{-4}-8v^{-3}\right)$.

27. $f(x)=\sqrt{x+1}+\sqrt{x-1}=(x+1)^{1/2}+(x-1)^{1/2}$, so

$f'(x)=\frac{1}{2}(x+1)^{-1/2}(1)+\frac{1}{2}(x-1)^{-1/2}(1)=\frac{1}{2}\left[(x+1)^{-1/2}+(x-1)^{-1/2}\right]$.

28. $f(u)=(2u+1)^{3/2}+\left(u^2-1\right)^{-3/2}$, so

$f'(u)=\frac{3}{2}(2u+1)^{1/2}(2)-\frac{3}{2}\left(u^2-1\right)^{-5/2}(2u)=3(2u+1)^{1/2}-3u\left(u^2-1\right)^{-5/2}$.

29. $f(x)=2x^2(3-4x)^4$, so

$$\begin{aligned}f'(x)&=2x^2(4)(3-4x)^3(-4)+(3-4x)^4(4x)=4x(3-4x)^3(-8x+3-4x)\\&=4x(3-4x)^3(-12x+3)=(-12x)(4x-1)(3-4x)^3.\end{aligned}$$

30. $h(t) = t^2(3t+4)^3$, so

$$h'(t) = 2t(3t+4)^3 + t^2(3)(3t+4)^2(3) = t(3t+4)^2[2(3t+4)+9t] = t(15t+8)(3t+4)^2.$$

31. $f(x) = (x-1)^2(2x+1)^4$, so

$$\begin{aligned} f'(x) &= (x-1)^2 \tfrac{d}{dx}(2x+1)^4 + (2x+1)^4 \tfrac{d}{dx}(x-1)^2 \quad \text{(by the Product Rule)} \\ &= (x-1)^2(4)(2x+1)^3 \tfrac{d}{dx}(2x+1) + (2x+1)^4(2)(x-1)\tfrac{d}{dx}(x-1) \\ &= 8(x-1)^2(2x+1)^3 + 2(x-1)(2x+1)^4 = 2(x-1)(2x+1)^3(4x-4+2x+1) \\ &= 6(x-1)(2x-1)(2x+1)^3. \end{aligned}$$

32. $g(u) = (u+1)^{1/2}(1-2u)^8$, so

$$\begin{aligned} g'(u) &= (u+1)^{1/2}(8)\left(1-2u^2\right)^7(-4u) + \left(1-2u^2\right)^8\left(\tfrac{1}{2}\right)(u+1)^{-1/2} \\ &= -\tfrac{1}{2}(u+1)^{-1/2}\left(1-2u^2\right)^7\left[64u(u+1) - \left(1-2u^2\right)\right] \\ &= -\frac{\left(66u^2+64u-1\right)\left(1-2u^2\right)^7}{2\sqrt{u+1}} = \frac{\left(2u^2-1\right)^7\left(66u^2+64u-1\right)}{2\sqrt{u+1}}. \end{aligned}$$

33. $f(x) = \left(\dfrac{x+3}{x-2}\right)^3$, so

$$\begin{aligned} f'(x) &= 3\left(\frac{x+3}{x-2}\right)^2 \frac{d}{dx}\left(\frac{x-3}{x-2}\right) = 3\left(\frac{x+3}{x-2}\right)^2\left[\frac{(x-2)(1)-(x+3)(1)}{(x-2)^2}\right] \\ &= 3\left(\frac{x+3}{x-2}\right)^2\left[-\frac{5}{(x-2)^2}\right] = -\frac{15(x+3)^2}{(x-2)^4}. \end{aligned}$$

34. $f(x) = \left(\dfrac{x+1}{x-1}\right)^5$, so $f'(x) = 5\left(\dfrac{x+1}{x-1}\right)^4\left[\dfrac{(x-1)(1)-(x+1)(1)}{(x-1)^2}\right] = -\dfrac{10(x+1)^4}{(x-1)^6}$.

35. $s(t) = \left(\dfrac{t}{2t+1}\right)^{3/2}$, so

$$\begin{aligned} s'(t) &= \frac{3}{2}\left(\frac{t}{2t+1}\right)^{1/2}\frac{d}{dt}\left(\frac{t}{2t+1}\right) = \frac{3}{2}\left(\frac{t}{2t+1}\right)^{1/2}\left[\frac{(2t+1)(1)-t(2)}{(2t+1)^2}\right] \\ &= \frac{3}{2}\left(\frac{t}{2t+1}\right)^{1/2}\left[\frac{1}{(2t+1)^2}\right] = \frac{3t^{1/2}}{2(2t+1)^{5/2}}. \end{aligned}$$

36. $g(s) = \left(s^2+\dfrac{1}{s}\right)^{3/2} = \left(s^2+s^{-1}\right)^{3/2}$, so

$$g'(s) = \tfrac{3}{2}\left(s^2+s^{-1}\right)^{1/2}\left(2s-s^{-2}\right) = \frac{3}{2}\left(s^2+\frac{1}{s}\right)^{1/2}\left(2s-\frac{1}{s^2}\right) = \frac{3}{2}\left(\frac{s^3+1}{s}\right)^{1/2}\left(\frac{2s^3-1}{s^2}\right).$$

37. $g(u) = \left(\dfrac{u+1}{3u+2}\right)^{1/2}$, so

$$\begin{aligned} g'(u) &= \frac{1}{2}\left(\frac{u+1}{3u+2}\right)^{-1/2}\frac{d}{du}\left(\frac{u+1}{3u+2}\right) = \frac{1}{2}\left(\frac{u+1}{3u+2}\right)^{-1/2}\left[\frac{(3u+2)(1)-(u+1)(3)}{(3u+2)^2}\right] \\ &= -\frac{1}{2\sqrt{u+1})(3u+2)^{3/2}}. \end{aligned}$$

38. $g(x) = \left(\dfrac{2x+1}{2x-1}\right)^{1/2}$, so

$$g'(x) = \frac{1}{2}\left(\frac{2x+1}{2x-1}\right)^{-1/2}\left[\frac{(2x-1)(2)-(2x+1)(2)}{(2x-1)^2}\right] = \frac{1}{2}\left(\frac{2x+1}{2x-1}\right)^{-1/2}\left(-\frac{4}{(2x-1)^2}\right)$$
$$= -\frac{2}{(2x+1)^{1/2}(2x-1)^{3/2}}.$$

39. $f(x) = \dfrac{x^2}{\left(x^2-1\right)^4}$, so

$$f'(x) = \frac{\left(x^2-1\right)^4 \dfrac{d}{dx}\left(x^2\right) - \left(x^2\right)\dfrac{d}{dx}\left(x^2-1\right)^4}{\left[\left(x^2-1\right)^4\right]^2} = \frac{\left(x^2-1\right)^4(2x) - x^2(4)\left(x^2-1\right)^3(2x)}{\left(x^2-1\right)^8}$$
$$= \frac{\left(x^2-1\right)^3(2x)\left(x^2-1-4x^2\right)}{\left(x^2-1\right)^8} = \frac{(-2x)\left(3x^2+1\right)}{\left(x^2-1\right)^5}.$$

40. $g(u) = \dfrac{2u^2}{\left(u^2+u\right)^3}$, so

$$g'(u) = \frac{\left(u^2+u\right)^3(4u) - \left(2u^2\right)3\left(u^2+u\right)^2(2u+1)}{\left(u^2+u\right)^6}$$
$$= \frac{2u\left(u^2+u\right)^2\left[2\left(u^2+u\right)-3u(2u+1)\right]}{\left(u^2+u\right)^6} = \frac{-2u\left(4u^2+u\right)}{\left(u^2+u\right)^4} = \frac{-2u^2(4u+1)}{u^4(u+1)^4} = -\frac{2(4u+1)}{u^2(u+1)^4}.$$

41. $h(x) = \dfrac{\left(3x^2+1\right)^3}{\left(x^2-1\right)^4}$, so

$$h'(x) = \frac{\left(x^2-1\right)^4(3)\left(3x^2+1\right)^2(6x) - \left(3x^2+1\right)^3(4)\left(x^2-1\right)^3(2x)}{\left(x^2-1\right)^8}$$
$$= \frac{2x\left(x^2-1\right)^3\left(3x^2+1\right)^2\left[9\left(x^2-1\right)-4\left(3x^2+1\right)\right]}{\left(x^2-1\right)^8} = -\frac{2x\left(3x^2+13\right)\left(3x^2+1\right)^2}{\left(x^2-1\right)^5}.$$

42. $g(t) = \dfrac{(2t-1)^2}{(3t+2)^4}$, so

$$g'(t) = \frac{(3t+2)^4(2)(2t-1)(2) - (2t-1)^2(4)(3t+2)^3(3)}{(3t+2)^8}$$
$$= \frac{4(3t+2)^3(2t-1)\left[(3t+2)-3(2t-1)\right]}{(3t+2)^8} = \frac{4(2t-1)(5-3t)}{(3t+2)^5}.$$

43. $f(x) = \dfrac{\sqrt{2x+1}}{x^2-1}$, so

$$f'(x) = \frac{\left(x^2-1\right)\left(\frac{1}{2}\right)(2x+1)^{-1/2}(2) - (2x+1)^{1/2}(2x)}{\left(x^2-1\right)^2} = \frac{(2x+1)^{-1/2}\left[\left(x^2-1\right)-(2x+1)(2x)\right]}{\left(x^2-1\right)^2}$$
$$= -\frac{3x^2+2x+1}{\sqrt{2x+1}\left(x^2-1\right)^2}.$$

44. $f(t) = \dfrac{4t^2}{\sqrt{2t^2+2t-1}} = \dfrac{4t^2}{\left(2t^2+2t-1\right)^{1/2}}$, so

$$\begin{aligned}
f'(t) &= \frac{\left(2t^2+2t-1\right)^{1/2}\dfrac{d}{dt}\left(4t^2\right) - 4t^2\dfrac{d}{dt}\left(2t^2+2t-1\right)^{1/2}}{\left[\left(2t^2+2t-1\right)^{1/2}\right]^2} \\
&= \frac{\left(2t^2+2t-1\right)^{1/2}(8t) - 4t^2\left(\frac{1}{2}\right)\left(2t^2+2t-1\right)^{-1/2}(4t+2)}{2t^2+2t-1} \\
&= \frac{4t\left(2t^2+2t-1\right)^{-1/2}\left[2\left(2t^2+2t-1\right) - t(2t+1)\right]}{2t^2+2t-1} = \frac{4t\left(2t^2+3t-2\right)}{\left(\sqrt{2t^2+2t-1}\right)^3}.
\end{aligned}$$

45. $g(t) = \dfrac{(t+1)^{1/2}}{\left(t^2+1\right)^{1/2}}$, so

$$\begin{aligned}
g'(t) &= \frac{\left(t^2+1\right)^{1/2}\dfrac{d}{dt}(t+1)^{1/2} - (t+1)^{1/2}\dfrac{d}{dt}\left(t^2+1\right)^{1/2}}{t^2+1} \\
&= \frac{\left(t^2+1\right)^{1/2}\left(\frac{1}{2}\right)(t+1)^{-1/2}(1) - (t+1)^{1/2}\left(\frac{1}{2}\right)\left(t^2+1\right)^{-1/2}(2t)}{t^2+1} \\
&= \frac{\frac{1}{2}(t+1)^{-1/2}\left(t^2+1\right)^{-1/2}\left[\left(t^2+1\right) - 2t(t+1)\right]}{t^2+1} = -\frac{t^2+2t-1}{2\sqrt{t+1}\left(t^2+1\right)^{3/2}}.
\end{aligned}$$

46. $f(x) = \dfrac{\left(x^2+1\right)^{1/2}}{\left(x^2-1\right)^{1/2}}$, so

$$\begin{aligned}
f'(x) &= \frac{\left(x^2-1\right)^{1/2}\frac{d}{dx}\left(x^2+1\right)^{1/2} - \left(x^2+1\right)^{1/2}\frac{d}{dx}\left(x^2-1\right)^{1/2}}{\left(x^2-1\right)} \\
&= \frac{\left(x^2-1\right)^{1/2}\left(\frac{1}{2}\right)\left(x^2+1\right)^{-1/2}(2x) - \left(x^2+1\right)^{1/2}\left(\frac{1}{2}\right)\left(x^2-1\right)^{-1/2}(2x)}{x^2-1} \\
&= \frac{x\left(x^2-1\right)^{-1/2}\left(x^2+1\right)^{-1/2}\left[\left(x^2-1\right) - \left(x^2+1\right)\right]}{x^2-1} = -\frac{2x}{\sqrt{x^2+1}\left(x^2-1\right)^{3/2}}.
\end{aligned}$$

47. $f(x) = (3x+1)^4\left(x^2-x+1\right)^3$, so

$$\begin{aligned}
f'(x) &= (3x+1)^4\frac{d}{dx}\left(x^2-x+1\right)^3 + \left(x^2-x+1\right)^3\frac{d}{dx}(3x+1)^4 \\
&= (3x+1)^4\cdot 3\left(x^2-x+1\right)^2(2x-1) + \left(x^2-x+1\right)^3\cdot 4(3x+1)^3\cdot 3 \\
&= 3(3x+1)^3\left(x^2-x+1\right)^2\left[(3x+1)(2x-1) + 4\left(x^2-x+1\right)\right] \\
&= 3(3x+1)^3\left(x^2-x+1\right)^2\left(6x^2-3x+2x-1+4x^2-4x+4\right) \\
&= 3(3x+1)^3\left(x^2-x+1\right)^2\left(10x^2-5x+3\right).
\end{aligned}$$

48. $g(t) = (2t+3)^2 \left(3t^2-1\right)^{-3}$, so

$$\begin{aligned} g'(t) &= (2t+3)^2 \frac{d}{dt}\left(3t^2-1\right)^{-3} + \left(3t^2-1\right)^{-3} \frac{d}{dt}(2t+3)^2 \\ &= (2t+3)^2(-3)\left(3t^2-1\right)^{-4}(6t) + \left(3t^2-1\right)^{-3}(2)(2t+3)(2) \\ &= 2(2t+3)\left(3t^2-1\right)^{-4}\left[-9t(2t+3) + 2\left(3t^2-1\right)\right] \\ &= 2(2t+3)\left(3t^2-1\right)^{-4}\left(-18t^2-27t+6t^2-2\right) \\ &= -2\left(12t^2+27t+2\right)(2t+3)\left(3t^2-1\right)^{-4}. \end{aligned}$$

49. $y = g(u) = u^{4/3}$, so $\frac{dy}{du} = \frac{4}{3}u^{1/3}$, and $u = f(x) = 3x^2-1$, so $\dfrac{du}{dx} = 6x$. Thus,
$\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = \frac{4}{3}u^{1/3}(6x) = \frac{4}{3}\left(3x^2-1\right)^{1/3}6x = 8x\left(3x^2-1\right)^{1/3}$.

50. $y = \sqrt{u}$ and $u = 7x-2x^2$, so $\dfrac{dy}{du} = \frac{1}{2}u^{-1/2}$ and $\dfrac{du}{dx} = 7-4x$. Thus, $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = \dfrac{7-4x}{2\sqrt{u}} = \dfrac{7-4x}{2\sqrt{7x-2x^2}}$.

51. $y = u^{-2/3}$ and $u = 2x^3-x+1$, so $\dfrac{dy}{du} = -\frac{2}{3}u^{-5/3} = -\dfrac{2}{3u^{5/3}}$ and $\dfrac{du}{dx} = 6x^2-1$. Thus,
$\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = -\dfrac{2\left(6x^2-1\right)}{3u^{5/3}} = -\dfrac{2\left(6x^2-1\right)}{3\left(2x^3-x+1\right)^{5/3}}$.

52. $y = 2u^2+1$ and $u = x^2+1$, so $\dfrac{dy}{du} = 4u$ and $\dfrac{du}{dx} = 2x$. Thus, $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = 4u(2x) = 8xu = 8x\left(x^2+1\right)$.

53. $y = \sqrt{u} + \dfrac{1}{\sqrt{u}}$ and $u = x^3-x$, so $\dfrac{dy}{du} = \frac{1}{2}u^{-1/2} - \frac{1}{2}u^{-3/2}$ and $\dfrac{du}{dx} = 3x^2-1$. Thus,
$\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = \left[\dfrac{1}{2\sqrt{x^3-x}} - \dfrac{1}{2\left(x^3-x\right)^{3/2}}\right]\left(3x^2-1\right) = \dfrac{\left(3x^2-1\right)\left(x^3-x-1\right)}{2\left(x^3-x\right)^{3/2}}$.

54. $y = \dfrac{1}{u}$ and $u = \sqrt{x}+1$, so $\dfrac{dy}{du} = -\dfrac{1}{u^2}$ and $\dfrac{du}{dx} = \dfrac{1}{2}x^{-1/2}$. Thus,
$\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = -\dfrac{1}{u^2}\cdot\left(\dfrac{1}{2}x^{-1/2}\right) = -\dfrac{1}{2\sqrt{x}\left(\sqrt{x}+1\right)^2}$.

55. $g(x) = f(2x+1)$. Let $u = 2x+1$, so $\dfrac{du}{dx} = 2$. Using the Chain Rule, we have
$g'(x) = f'(u)\dfrac{du}{dx} = f'(2x+1)\cdot 2 = 2f'(2x+1)$.

56. $h(x) = f\left(-x^3\right)$. Let $u = -x^3$, so $\dfrac{du}{dx} = -3x^2$. Using the Chain Rule, we have
$h'(x) = f'(u)\dfrac{du}{dx} = f'\left(-x^3\right)\left(-3x^2\right) = -3x^2 f'\left(-x^3\right)$.

57. $F(x) = g(f(x))$, so $F'(x) = g'(f(x))f'(x)$. Thus, $F'(2) = g'(3)(-3) = (4)(-3) = -12$.

58. $h = f(g(x))$, so $h'(0) = f'(g(0))g'(0) = f'(5)\cdot 3 = -2\cdot 3 = -6$.

59. Let $g(x) = x^2 + 1$. Then $F(x) = f(g(x))$. Next, $F'(x) = f'(g(x))\,g'(x)$ and $F'(1) = f'(2)(2x) = (3)(2) = 6$.

60. No. Let $F(x) = f(f(x))$. Then $F'(x) = f'(f(x))\,f'(x)$. Then let $f(x) = x^2$. $f(f(x)) = f\left(x^2\right) = x^4$ and $F'(x) = 4x^3$, but $f'(x) = 2x$, so $\left[f'(x)\right]^2 = 4x^2 \neq 4x^3$.

61. No. Suppose $h = g(f(x))$. Let $f(x) = x$ and $g(x) = x^2$. Then $h = g(f(x)) = g(x) = x^2$ and $h'(x) = 2x \neq g'\left(f'(x)\right) = g'(1) = 2(1) = 2$.

62. $h = f(g(x))$, so $h' = f'(g(x))\,g'(x)$ and $f' \circ g = f'(g(x))$, but $\left(f' \circ g\right) g' = f'(g(x))\,g'(x)$.

63. $f(x) = (1-x)\left(x^2-1\right)^2$, so

$f'(x) = (1-x)\,2\left(x^2-1\right)(2x) + (-1)\left(x^2-1\right)^2 = \left(x^2-1\right)\left(4x - 4x^2 - x^2 + 1\right) = \left(x^2-1\right)\left(-5x^2 + 4x + 1\right)$.

Therefore, the slope of the tangent line at $(2, -9)$ is $f'(2) = \left[(2)^2 - 1\right]\left[-5(2)^2 + 4(2) + 1\right] = -33$. Then an equation of the line is $y + 9 = -33(x-2)$, or $y = -33x + 57$.

64. $f(x) = \left(\dfrac{x+1}{x-1}\right)^2$, so $f'(x) = 2\left(\dfrac{x+1}{x-1}\right)\left[\dfrac{(x-1)(1)-(x+1)(1)}{(x-1)^2}\right] = 2\left(\dfrac{x+1}{x-1}\right)\left[-\dfrac{2}{(x-1)^2}\right]$. The slope of the tangent line at $x = 3$ is $f'(3) = 2\left(\frac{4}{2}\right)\left(-\frac{2}{4}\right) = -2$, so an equation is $y - 4 = -2(x-3)$, or $y = -2x + 10$.

65. $f(x) = x\sqrt{2x^2+7}$, so $f'(x) = \sqrt{2x^2+7} + x\left(\frac{1}{2}\right)\left(2x^2+7\right)^{-1/2}(4x)$. The slope of the tangent line at $x = 3$ is $f'(3) = \sqrt{25} + \left(\frac{3}{2}\right)(25)^{-1/2}(12) = \frac{43}{5}$, so an equation is $y - 15 = \frac{43}{5}(x-3)$, or $y = \frac{43}{5}x - \frac{54}{5}$.

66. $f(x) = \dfrac{8}{\sqrt{x^2+6x}} = 8\left(x^2+6x\right)^{-1/2}$, so $f'(x) = -\frac{1}{2}(8)\left(x^2+6x\right)^{-3/2}(2x+6) = -4(2x+6)\left(x^2+6x\right)^{-3/2}$.

Therefore, the slope of the tangent line at $(2, 2)$ is

$f'(2) = -4(10)(4+12)^{-3/2} = -4(10)(16)^{-3/2} = -40\left(\frac{1}{64}\right) = -\frac{5}{8}$, and an equation is $y - 2 = -\frac{5}{8}(x-2)$, or $y = -\frac{5}{8}x + \frac{13}{4}$.

67. $N(t) = (60+2t)^{2/3}$, so $N'(t) = \frac{2}{3}(60+2t)^{-1/3}\frac{d}{dt}(60+2t) = \frac{4}{3}(60+2t)^{-1/3}$. The rate of increase at the end of the second week is $N'(2) = \frac{4}{3}(64)^{-1/3} = \frac{1}{3}$, or $\frac{1}{3}$ million/week. At the end of the 12th week, $N'(12) = \frac{4}{3}(84)^{-1/3} \approx 0.3$, or 0.3 million/week. The number of viewers in the 2nd week is $N(2) = (60+4)^{2/3} = 16$, or 16 million, and the number of viewers in the 24th week is $N(24) = (60+48)^{2/3} \approx 22.7$, or approximately 22.7 million.

68. **a.** The number of gigabytes of information being created monthly at the beginning of 2008 was $f(0) = 400\left(\frac{0}{12}+1\right)^{1.09} = 400$ billion.

b. $f'(t) = 400(1.09)\left(\dfrac{t}{12}+1\right)^{0.09}\left(\dfrac{1}{12}\right) = \dfrac{109}{3}\left(\dfrac{t}{12}+1\right)^{0.09}$, so the rate at which the rate of digital information creation was changing at the beginning of 2010 was approximately $f'(24) = \dfrac{109}{3}\left(\dfrac{24}{12}+1\right)^{0.09} \approx 40.1094$, or approximately 40.109 billion gigabytes per month.

69. $N(t) = -0.05(t+1.1)^{2.2} + 0.7t + 0.9$, so in 2008, the cumulative number of jobs that were outsourced was changing at the rate of $N'(3) = \left[-0.05(2.2)(t+1.1)^{1.2} + 0.7\right]_{t=3} = -0.05(2.2)(3+1.1)^{1.2} + 0.7 \approx 0.101956$, or approximately 102,000 per year.

70. $A'(t) = 699\dfrac{d}{dt}(t+1)^{-0.94} = -657.06(t+1)^{-1.94}$. At the beginning of 2002, $A'(0) = -657.06$, so it is falling at the rate of \$657.06 per year. At the beginning of 2012, $A'(10) = -6.27$, so it is falling at the rate of \$6.27 per year.

71. a. $P(1) \approx 30.0$ and $P(2) \approx 10.3$. The probability of survival at the moment of diagnosis is 100%, the probability of survival 1 year after diagnosis is approximately 30%, and the probability after 2 years is approximately 10.3%.

b. $P'(t) = \dfrac{d}{dt}\left[100(1+0.14t)^{-9.2}\right] = 100(-9.2)(1+0.14t)^{-10.2}(0.14) = -\dfrac{920 \cdot 0.14}{(1+0.14t)^{10.2}} = -\dfrac{128.8}{(1+0.14t)^{10.2}}$.

Thus, $P'(1) \approx -33.84$ and $P'(2) \approx 10.38$. After 1 year, the probability of survival is dropping at the rate of approximately 34% per year, and after 2 years, it is dropping at approximately 10.4% per year.

72. $f(t) = 10.72(0.9t+10)^{0.3}$. The rate of change at any time t is given by $f'(t) = 10.72(0.3)(0.9t+10)^{-0.7}(0.9) = 2.8944(0.9t+10)^{-0.7}$. At the beginning of 2000, we find $f'(0) = 2.8944(10)^{-0.7} \approx 0.5775$, or 0.6%/yr. At the beginning of 2015, we have $f'(15) = 2.8944(13.5+10)^{-0.7} \approx 0.3175$, or 0.3%/yr. The percent of the population of Americans age 55 or over in 2015 is $f(15) = 10.72(13.5+10)^{0.3} \approx 27.64$, or 27.6%.

73. $C(t) = 0.01\left(0.2t^2+4t+64\right)^{2/3}$.

a.

$$\begin{aligned} C'(t) &= 0.01\left(\tfrac{2}{3}\right)\left(0.2t^2+4t+64\right)^{-1/3}\frac{d}{dt}\left(0.2t^2+4t+64\right) \\ &= (0.01)(0.667)(0.4t+4)\left(0.2t^2+4t+64\right)^{-1/3} \approx 0.027(0.1t+1)\left(0.2t^2+4t+64\right)^{-1/3}. \end{aligned}$$

b. $C'(5) = 0.027(0.5+1)[0.2(25)+4(5)+64]^{-1/3} \approx 0.009$, or 0.009 parts per million per year.

74. $N(t) = -\dfrac{20{,}000}{\sqrt{1+0.2t}} + 21{,}000$, so $N'(t) = -20{,}000\left(-\tfrac{1}{2}\right)(1+0.2t)^{-3/2}(0.2) = \dfrac{2000}{(1+0.2t)^{3/2}}$.

$N'(1) = \dfrac{2000}{[1+0.2(1)]^{3/2}} \approx 1521$, or 1521 students/yr. $N'(5) = \dfrac{2000}{[1+0.2(5)]^{3/2}} \approx 707$, or 707 students/yr.

75. a. $A(t) = 0.03t^3(t-7)^4 + 60.2$, so
$A'(t) = 0.03\left[3t^2(t-7)^4 + t^3(4)(t-7)^3\right] = 0.03t^2(t-7)^3[3(t-7)+4t] = 0.21t^2(t-3)(t-7)^3$.

b. $A'(1) = 0.21(-2)(-6)^3 = 90.72$, $A'(3) = 0$, and $A'(4) = 0.21(16)(1)(-3)^3 = -90.72$. The amount of pollutant is increasing at the rate of 90.72 units/hr at 8 a.m. The rate of change is 0 units/hr at 10 a.m. and -90.72 units/hr at 11 a.m.

76. $N(x) = \left(10{,}000 - 40x - 0.02x^2\right)^{1/2}$, so $N'(x) = \frac{1}{2}\left(10{,}000 - 40x - 0.02x^2\right)^{-1/2}(-40-0.04x)$. Thus, $N'(10) = \frac{1}{2}(10{,}000 - 400 - 2)^{-1/2}(-40-0.4) = \frac{1}{2}(9598)^{-1/2}(-40.4) \approx -0.2062$, representing a 20.6% drop in consumption per 10% tax increase. Similarly, $N'(100) \approx -0.2889$, a drop of 28.9%, and $N'(150) \approx -0.3860$, a drop of 38.6%.

77. $P(t) = \dfrac{300\sqrt{\frac{1}{2}t^2 + 2t + 25}}{t+25} = \dfrac{300\left(\frac{1}{2}t^2 + 2t + 25\right)^{1/2}}{t+25}$, so

$$P'(t) = 300\left[\frac{(t+25)\frac{1}{2}\left(\frac{1}{2}t^2+2t+25\right)^{-1/2}(t+2) - \left(\frac{1}{2}t^2+2t+25\right)^{1/2}(1)}{(t+25)^2}\right]$$

$$= 300\left[\frac{\left(\frac{1}{2}t^2+2t+25\right)^{-1/2}\left[(t+25)(t+2) - 2\left(\frac{1}{2}t^2+2t+25\right)\right]}{2(t+25)^2}\right] = \frac{3450t}{(t+25)^2\sqrt{\frac{1}{2}t^2+2t+25}}.$$

Ten seconds into the run, the athlete's pulse rate is increasing at $P'(10) = \dfrac{3450(10)}{(35)^2\sqrt{50+20+25}} \approx 2.9$, or approximately 2.9 beats per minute per second. Sixty seconds into the run, it is increasing at $P'(60) = \dfrac{3450(60)}{(85)^2\sqrt{1800+120+25}} \approx 0.65$, or approximately 0.7 beats per minute per second. Two minutes into the run, it is increasing at $P'(120) = \dfrac{3450(120)}{(145)^2\sqrt{7200+240+25}} \approx 0.23$, or approximately 0.2 beats per minute per second. The pulse rate two minutes into the run is given by $P(120) = \dfrac{300\sqrt{7200+240+25}}{120+25} \approx 178.8$, or approximately 179 beats per minute.

78. a. $\dfrac{dT}{dn} = A(n-b)^{1/2} + An\left(\frac{1}{2}\right)(n-b)^{-1/2} = \frac{1}{2}A(n-b)^{-1/2}[2(n-b)+n] = \dfrac{(3n-2b)A}{2\sqrt{n-b}}$, and this gives the rate of change of the learning time with respect to the length of the list.

b. $\dfrac{dT}{dn} = \dfrac{(3n-8)4}{2\sqrt{n-4}}$ if $A = 4$ and $b = 4$. $f'(13) = \dfrac{(39-8)4}{2\sqrt{9}} \approx 20.7$, or approximately 21 units of time per unit increase in the word list. $f'(29) = \dfrac{(87-8)4}{2\sqrt{25}} \approx 31.6$, or approximately 32 units of time per unit increase in the word list.

79. The area is given by $A = \pi r^2$. The rate at which the area is increasing is given by dA/dt, that is, $\dfrac{dA}{dt} = \dfrac{d}{dt}\left(\pi r^2\right) = \dfrac{d}{dt}\left(\pi r^2\right)\dfrac{dr}{dt} = 2\pi r\dfrac{dr}{dt}$. If $r = 40$ and $dr/dt = 2$, then $\dfrac{dA}{dt} = 2\pi(40)(2) = 160\pi$, that is, it is increasing at the rate of 160π, or approximately 503 ft^2/sec.

80. $g(t) = 0.5t^2\left(t^2+10\right)^{-1}$, so

$g'(t) = 0.5(2t)\left(t^2+10\right)^{-1} + 0.5t^2(-1)\left(t^2+10\right)^{-2}(2t) = t\left(t^2+10\right)^{-2}\left[\left(t^2+10\right) - t^2\right] = \dfrac{10t}{\left(t^2+10\right)^2}$. In particular, $g'(5) = \dfrac{50}{(35)^2} \approx 0.04$ cm/yr.

81. $N(x) = 1.42x$ and $x(t) = \dfrac{7t^2 + 140t + 700}{3t^2 + 80t + 550}$. The number of construction jobs as a function of time is $n(t) = N(x(t))$. Using the Chain Rule,

$$n'(t) = \frac{dN}{dx} \cdot \frac{dx}{dt} = 1.42\frac{dx}{dt} = (1.42)\left[\frac{\left(3t^2 + 80t + 550\right)(14t + 140) - \left(7t^2 + 140t + 700\right)(6t + 80)}{\left(3t^2 + 80t + 550\right)^2}\right]$$

$$= \frac{1.42\left(140t^2 + 3500t + 21000\right)}{\left(3t^2 + 80t + 550\right)^2}.$$

$n'(12) = \dfrac{1.42\left[140(12)^2 + 3500(12) + 21000\right]}{\left[3(12)^2 + 80(12) + 550\right]^2} \approx 0.0313115$, or approximately 31,312 jobs/year/month.

82. $r(t) = \frac{10}{81}t^3 - \frac{10}{3}t^2 + \frac{200}{9}t + 56.2$ and $R(r) = -\frac{3}{5000}r^3 + \frac{9}{50}r^2$.

a. The rate of change of Wonderland's occupancy rate with respect to time is given by $r'(t) = \frac{10}{27}t^2 - \frac{20}{3}t + \frac{200}{9}$.

b. The rate of change of Wonderlands' monthly revenue with respect to the occupancy rate is given by $R'(r) = -\frac{9}{5000}r^2 + \frac{9}{25}r$.

c. When $t = 0$, $r(0) = \frac{10}{81}(0)^3 - \frac{10}{3}(0)^2 + \frac{200}{9}(0) + 56.2 = 56.2$, $r'(0) = \frac{10}{27}(0)^2 - \frac{20}{3}(0) + \frac{200}{9} = \frac{200}{9}$, $R'(56.2) = -\frac{9}{5000}(56.2)^2 + \frac{9}{25}(56.2) \approx 14.55$, and $R'(r(0))\,r'(0) \approx 14.55\left(\frac{200}{9}\right) \approx 323.3$. The rate of change of Wonderland's monthly revenue with respect to time at the beginning of January is approximately \$323,300/month. Next, when $t = 6$, $r(6) = \frac{10}{81}(6)^3 - \frac{10}{3}(6)^2 + \frac{200}{9}(6) + 56.2 = 96.2$, $r'(6) = \frac{30}{81}(6)^2 - \frac{20}{3}(6) + \frac{200}{9} = -4.44$, $R'(96.2) = -\frac{9}{5000}(96.2)^2 + \frac{9}{25}(96.2) \approx 17.97$, and $R'(r(6))\,r'(6) \approx (17.97)(-4.44) \approx -79.79$, so the rate of change of Wonderland's monthly revenue with respect to time at the beginning of July is approximately −\$79,790/month; that is, the revenue is decreasing at the rate of \$79,790/month.

83. $x = f(p) = \frac{100}{9}\sqrt{810{,}000 - p^2}$ and $p(t) = \dfrac{400}{1 + \frac{1}{8}\sqrt{t}} + 200$. We want to find $\dfrac{dx}{dt} = \dfrac{dx}{dp} \cdot \dfrac{dp}{dt}$. But $\dfrac{dx}{dp} = \dfrac{100}{9}\left(\dfrac{1}{2}\right)\left(810{,}000 - p^2\right)^{-1/2}(-2p) = -\dfrac{100p}{9\sqrt{810{,}000 - p^2}}$ and

$$\frac{dp}{dt} = 400\frac{d}{dt}\left(1 + \tfrac{1}{8}t^{1/2}\right)^{-1} + \frac{d}{dt}(200) = -400\left(1 + \tfrac{1}{8}t^{1/2}\right)^{-2}\left(\tfrac{1}{8}\right)\left(\tfrac{1}{2}t^{-1/2}\right) = -\frac{25}{\sqrt{t}\left(1 + \frac{1}{8}\sqrt{t}\right)^2},$$

so $\dfrac{dx}{dt} = \dfrac{2500p}{9\sqrt{t}\sqrt{810{,}000 - p^2}\left(1 + \frac{1}{8}\sqrt{t}\right)^2}$ and when $t = 16$, $p = \dfrac{400}{1 + \frac{1}{8}\sqrt{16}} + 200 = \frac{1400}{3}$. Therefore,

$\dfrac{dx}{dt} = \dfrac{2500\left(\frac{1400}{3}\right)}{9\sqrt{16}\sqrt{810{,}000 - \left(\frac{1400}{3}\right)^2}\left(1 + \frac{1}{8}\sqrt{16}\right)^2} \approx 18.7$. The quantity demanded will be changing at the rate of approximately 19 computers/month.

84. $p = f(t) = 50\left(\dfrac{t^2+2t+4}{t^2+4t+8}\right)$ and $R(p) = 1000\left(\dfrac{p+4}{p+2}\right)$. We want $\dfrac{dR}{dt} = \dfrac{dR}{dp}\cdot\dfrac{dp}{dt}$. Now $\dfrac{dR}{dp} = 1000\left[\dfrac{(p+2)(1)-(p+4)(1)}{(p+2)^2}\right] = -\dfrac{2000}{(p+2)^2}$ and $\dfrac{dp}{dt} = 50\left[\dfrac{\left(t^2+4t+8\right)(2t+2)-\left(t^2+2t+4\right)(2t+4)}{\left(t^2+4t+8\right)^2}\right] = \dfrac{100t(t+4)}{\left(t^2+4t+8\right)^2}$. When $t = 2$, $p = 50\left(\dfrac{4+4+4}{4+8+8}\right) = 30$, and $\dfrac{dR}{dt} = -\dfrac{2000}{(p+2)^2}\cdot\dfrac{100t(t+4)}{\left(t^2+4t+8\right)^2}\Bigg|_{t=2} = -\dfrac{2000}{(32)^2}\cdot\dfrac{100(2)(6)}{(4+8+8)^2} \approx -5.86$; that is, the passage will decrease at the rate of approximately \$5.86 per passenger per year.

85. True. This is just the statement of the Chain Rule.

86. True. $\frac{d}{dx}\left[f(cx)\right] = f'(cx)\frac{d}{dx}(cx) = f'(cx)\cdot c$.

87. True. $\frac{d}{dx}\sqrt{f(x)} = \frac{d}{dx}\left[f(x)\right]^{1/2} = \frac{1}{2}\left[f(x)\right]^{-1/2} f'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$.

88. False. Let $f(x) = x$. Then $f\left(\dfrac{1}{x}\right) = \dfrac{1}{x}$ and so $f'(x) = -\dfrac{1}{x^2}$. But $f'(x) = 1$, so $f'\left(\dfrac{1}{x}\right) = 1$.

89. Let $f(x) = x^{1/n}$ so that $\left[f(x)\right]^n = x$. Differentiating both sides with respect to x, we get $n\left[f(x)\right]^{n-1} f'(x) = 1$, so $f'(x) = \dfrac{1}{n\left[f(x)\right]^{n-1}} = \dfrac{1}{n\left[x^{1/n}\right]^{n-1}} = \dfrac{1}{nx^{1-(1/n)}} = \dfrac{1}{n}x^{(1/n)-1}$, as was to be shown.

90. Let $f(x) = x^r = x^{m/n} = (x^m)^{1/n}$. Then $\left[f(x)\right]^n = x^m$. Therefore,

$$n\left[f(x)\right]^{n-1} f'(x) = \frac{m}{n}\left[f(x)\right]^{-n+1} x^{m-1} = \frac{m}{n}\left(x^{m/n}\right)^{-n+1} x^{m-1} = \frac{m}{n}x^{[m(-n+1)/n]+m-1}$$
$$= \frac{m}{n}x^{(m-n)/n} = \frac{m}{n}x^{(m/n)-1} = rx^{r-1}.$$

Using Technology page 198

1. 0.5774

2. 1.4364

3. 0.9390

4. 3.9051

5. −4.9498

6. 0.1056

7. **a.** Using the numerical derivative operation, we find that $N'(0) = 5.41450$, so the rate of change of the number of people watching TV on mobile phones at the beginning of 2007 is approximately 5.415 million/year.

b. $N'(4) \approx 2.5136$, so the corresponding rate of change at the beginning of 2011 is expected to be approximately 2.5136 million/year.

8. **a.** 43.6 million

b. 0.432745 million/year

3.4 Marginal Functions in Economics

Concept Questions page 209

1. **a.** The marginal cost function is the derivative of the cost function.

   b. The average cost function is equal to the total cost function divided by the total number of the commodity produced.

   c. The marginal average cost function is the derivative of the average cost function.

   d. The marginal revenue function is the derivative of the revenue function.

   e. The marginal profit function is the derivative of the profit function.

2. **a.** The elasticity of demand at a price P is $E(p) = -\dfrac{pf'(p)}{f(p)}$, where f is the demand function $x = f(p)$.

   b. The elasticity of demand is elastic if $E(p) > 1$, unitary if $E(p) = 1$, and inelastic if $E(p) < 1$. If $E(p) > 1$, then an increase in the unit price will cause the revenue to decrease, whereas a decrease in the unit price will cause the revenue to increase. If $E(p) = 1$, then an increase in the unit price will have no effect on the revenue. If $E(p) < 1$, then an increase in the unit price will cause the revenue to increase, and a decrease in the unit price will cause the revenue to decrease.

3. $P(x) = R(x) - C(x)$, so $P'(x) = R'(x) - C'(x)$. Using the given information, we find $P'(500) = R'(500) - C'(500) = 3 - 2.8 = 0.2$. Thus, if the level of production is 500, then the marginal profit is \$0.20 per unit. This tells us that the proprietor should increase production in order to increase the company's profit.

Exercises page 209

1. **a.** $C(x)$ is always increasing because as x, the number of units produced, increases, the amount of money that must be spent on production also increases.

   b. This occurs at $x = 4$, a production level of 4000. You can see this by looking at the slopes of the tangent lines for x less than, equal to, and a little larger then $x = 4$.

2. **a.** If very few units of the commodity are produced then the cost per unit of production will be very large. If x is very large, the typical total cost is very large due to overtime, excessive cost of raw material, machinery breakdown, etc., so that $A(x)$ is very large as well; in fact, for a typical total cost function $C(x)$, ultimately $A(x)$ grows faster than x, that is, $\lim_{x\to\infty} \dfrac{C(x)}{x} = \infty$.

   b. The average cost per unit is smallest (\$$y_0$) when the level of production is x_0 units.

3. **a.** The actual cost incurred in the production of the 1001st disc is given by

$$\begin{aligned} C(1001) - C(1000) &= \left[2000 + 2(1001) - 0.0001(1001)^2\right] - \left[2000 + 2(1000) - 0.0001(1000)^2\right] \\ &= 3901.7999 - 3900 = 1.7999, \text{ or approximately } \$1.80. \end{aligned}$$

   The actual cost incurred in the production of the 2001st disc is given by

$$\begin{aligned} C(2001) - C(2000) &= \left[2000 + 2(2001) - 0.0001(2001)^2\right] - \left[2000 + 2(2000) - 0.0001(2000)^2\right] \\ &= 5601.5999 - 5600 = 1.5999, \text{ or approximately } \$1.60. \end{aligned}$$

   b. The marginal cost is $C'(x) = 2 - 0.0002x$. In particular, $C'(1000) = 2 - 0.0002(1000) = 1.80$ and $C'(2000) = 2 - 0.0002(2000) = 1.60$.

4. **a.** $$C(101) - C(100) = \left[0.0002(101)^3 - 0.06(101)^2 + 120(101) + 5000\right] - \left[0.0002(100)^3 - 0.06(100)^2 + 120(100) + 5000\right] \approx 114,$$ or approximately \$114.

Similarly, we find $C(201) - C(200) \approx \$120.06$ and $C(301) - C(300) \approx \$138.12$.

b. We compute $C'(x) = 0.0006x^2 - 0.12x + 120$. Thus, the required quantities are $C'(100) = 0.0006(100)^2 - 0.12(100) + 120 = 114$, or \$114; $C'(200) = 0.0006(200)^2 - 0.12(200) + 120 = 120$, or \$120; and $C'(300) = 0.0006(300)^2 - 0.12(300) + 120 = 138$, or \$138.

5. **a.** $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{100x + 200{,}000}{x} = 100 + \dfrac{200{,}000}{x}$.

b. $\overline{C}'(x) = \dfrac{d}{dx}(100) + \dfrac{d}{dx}\left(200{,}000x^{-1}\right) = -200{,}000x^{-2} = -\dfrac{200{,}000}{x^2}$.

c. $\displaystyle\lim_{x\to\infty} \overline{C}(x) = \lim_{x\to\infty}\left(100 + \frac{200{,}000}{x}\right) = 100$. This says that the average cost approaches \$100 per unit if the production level is very high.

6. **a.** $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{5000}{x} + 2$.

b. $\overline{C}'(x) = -\dfrac{5000}{x^2}$.

c. Because the marginal average cost function is negative for $x > 0$, the rate of change of the average cost function is negative for all $x > 0$.

7. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{2000 + 2x - 0.0001x^2}{x} = \dfrac{2000}{x} + 2 - 0.0001x$, so $\overline{C}'(x) = -\dfrac{2000}{x^2} + 0 - 0.0001 = -\dfrac{2000}{x^2} - 0.0001$.

8. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{0.0002x^3 - 0.06x^2 + 120x + 5000}{x} = 0.0002x^2 - 0.06x + 120 + \dfrac{5000}{x}$, so $\overline{C}'(x) = 0.0004x - 0.06 - \dfrac{5000}{x^2}$.

9. **a.** $R'(x) = \dfrac{d}{dx}\left(8000x - 100x^2\right) = 8000 - 200x$.

b. $R'(39) = 8000 - 200(39) = 200$, $R'(40) = 8000 - 200(40) = 0$, and $R'(41) = 8000 - 200(41) = -200$.

c. This suggests the total revenue is maximized if the price charged per passenger is \$40.

10. **a.** $R(x) = px = x(-0.04x + 800) = -0.04x^2 + 800x$.

b. $R'(x) = -0.08x + 800$.

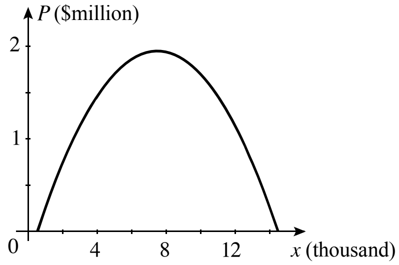
c. $R'(5000) = -0.08(5000) + 800 = 400$. This says that when the level of production is 5000 units, the production of the next speaker system will bring an additional revenue of \$400.

11. **a.** $P(x) = R(x) - C(x) = \left(-0.04x^2 + 800x\right) - (200x + 300{,}000) = -0.04x^2 + 600x - 300{,}000$.

b. $P'(x) = -0.08x + 600$.

c. $P'(5000) = -0.08(5000) + 600 = 200$ and $P'(8000) = -0.08(8000) + 600 = -40$.

d.



The profit realized by the company increases as production increases, peaking at a production level of 7500 units. Beyond this level of production, the profit begins to fall.

12. **a.** $P(x) = -10x^2 + 1760x - 50{,}000$. To find the actual profit realized from renting the 51st unit, assuming that 50 units have already been rented, we calculate

$$P(51) - P(50) = \left[-10(51)^2 + 1760(51) - 50{,}000\right] - \left[-10(50)^2 + 1760(50) - 50{,}000\right]$$
$$= -26{,}010 + 89{,}760 - 50{,}000 + 25{,}000 - 88{,}000 + 50{,}000 = 750, \text{ or } \$750.$$

b. The marginal profit is given by $P'(x) = -20x + 1760$. When $x = 50$, $P'(50) = -20(50) + 1760 = 760$, or \$760.

13. **a.** The revenue function is $R(x) = px = (600 - 0.05x)x = 600x - 0.05x^2$ and the profit function is

$$P(x) = R(x) - C(x) = \left(600x - 0.05x^2\right) - \left(0.000002x^3 - 0.03x^2 + 400x + 80{,}000\right)$$
$$= -0.000002x^3 - 0.02x^2 + 200x - 80{,}000.$$

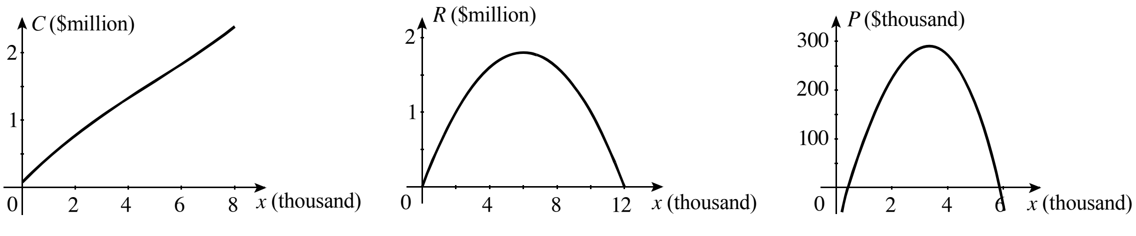
b. $C'(x) = \dfrac{d}{dx}\left(0.000002x^3 - 0.03x^2 + 400x + 80{,}000\right) = 0.000006x^2 - 0.06x + 400$,

$R'(x) = \dfrac{d}{dx}\left(600x - 0.05x^2\right) = 600 - 0.1x$, and

$P'(x) = \dfrac{d}{dx}\left(-0.000002x^3 - 0.02x^2 + 200x - 80{,}000\right) = -0.000006x^2 - 0.04x + 200$.

c. $C'(2000) = 0.000006(2000)^2 - 0.06(2000) + 400 = 304$, and this says that at a production level of 2000 units, the cost for producing the 2001st unit is \$304. $R'(2000) = 600 - 0.1(2000) = 400$, and this says that the revenue realized in selling the 2001st unit is \$400. $P'(2000) = R'(2000) - C'(2000) = 400 - 304 = 96$, and this says that the revenue realized in selling the 2001st unit is \$96.

d.



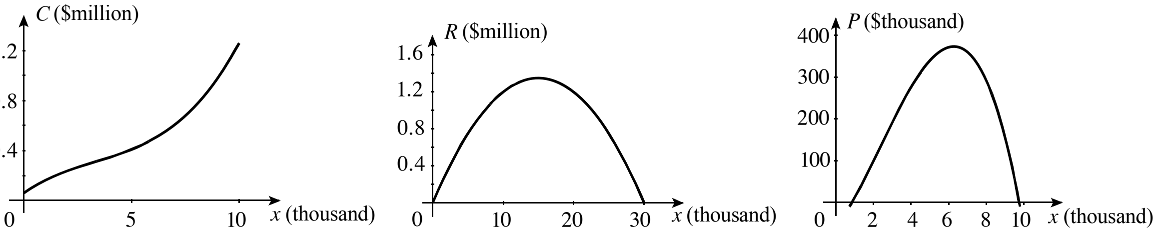
14. **a.** $R(x) = xp(x) = -0.006x^2 + 180x$ and

$$P(x) = R(x) - C(x) = \left(-0.006x^2 + 180x\right) - \left(0.000002x^3 - 0.02x^2 + 120x + 60{,}000\right)$$
$$= -0.000002x^3 + 0.014x^2 + 60x - 60{,}000.$$

b. $C'(x) = 0.000006x^2 - 0.04x + 120$, $R'(x) = -0.012x + 180$, and $P'(x) = -0.000006x^2 + 0.028x + 60$.

c. $C'(2000) = 0.000006(2000)^2 - 0.04(2000) + 120 = 64$, $R'(2000) = -0.012(2000) + 180 = 156$, and $P'(2000) = -0.000006(2000)^2 + 0.028(2000) + 60 = 92$.

d.



15. a. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{0.000002x^3 - 0.03x^2 + 400x + 80{,}000}{x} = 0.000002x^2 - 0.03x + 400 + \dfrac{80{,}000}{x}$.

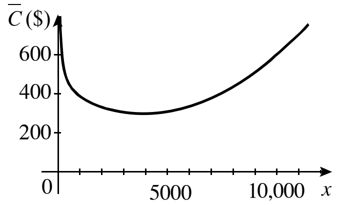
b. $\overline{C}'(x) = 0.000004x - 0.03 - \dfrac{80{,}000}{x^2}$.

c. $\overline{C}'(5000) = 0.000004(5000) - 0.03 - \dfrac{80{,}000}{5000^2} \approx -0.0132$, and this says that at a production level of 5000 units, the average cost of production is dropping at the rate of approximately a penny per unit.

$\overline{C}'(10{,}000) = 0.000004(10000) - 0.03 - \dfrac{80{,}000}{10{,}000^2} \approx 0.0092$,

and this says that, at a production level of 10,000 units, the average cost of production is increasing at the rate of approximately a penny per unit.

d.



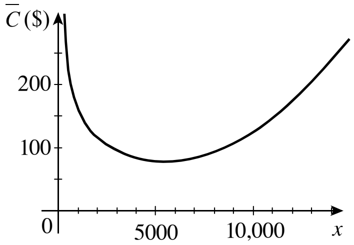
16. a. $C(x) = 0.000002x^3 - 0.02x^2 + 120x + 60{,}000$, so $\overline{C}(x) = 0.000002x^2 - 0.02x + 120 + \dfrac{60{,}000}{x}$.

b. The marginal average cost function is given by $\overline{C}'(x) = 0.000004x - 0.02 - \dfrac{60{,}000}{x^2}$.

c.
$$\overline{C}'(5000) = 0.000004(5000) - 0.02 - \frac{60{,}000}{(5000)^2}$$
$$= 0.02 - 0.02 - 0.0024 = -0.0024 \text{ and}$$
$$\overline{C}'(10{,}000) = 0.000004(10{,}000) - 0.02 - \frac{60{,}000}{(10000)^2}$$
$$= 0.04 - 0.02 - 0.0006 = 0.0194.$$

We conclude that the average cost is decreasing when 5000 TV sets are produced and increasing when 10,000 units are produced.

d.



17. a. $R(x) = px = \dfrac{50x}{0.01x^2 + 1}$.

b. $R'(x) = \dfrac{\left(0.01x^2 + 1\right)50 - 50x(0.02x)}{\left(0.01x^2 + 1\right)^2} = \dfrac{50 - 0.5x^2}{\left(0.01x^2 + 1\right)^2}$.

c. $R'(2) = \dfrac{50 - 0.5(4)}{[0.01(4) + 1]^2} \approx 44.379$. This result says that at a sales level of 2000 units, the revenue increases at the rate of approximately $44,379 per 1000 units.

18. $\dfrac{dC}{dx} = \dfrac{d}{dx}(0.712x + 95.05) = 0.712$.

19. $C(x) = 0.873x^{1.1} + 20.34$, so $C'(x) = 0.873(1.1)x^{0.1}$. $C'(10) = 0.873(1.1)(10)^{0.1} = 1.21$.

20. $\dfrac{dS}{dx} = \dfrac{d}{dx}\left[x - C(x)\right] = 1 - \dfrac{dC}{dx}$.

21. The consumption function is given by $C(x) = 0.712x + 95.05$. The marginal propensity to consume is given by $\frac{dC}{dx} = 0.712$. The marginal propensity to save is given by $\frac{dS}{dx} = 1 - \frac{dC}{dx} = 1 - 0.712 = 0.288$.

22. Here $C(x) = 0.873x^{1.1} + 20.34$, so $C'(x) = 0.9603x^{0.1}$ and $\frac{dS}{dx} = 1 - \frac{dC}{dx} = 1 - 0.9603x^{0.1}$. When $x = 10$, we have $\frac{dS}{dx} = 1 - 0.9603(10)^{0.1} \approx -0.209$, or approximately $-\$0.209$ billion per billion dollars.

23. $f(x) = 2x^2 + x + 1$, so $f'(x) = 4x + 1$. The percentage rate of change of f at $x = 2$ is $100\dfrac{f'(2)}{f(2)} = 100\left[\dfrac{4x+1}{2x^2+x+1}\right]_{x=2} = 100\left(\dfrac{8+1}{8+2+1}\right) \approx 81.82$ (percent per unit change in x).

24. $f(x) = \left(2x^2+7\right)^{1/2}$, so $f'(x) = \frac{1}{2}\left(2x^2+7\right)^{-1/2}(4x)$. Then $f(3) = 25^{1/2}$ and $f'(3) = 6(25)^{-1/2}$, so the percentage rate of change of f at $x = 3$ is $100\dfrac{f'(3)}{f(3)} = 100 \cdot \dfrac{6(25)^{-1/2}}{25^{1/2}} = \dfrac{100 \cdot 6}{25} = 24$ (percent per unit change in x).

25. $f(x) = \dfrac{x+1}{x^3+x+1}$, so $f'(x) = \dfrac{\left(x^3+x+1\right)(1) - (x+1)\left(3x^2+1\right)}{\left(x^3+x+1\right)^2}$. Then $f(2) = \frac{3}{11}$ and $f'(2) = -\frac{28}{121}$, so the percentage rate of change of f at $x = 2$ is $100 \cdot \dfrac{-\frac{28}{121}}{\frac{3}{11}} \approx -84.85$ (percent per unit change in x).

26. $f(x) = \left(\dfrac{x}{x^2+3x+12}\right)^{3/2}$, so

$$\begin{aligned} f'(x) &= \frac{3}{2}\left(\frac{x}{x^2+3x+12}\right)^{1/2}\frac{d}{dx}\left(\frac{x}{x^2+3x+12}\right) \\ &= \frac{3}{2}\left(\frac{x}{x^2+3x+12}\right)^{1/2}\left[\frac{\left(x^2+3x-12\right)(1) - x(2x+3)}{\left(x^2+3x+12\right)^2}\right]. \end{aligned}$$

Then $f(1) = \left(\frac{1}{16}\right)^{3/2} = \frac{1}{64}$ and $f'(1) = \frac{3}{2}\left(\frac{1}{16}\right)^{1/2}\left(\frac{11}{256}\right) = \frac{33}{2048}$, so the percentage rate of change of f at $x = 1$ is $100 \cdot \dfrac{\frac{33}{2048}}{\frac{1}{64}} = 103.125$ (percent per unit change in x).

27. Here $x = f(p) = -\frac{5}{4}p + 20$ and so $f'(p) = -\frac{5}{4}$. Therefore, $E(p) = -\dfrac{pf'(p)}{f(p)} = -\dfrac{p\left(-\frac{5}{4}\right)}{-\frac{5}{4}p + 20} = \dfrac{5p}{80 - 5p}$. $E(10) = \dfrac{5(10)}{80 - 5(10)} = \dfrac{50}{30} = \dfrac{5}{3} > 1$, and so the demand is elastic.

28. $f(p) = -\frac{3}{2}p + 9$, so $f'(p) = -\frac{3}{2}$. Then the elasticity of demand is given by $E(p) = -\dfrac{pf'(p)}{f(p)} = -\dfrac{p\left(-\frac{3}{2}\right)}{-\frac{3}{2}p + 9}$.

Therefore, when $p = 2$, $E(2) = -\dfrac{2\left(-\frac{3}{2}\right)}{-\frac{3}{2}(2) + 9} = \dfrac{3}{6} = \dfrac{1}{2} < 1$, and we conclude that the demand is inelastic at this price.

29. $f(p) = -\frac{1}{3}p + 20$, so $f'(p) = -\frac{1}{3}$. Then the elasticity of demand is given by $E(p) = -\dfrac{p\left(-\frac{1}{3}\right)}{-\frac{1}{3}p + 20}$, and

$E(30) = -\dfrac{30\left(-\frac{1}{3}\right)}{-\frac{1}{3}(30) + 20} = 1$, and we conclude that the demand is unitary at this price.

30. Solving the demand equation for x, we find $x^2 = 144 - p$, or $x = \sqrt{144 - p}$ (because x must be nonnegative). With $x = f(p) = (144 - p)^{1/2}$, we have $f'(p) = \frac{1}{2}(144 - p)^{-1/2}(-1) = -\dfrac{1}{2\sqrt{144 - p}}$. Therefore,

$$E(p) = -\frac{pf'(p)}{f(p)} = -\frac{p\left(-\frac{1}{2\sqrt{144-p}}\right)}{\sqrt{144 - p}} = \frac{p}{2(144 - p)}.$$

$E(96) = \frac{96}{2(48)} = 1$, and so the demand equation is unitary.

31. $x^2 = 169 - p$ and $f(p) = (169 - p)^{1/2}$. Next, $f'(p) = \frac{1}{2}(169 - p)^{-1/2}(-1) = -\frac{1}{2}(169 - p)^{-1/2}$. Then the elasticity of demand is given by $E(p) = -\dfrac{pf'(p)}{f(p)} = -\dfrac{p\left(-\frac{1}{2}\right)(169 - p)^{-1/2}}{(169 - p)^{1/2}} = \dfrac{\frac{1}{2}p}{169 - p}$. Therefore, when $p = 29$, $E(p) = \dfrac{\frac{1}{2}(29)}{169 - 29} = \dfrac{14.5}{140} \approx 0.104$. Because $E(p) < 1$, we conclude that demand is inelastic at this price.

32. $I(t) = -0.02t^3 + 0.4t^2 + 120$, so $I'(t) = -0.06t^2 + 0.8t$. Then $I(1) = -0.02 + 0.4 + 120 = 120.38$ and $I'(1) = -0.06 + 0.8 = 0.74$, so the annual percentage rate of inflation in the CPI of the country at the beginning of 2014 ($t = 1$) is $100 \cdot \dfrac{I'(1)}{I(1)} = 100 \cdot \dfrac{0.74}{120.38} \approx 0.6147$, or approximately 0.615%.

33. **a.** The percentage rate of change in per capita income in year t is

$$100\frac{C'(t)}{C(t)} = 100 \cdot \frac{\dfrac{d}{dt}\left[\dfrac{I(t)}{P(t)}\right]}{C(t)} = 100 \cdot \frac{P(t)I'(t) - I(t)P'(t)}{[P(t)]^2} \cdot \frac{P(t)}{I(t)} = 100 \cdot \frac{P(t)I'(t) - I(t)P'(t)}{P(t)I(t)}.$$

b. Here $I(t) = 10^9(300 + 12t)$ and $P(t) = 2 \times 10^7 e^{0.02t}$, so $I'(t) = 12 \times 10^9$ and $P'(t) = 4 \cdot 10^5 e^{0.02t}$. Therefore, the percentage rate of change in per capita income in year t is

$$\begin{aligned} 100 \cdot \frac{3 \times 10^7 e^{0.02t}\left(12 \times 10^9\right) - 10^9(300 + 12t)\left(4 \cdot 10^5 e^{0.02t}\right)}{\left(2 \cdot 10^7 e^{0.02t}\right)10^9(300 + 12t)} &= \frac{2400 \times 10^{16} e^{0.02t} - 48 \times 10^{16} e^{0.02t}(25 + t)}{24 \cdot 10^{16} e^{0.02t}(25 + t)} \\ &= \frac{50 - 2t}{25 + t}. \end{aligned}$$

c. The percentage rate of change in per capita income 2 years from now is projected to be $\dfrac{50 - 2(2)}{25 + 2} = \dfrac{46}{27}$, or approximately 1.7%/yr.

34. The percentage growth rate at $t = a$ is $100 \cdot \dfrac{R'(a)}{R(a)} = \dfrac{100\left[f'(a) + g'(a)\right]}{f(a) + g(a)}$. Taking $f'(a) = 0.24$, $g'(a) = 0.30$, $f(a) = 3.2$, and $g(a) = 2.6$, we have $100 \cdot \dfrac{R'(a)}{R(a)} = \dfrac{100(0.24 + 0.30)}{3.2 + 2.6} \approx 9.31$. Thus, the percentage growth rate is approximately 9.31% per year.

35. $f(p) = \frac{1}{5}\left(225 - p^2\right)$, so $f'(p) = \frac{1}{5}(-2p) = -\frac{2}{5}p$. Then the elasticity of demand is given by

$$E(p) = -\frac{pf'(p)}{f(p)} = -\frac{p\left(-\frac{2}{5}p\right)}{\frac{1}{5}\left(225 - p^2\right)} = \frac{2p^2}{225 - p^2}.$$

a. When $p = 8$, $E(8) = \dfrac{2(64)}{225 - 64} \approx 0.8 < 1$ and the demand is inelastic. When $p = 10$,

$E(10) = \dfrac{2(100)}{225 - 100} = 1.6 > 1$ and the demand is elastic.

b. The demand is unitary when $E = 1$. Solving $\dfrac{2p^2}{225 - p^2} = 1$, we find $2p^2 = 225 - p^2$, $3p^2 = 225$, and $p \approx 8.66$. So the demand is unitary when $p \approx 8.66$.

c. Because demand is elastic when $p = 10$, lowering the unit price will cause the revenue to increase.

d. Because the demand is inelastic at $p = 8$, a slight increase in the unit price will cause the revenue to increase.

36. $f(p) = (144 - p)^{1/2}$, so $f'(p) = \frac{1}{2}(144 - p)^{-1/2}(-1)$. Then the elasticity of demand is given by

$$E(p) = -\frac{pf'(p)}{f(p)} = -\frac{p\left(-\frac{1}{2}\right)(144 - p)^{-1/2}}{(144 - p)^{1/2}} = \frac{p}{2(144 - p)}.$$

a. $E(63) = \dfrac{63}{2(144 - 63)} \approx 0.39$, $E(96) = \dfrac{96}{2(144 - 96)} = 1$, and $E(108) = \dfrac{108}{2(144 - 108)} = 1.5$.

b. At a unit price of \$63, an unit price increase of \$1 will result in a decrease of approximately 0.39% in demand as well as increased revenue. When the unit price is set at \$96, a price increase of \$1 will not cause any change in demand or revenue. When the price is set at \$108, a price increase of \$1 will cause a decrease of approximately 1.5% in demand as well as decreased revenue.

c. The demand is inelastic when $p = 63$, unitary when $p = 96$, and elastic when $p = 108$.

37. $f(p) = \frac{2}{3.}\left(36 - p^2\right)^{1/2}$. $f'(p) = \frac{2}{3}\left(\frac{1}{2}\right)\left(36 - p^2\right)^{-1/2}(-2p) = -\frac{2}{3}p\left(36 - p^2\right)^{-1/2}$. Then the elasticity of demand is given by $E(p) = -\dfrac{pf'(p)}{f(p)} = -\dfrac{-\frac{2}{3}p\left(36 - p^2\right)^{-1/2}p}{\frac{2}{3}\left(36 - p^2\right)^{1/2}} = \dfrac{p^2}{36 - p^2}$.

a. When $p = 2$, $E(2) = \dfrac{4}{36 - 4} = \dfrac{1}{8} < 1$, and we conclude that the demand is inelastic.

b. Because the demand is inelastic, the revenue will increase when the rental price is increased.

38. Here $x = f(p) = \sqrt{400 - 5p} = (400 - 5p)^{1/2}$. Therefore, $f'(p) = \frac{1}{2}(400 - 5p)^{-1/2}(-5) = -\dfrac{5}{2\sqrt{400 - 5p}}$,

and so $E(p) = -\dfrac{pf'(p)}{f(p)} = -\dfrac{p\left(-\frac{5}{2\sqrt{400-5p}}\right)}{\sqrt{400 - 5p}} = \dfrac{5p}{2(400 - 5p)}$.

a. $E(40) = \dfrac{5(40)}{2[400 - 5(40)]} = 0.5$, and so the demand is inelastic when $p = 40$.

$E(60) = \dfrac{5(60)}{2[400 - 5(60)]} = 1.5$, and so the demand is elastic when $p = 60$.

b. The demand is unitary if $\dfrac{5p}{2(400 - 5p)} = 1$, or $5p = 800 - 10p$, that is, when $p = 53\frac{1}{3}$. (This also follows from part (a).)

c. Because the demand is elastic at $p = 60$, lowering the unit price a little will cause the revenue to increase.

d. Because the demand is inelastic at $p = 40$, a slight increase of the unit price will cause the revenue to increase.

39. We first solve the demand equation for x in terms of p. Thus, $p = \sqrt{9 - 0.02x}$, and $p^2 = 9 - 0.02x$, or $x = -50p^2 + 450$. With $f(p) = -50p^2 + 450$, we find $E(p) = -\frac{pf'(p)}{f(p)} = -\frac{p(-100p)}{-50p^2 + 450} = \frac{2p^2}{9 - p^2}$. Setting $E(p) = 1$ gives $2p^2 = 9 - p^2$, so $p = \sqrt{3}$. So the demand is inelastic in $\left(0, \sqrt{3}\right)$, unitary when $p = \sqrt{3}$, and elastic in $\left(\sqrt{3}, 3\right)$.

40. $f(p) = 10\left(\frac{50 - p}{p}\right)^{1/2} = 10\left(\frac{50}{p} - 1\right)^{1/2}$, so

$f'(p) = 10\left(\frac{1}{2}\right)\left(\frac{50}{p} - 1\right)^{-1/2}\left(-\frac{50}{p^2}\right) = -\frac{250}{p^2}\left(\frac{50}{p} - 1\right)^{-1/2}$. Then the elasticity of demand is given by

$E(p) = -\frac{pf'(p)}{f(p)} = -\frac{p\left(-\frac{250}{p^2}\right)\left(\frac{50}{p} - 1\right)^{-1/2}}{10\left(\frac{50}{p} - 1\right)^{1/2}} = -\frac{\frac{250}{p}}{10\left(\frac{50}{p} - 1\right)} = \frac{25}{p\left(\frac{50 - p}{p}\right)} = \frac{25}{50 - p}$. Setting $E = 1$ gives $1 = \frac{25}{50 - p}$, and so $25 = 50 - p$, and $p = 25$. Thus, if $p > 25$, then $E > 1$, and the demand is elastic; if $p = 25$, then $E = 1$ and the demand is unitary; and if $p < 25$, then $E < 1$ and the demand is inelastic.

41. True. $\overline{C}'(x) = \frac{d}{dx}\left[\frac{C(x)}{x}\right] = \frac{xC'(x) - C(x)\frac{d}{dx}(x)}{x^2} = \frac{xC'(x) - C(x)}{x^2}$.

42. False. In fact, it makes good sense to *increase* the level of production since, in this instance, the profit increases by $f'(a)$ units per unit increase in x.

3.5 Higher-Order Derivatives

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1. a. The second derivative of f is the derivative of f'.

b. To find the second derivative of f, we differentiate f'.

2. $f'(t)$ measures its velocity at time t, and $f''(t)$ measures its acceleration at time t.

3. a. $f'(t) > 0$ and $f''(t) > 0$ in (a, b).

b. $f'(t) > 0$ and $f''(t) < 0$ in (a, b).

c. $f'(t) < 0$ and $f''(t) < 0$ in (a, b).

d. $f'(t) < 0$ and $f''(t) > 0$ in (a, b).

4. a. $f'(t) > 0$ and $f''(t) = 0$ in (a, b).

b. $f'(t) < 0$ and $f''(t) = 0$ in (a, b).

c. $f'(t) = 0$ and $f''(t) = 0$ in (a, b).

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1. $f(x) = 4x^2 - 2x + 1$, so $f'(x) = 8x - 2$ and $f''(x) = 8$.

2. $f(x) = -0.2x^2 + 0.3x + 4$, so $f'(x) = -0.4x + 0.3$ and $f''(x) = -0.4$.

3. $f(x) = 2x^3 - 3x^2 + 1$, so $f'(x) = 6x^2 - 6x$ and $f''(x) = 12x - 6 = 6(2x - 1)$.

4. $g(x) = -3x^3 + 24x^2 + 6x - 64$, so $g'(x) = -9x^2 + 48x + 6$ and $g''(x) = -18x + 48$.

5. $h(t) = t^4 - 2t^3 + 6t^2 - 3t + 10$, so $h'(t) = 4t^3 - 6t^2 + 12t - 3$ and $h''(t) = 12t^2 - 12t + 12 = 12\left(t^2 - t + 1\right)$.

6. $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$, so $f'(x) = 5x^4 - 4x^3 + 3x^2 - 2x + 1$ and $f''(x) = 20x^3 - 12x^2 + 6x - 2$.

7. $f(x) = \left(x^2+2\right)^5$, so $f'(x) = 5\left(x^2+2\right)^4(2x) = 10x\left(x^2+2\right)^4$ and
$f''(x) = 10\left(x^2+2\right)^4 + 10x(4)\left(x^2+2\right)^3(2x) = 10\left(x^2+2\right)^3\left[\left(x^2+2\right)+8x^2\right] = 10\left(9x^2+2\right)\left(x^2+2\right)^3$.

8. $g(t) = t^2(3t+1)^4$, so

$$g'(t) = 2t(3t+1)^4 + t^2(4)(3t+1)^3(3) = 2t(3t+1)^3[(3t+1)+6t] = (3t+1)^3\left(18t^2+2t\right) \text{ and}$$

$$\begin{aligned} g''(t) &= 2t(9t+1)(3)(3t+1)^2(3) + (3t+1)^3(36t+2) = 2(3t+1)^2[9t(9t+1)+(3t+1)(18t+1)] \\ &= 2(3t+1)^2\left(81t^2+9t+54t^2+3t+18t+1\right) = 2\left(135t^2+30t+1\right)(3t+1)^2. \end{aligned}$$

9. $g(t) = \left(2t^2-1\right)^2\left(3t^2\right)$, so

$$\begin{aligned} g'(t) &= 2\left(2t^2-1\right)(4t)\left(3t^2\right) - \left(2t^2-1\right)^2(6t) = 6t\left(2t^2-1\right)\left[4t^2+\left(2t^2-1\right)\right] = 6t\left(2t^2-1\right)\left(6t^2-1\right) \\ &= 6t\left(12t^4-8t^2+1\right) = 72t^5-48t^3+6t \end{aligned}$$

and $g''(t) = 360t^4 - 144t^2 + 6 = 6\left(60t^4 - 24t^2 + 1\right)$.

10. $h(x) = \left(x^2+1\right)^2(x-1)$, so $h'(x) = 2\left(x^2+1\right)(2x)(x-1) + \left(x^2+1\right)^2(1) = \left(x^2+1\right)\left[4x(x-1)+\left(x^2+1\right)\right] = \left(x^2+1\right)\left(5x^2-4x+1\right)$ and

$$\begin{aligned} h''(x) &= 2x\left(5x^2-4x+1\right) + \left(x^2+1\right)(10x-4) = 10x^3-8x^2+2x+10x^3-4x^2+10x-4 \\ &= 20x^3-12x^2+12x-4 = 4\left(5x^3-3x^2+3x-1\right). \end{aligned}$$

11. $f(x) = \left(2x^2+2\right)^{7/2}$, so $f'(x) = \frac{7}{2}\left(2x^2+2\right)^{5/2}(4x) = 14x\left(2x^2+2\right)^{5/2}$ and

$$\begin{aligned} f''(x) &= 14\left(2x^2+2\right)^{5/2} + 14x\left(\tfrac{5}{2}\right)\left(2x^2+2\right)^{3/2}(4x) = 14\left(2x^2+2\right)^{3/2}\left[\left(2x^2+2\right)+10x^2\right] \\ &= 28\left(6x^2+1\right)\left(2x^2+2\right)^{3/2}. \end{aligned}$$

12. $h(w) = \left(w^2+2w+4\right)^{5/2}$, so $h'(w) = \frac{5}{2}\left(w^2+2w+4\right)^{3/2}(2w+2) = 5(w+1)\left(w^2+2w+4\right)^{3/2}$ and

$$\begin{aligned} h''(w) &= 5\left(w^2+2w+4\right)^{3/2} + 5(w+1)\left(\tfrac{3}{2}\right)\left(w^2+2w+4\right)^{1/2}(2w+2) \\ &= 5\left(w^2+2w+4\right)^{1/2}\left[\left(w^2+2w+4\right)+3(w+1)^2\right] = 5\left(4w^2+8w+7\right)\left(w^2+2w+4\right)^{1/2}. \end{aligned}$$

13. $f(x) = x\left(x^2+1\right)^2$, so
$f'(x) = \left(x^2+1\right)^2 + x(2)\left(x^2+1\right)(2x) = \left(x^2+1\right)\left[\left(x^2+1\right)+4x^2\right] = \left(x^2+1\right)\left(5x^2+1\right)$ and
$f''(x) = 2x\left(5x^2+1\right) + \left(x^2+1\right)(10x) = 2x\left(5x^2+1+5x^2+5\right) = 4x\left(5x^2+3\right)$.

14. $g(u) = u(2u-1)^3$, so $g'(u) = (2u-1)^3 + u(3)(2u-1)^2(2) = (2u-1)^2[(2u-1)+6u] = (8u-1)(2u-1)^2$ and $g''(u) = 8(2u-1)^2 + (8u-1)(2)(2u-1)(2) = 4(2u-1)[2(2u-1)+(8u-1)] = 12(2u-1)(4u-1)$.

15. $f(x) = \dfrac{x}{2x+1}$, so $f'(x) = \dfrac{(2x+1)(1) - x(2)}{(2x+1)^2} = \dfrac{1}{(2x+1)^2}$ and

$f''(x) = \dfrac{d}{dx}(2x+1)^{-2} = -2(2x+1)^{-3}(2) = -\dfrac{4}{(2x+1)^3}$.

16. $g(t) = \dfrac{t^2}{t-1}$, so $g'(t) = \dfrac{(t-1)(2t) - t^2(1)}{(t-1)^2} = \dfrac{t^2-2t}{(t-1)^2} = \dfrac{t(t-2)}{(t-1)^2}$ and

$g''(t) = \dfrac{(t-1)^2(2t-2) - t(t-2)\,2(t-1)}{(t-1)^4} = \dfrac{2(t-1)\left[(t-1)^2 - t(t-2)\right]}{(t-1)^4} = \dfrac{2}{(t-1)^3}$.

17. $f(s) = \dfrac{s-1}{s+1}$, so $f'(s) = \dfrac{(s+1)(1) - (s-1)(1)}{(s+1)^2} = \dfrac{2}{(s+1)^2}$ and

$f''(s) = 2\dfrac{d}{ds}(s+1)^{-2} = -4(s+1)^{-3} = -\dfrac{4}{(s+1)^3}$.

18. $f(u) = \dfrac{u}{u^2+1}$, so $f'(u) = \dfrac{\left(u^2+1\right)(1) - (u)(2u)}{\left(u^2+1\right)^2} = \dfrac{-u^2+1}{\left(u^2+1\right)^2}$ and

$f''(u) = \dfrac{\left(u^2+1\right)^2(-2u) - \left(-u^2+1\right)(2)\left(u^2+1\right)(2u)}{\left(u^2+1\right)^4} = \dfrac{2u\left(u^2+1\right)\left(-u^2-1+2u^2-2\right)}{\left(u^2+1\right)^4} = \dfrac{2u\left(u^2-3\right)}{\left(u^2+1\right)^3}$.

19. $f(u) = \sqrt{4-3u} = (4-3u)^{1/2}$, so $f'(u) = \tfrac{1}{2}(4-3u)^{-1/2}(-3) = -\dfrac{3}{2\sqrt{4-3u}}$ and

$f''(u) = -\dfrac{3}{2}\cdot\dfrac{d}{du}(4-3u)^{-1/2} = -\dfrac{3}{2}\left(-\dfrac{1}{2}\right)(4-3u)^{-3/2}(-3) = -\dfrac{9}{4(4-3u)^{3/2}}$.

20. $f(x) = \sqrt{2x-1} = (2x-1)^{1/2}$, so $f'(x) = \tfrac{1}{2}(2x-1)^{-1/2}(2) = (2x-1)^{-1/2} = \dfrac{1}{\sqrt{2x-1}}$ and

$f''(x) = -\tfrac{1}{2}(2x-1)^{-3/2}(2) = -(2x-1)^{-3/2} = -\dfrac{1}{\sqrt{(2x-1)^3}}$.

21. $f(x) = 3x^4 - 4x^3$, so $f'(x) = 12x^3 - 12x^2$, $f''(x) = 36x^2 - 24x$, and $f'''(x) = 72x - 24$.

22. $f(x) = 3x^5 - 6x^4 + 2x^2 - 8x + 12$, so $f'(x) = 15x^4 - 24x^3 + 4x - 8$, $f''(x) = 60x^3 - 72x^2 + 4$, and $f'''(x) = 180x^2 - 144x$.

23. $f(x) = \dfrac{1}{x}$, so $f'(x) = \dfrac{d}{dx}\left(x^{-1}\right) = -x^{-2} = -\dfrac{1}{x^2}$, $f''(x) = 2x^{-3} = \dfrac{2}{x^3}$, and $f'''(x) = -6x^{-4} = -\dfrac{6}{x^4}$.

24. $f(x) = \dfrac{2}{x^2}$, so $f'(x) = 2\dfrac{d}{dx}\left(x^{-2}\right) = -4x^{-3} = -\dfrac{4}{x^3}$, $f''(x) = 12x^{-4} = \dfrac{12}{x^4}$, and $f'''(x) = -48x^{-5} = -\dfrac{48}{x^5}$.

25. $g(s) = (3s-2)^{1/2}$, so $g'(s) = \tfrac{1}{2}(3s-2)^{-1/2}(3) = \dfrac{3}{2(3s-2)^{1/2}}$,

$g''(s) = \tfrac{3}{2}\left(-\tfrac{1}{2}\right)(3s-2)^{-3/2}(3) = -\tfrac{9}{4}(3s-2)^{-3/2} = -\dfrac{9}{4(3s-2)^{3/2}}$, and

$g'''(s) = \tfrac{27}{8}(3s-2)^{-5/2}(3) = \tfrac{81}{8}(3s-2)^{-5/2} = \dfrac{81}{8(3s-2)^{5/2}}$.

26. $g(t) = \sqrt{2t+3}$, so $g'(t) = \frac{1}{2}(2t+3)^{-1/2}(2) = (2t+3)^{-1/2}$, $g''(t) = -\frac{1}{2}(2t+3)^{-3/2}(2) = -(2t+3)^{-3/2}$, and $g'''(t) = \frac{3}{2}(2t+3)^{-5/2}(2) = \dfrac{3}{(2t+3)^{5/2}}$.

27. $f(x) = (2x-3)^4$, so $f'(x) = 4(2x-3)^3(2) = 8(2x-3)^3$, $f''(x) = 24(2x-3)^2(2) = 48(2x-3)^2$, and $f'''(x) = 96(2x-3)(2) = 192(2x-3)$.

28. $g(t) = \left(\frac{1}{2}t^2-1\right)^5$, so $g'(t) = 5\left(\frac{1}{2}t^2-1\right)^4(t) = 5t\left(\frac{1}{2}t^2-1\right)^4$,

$$g''(t) = 5\left(\tfrac{1}{2}t^2-1\right)^4 + 5t(4)\left(\tfrac{1}{2}t^2-1\right)^3(t) = 5\left(\tfrac{1}{2}t^2-1\right)^3\left[\left(\tfrac{1}{2}t^2-1\right)+4t^2\right] = \tfrac{5}{2}\left(9t^2-2\right)\left(\tfrac{1}{2}t^2-1\right)^3, \text{ and}$$

$$g'''(t) = \tfrac{5}{2}\left[18t\left(\tfrac{1}{2}t^2-1\right)^3 + \left(9t^2-2\right)3\left(\tfrac{1}{2}t^2-1\right)^2(t)\right] = \tfrac{15}{2}t\left(\tfrac{1}{2}t^2-1\right)^2\left[6\left(\tfrac{1}{2}t^2-1\right)+\left(9t^2-2\right)\right]$$

$$= 30t\left(3t^2-2\right)\left(\tfrac{1}{2}t^2-1\right)^2.$$

29. Its velocity at any time t is $v(t) = \frac{d}{dt}\left(16t^2\right) = 32t$. The hammer strikes the ground when $16t^2 = 256$ or $t = 4$ (we reject the negative root). Therefore, its velocity at the instant it strikes the ground is $v(4) = 32(4) = 128$ ft/sec. Its acceleration at time t is $a(t) = \frac{d}{dt}(32t) = 32$. In particular, its acceleration at $t = 4$ is 32 ft/sec^2.

30. $s(t) = 20t + 8t^2 - t^3$, so $s'(t) = 20 + 16t - 3t^2$ and $s''(t) = 16 - 6t$. In particular, $s''\left(\frac{8}{3}\right) = 16 - 6\left(\frac{8}{3}\right) = 16 - \frac{48}{3} = 0$. We conclude that the acceleration of the car at $t = \frac{8}{3}$ seconds is zero and that the car will start to decelerate at that point in time.

31. $P(t) = 0.38t^2 + 1.3t + 3$.

- **a.** The projected percentage is $P(5) = 0.38(5)^2 + 1.3(5) + 3 = 19$, or 19%.
- **b.** $P'(t) = 0.76t + 1.3$, so the percentage of vehicles with transmissions that have 7 or more speeds is projected to be changing at the rate of $P'(5) = 0.76(5) + 1.3 = 5.1$, or 5.1% per year (in 2015).
- **c.** $P''(15) = 0.76$, so the rate of increase in vehicles with such transmissions is itself increasing at the rate of 0.76% per year per year in 2025.

32. $N(t) = 0.00525t^2 + 0.075t + 4.7$.

- **a.** The projected number of people of age 65 and over with Alzheimer's disease in the U.S. is projected to be $N(2) = 0.00525(2)^2 + 0.075(2) + 4.7 = 4.871$, or 4.871 million, in 2030.
- **b.** $N'(t) = 0.0105t + 0.075$, so the number of patients is projected to be growing at the rate of $N'(2) = 0.0105(2) + 0.075 = 0.096$, or 96,000 per decade, in 2030.
- **c.** $N''(t) = 0.0105$, so the rate of growth is projected to be growing at the rate of 10,500 per decade per decade.

33. $N(t) = -0.1t^3 + 1.5t^2 + 100$.

- **a.** $N'(t) = -0.3t^2 + 3t = 0.3t(10-t)$. Because $N'(t) > 0$ for $t = 0, 1, 2, \ldots, 8$, it is evident that $N(t)$ (and therefore the crime rate) was increasing from 2006 through 2014.
- **b.** $N''(t) = -0.6t + 3 = 0.6(5-t)$. Now $N''(4) = 0.6 > 0$, $N''(5) = 0$, $N''(6) = -0.6 < 0$, $N''(7) = -1.2 < 0$, and $N''(8) = -1.8 < 0$. This shows that the rate of the rate of change was decreasing beyond $t = 5$ (in the year 2011). This indicates that the program was working.

34. $G(t) = -0.2t^3 + 2.4t^2 + 60$.

a. $G'(t) = -0.6t^2 + 4.8t = 0.6t(8 - t)$, so $G'(1) = 4.2$, $G'(2) = 7.2$, $G'(3) = 9$, $G'(4) = 9.6$, $G'(5) = 9$, $G'(6) = 7.2$, $G'(7) = 4.2$, and $G'(8) = 0$.

b. $G''(t) = -1.2t + 4.8 = 1.2(4 - t)$, so $G''(1) = 3.6$, $G''(2) = 2.4$, $G''(3) = 1.2$, $G''(4) = 0$, $G''(5) = -1.2$, $G''(6) = -2.4$, $G''(7) = -3.6$, and $G''(8) = -4.8$.

c. Our calculations show that the GDP is increasing at an increasing rate in the first five years. Even though the GDP continues to rise from that point on, the negativity of $G''(t)$ shows that the rate of increase is slowing down.

35. $S(t) = 4t^3 + 2t^2 + 300t$, so $S(6) = 4(6)^3 + 2(6)^2 + 300(6) = 2736$. This says that 6 months after the grand opening of the store, monthly LP sales are projected to be 2736 units.
$S'(t) = 12t^2 + 4t + 300$, so $S'(6) = 12(6)^2 + 4(6) + 300 = 756$. Thus, monthly sales are projected to be increasing by 756 units per month.
$S''(t) = 24t + 4$, so $S''(6) = 24(6) + 4 = 148$. This says that the rate of increase of monthly sales is itself increasing at the rate of 148 units per month per month.

36. a. $f(t) = -0.2176t^3 + 1.962t^2 - 2.833t + 29.4$, so the median age of the population in the year 2000 was $f(4) = -0.2176(4)^3 + 1.962(4)^2 - 2.833(4) + 29.4 \approx 35.53$, or approximately 35.5 years old.

b. $f'(t) = -0.6528t^2 + 3.924t - 2.833$, so the median age of the population in the year 2000 was changing at the rate of $-0.6528(4)^2 + 3.924(4) - 2.833 = 2.4182$; that is, it was increasing at the rate of approximately 2.4 years of age per decade.

c. $f''(t) = -1.3056t + 3.924$, so the rate at which the median age of the population was changing in the year was $f''(4) = -1.3056(4) + 3.924 = -1.2984$. That is, the rate of change was decreasing at the rate of approximately 1.3 years of age per decade per decade.

37. $P(t) = 0.0004t^3 + 0.0036t^2 + 0.8t + 12$, so $P'(t) = 0.00012t^2 + 0.0072t + 0.8$. Thus, $P'(t) \geq 0.8$ for $0 \leq t \leq 13$. $P''(t) = 0.00024t + 0.0072$, and for $0 \leq t \leq 13$, $P''(t) > 0$. This means that the proportion of the U.S. population that was obese was increasing at an increasing rate from 1991 through 2004.

38. a. $h(t) = \frac{1}{16}t^4 - t^3 + 4t^2$, so $h'(t) = \frac{1}{4}t^3 - 3t^2 + 8t$.

b. $h'(0) = 0$, or 0 ft/sec. $h'(4) = \frac{1}{4}(64) - 3(16) + 8(4) = 0$, or 0 ft/sec, and $h'(8) = \frac{1}{4}(8)^3 - 3(64) + 8(8) = 0$, or 0 ft/sec.

c. $h''(t) = \frac{3}{4}t^2 - 6t + 8$.

d. $h''(0) = 8$ ft/sec^2, $h''(4) = \frac{3}{4}(16) - 6(4) + 8 = -4$ ft/sec^2, and $h''(8) = \frac{3}{4}(64) - 6(8) + 8 = 8$ ft/sec^2.

e. $h(0) = 0$ ft, $h(4) = \frac{1}{16}(4)^4 - (4)^3 + 4(4)^2 = 16$ ft, and $h(8) = \frac{1}{16}(8)^4 - (8)^3 + 4(8)^2 = 0$ ft.

39. $A(t) = -0.00006t^5 + 0.00468t^4 - 0.1316t^3 + 1.915t^2 - 17.63t + 100$, so $A'(t) = -0.0003t^4 + 0.01872t^3 - 0.3948t^2 + 3.83t - 17.63$ and $A''(t) = -0.0012t^3 + 0.05616t^2 - 0.7896t + 3.83$. Thus, $A'(10) = -3.09$ and $A''(10) = 0.35$. Our calcutations show that 10 minutes after the start of the test, the smoke remaining is decreasing at a rate of 3.09% per minute, but the rate at which the rate of smoke is decreasing is decreasing at the rate of 0.35 percent per minute per minute.

40. $P(t) = 33.55(t + 5)^{0.205}$, so $P'(t) = 33.55(0.205)(t + 5)^{-0.795} = 6.87775(t + 5)^{-0.795}$ and $P''(t) = 6.87775(-0.795)(t + 5)^{-1.795} = -5.46781125(t + 5)^{-1.795}$. Thus, $P''(20) = -5.46781125(20 + 5)^{-1.795} \approx -0.017$, which says that the rate of the rate of change of such mothers is decreasing at the rate of 0.02%/yr^2.

41. $f(t) = 10.72\,(0.9t + 10)^{0.3}$, so $f'(t) = 10.72\,(0.3)\,(0.9t + 10)^{-0.7}\,(0.9) = 2.8944\,(0.9t + 10)^{-0.7}$ and $f''(t) = 2.8944\,(-0.7)\,(0.9t + 10)^{-1.7}\,(0.9) = -1.823472\,(0.9t + 10)^{-1.7}$. Thus, $f''(10) = -1.823472\,(19)^{-1.7} \approx -0.01222$, which says that the rate of the rate of change of the population is decreasing at the rate of 0.01%/yr^2.

42. False. If f has derivatives of order two at $x = a$, then $f''(a) = \left[f'(x)\right]'\Big|_{x=a}$.

43. True. If $h = fg$ where f and g have derivatives of order 2, then $h"(x) = f"(x)\,g(x) + 2f'(x)\,g'(x) + f(x)\,g"(x)$.

44. True. If $f(x)$ is a polynomial function of degree n, then $f^{(n+1)}(x) = 0$.

45. True. Suppose $P(t)$ represents the population of bacteria at time t and suppose $P'(t) > 0$ and $P''(t) < 0$. Then the population is increasing at time t, but at a decreasing rate.

46. True. Using the Chain Rule, $h'(x) = f'(2x) \cdot \frac{d}{dx}(2x) = f'(2x) \cdot 2 = 2f'(2x)$. Using the Chain Rule again, $h''(x) = 2f''(2x) \cdot 2 = 4f''(2x)$.

47. $\overline{C}(x) = \dfrac{C(x)}{x}$, so $\overline{C}'(x) = \dfrac{xC'(x) - C(x) \cdot 1}{x^2} = \dfrac{xC'(x) - C(x)}{x^2}$ and

$$\overline{C}''(x) = \frac{x^2\left[xC''(x) + C'(x) - C'(x)\right] - \left[xC'(x) - C(x)\right]2x}{x^4} = \frac{x^3C''(x) - 2x^2C'(x) + 2xC(x)}{x^4}$$
$$= \frac{C''(x)}{x} - \frac{2C'(x)}{x^2} + \frac{2C(x)}{x^3}.$$

48. $f'(x) = \frac{7}{3}x^{5/3}$ and $f''(x) = \frac{35}{9}x^{2/3}$, so f' and f''exist everywhere. However, $f'''(x) = \frac{70}{27}x^{-1/3} = \dfrac{70}{27x^{1/3}}$ is not defined at $x = 0$.

49. Consider the function $f(x) = x^{(2n+1)/2} = x^{n+(1/2)}$. We calculate $f'(x) = \left(n + \frac{1}{2}\right)x^{n-(1/2)}$, $f''(x) = \left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right)x^{n-(3/2)}, \ldots, f^{(n)}(x) = \left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right)\cdots\frac{3}{2}x^{1/2}$, and $f^{(n+1)}(x) = \left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right)\cdots\frac{1}{2}x^{-1/2}$. The first n derivatives exist at $x = 0$, but the $(n + 1)$st derivative fails to be defined there.

50. Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$. Then $P'(x) = na_nx^{n-1} + (n-1)\,a_{n-1}x^{n-2} + \cdots a_1$. Eventually, we calculate $P^{(n)}(x) = a_n$, and so $P^{(n+1)}(x) = P^{(n+2)}(x) = P^{(n+3)}(x) = \cdots = 0$. Thus, P has derivatives of all orders.

Using Technology page 221

1. -18

2. 425.25

3. 15.2762

4. 128.7540

5. -0.6255

6. -13.9463

7. 0.1973

8. -0.0163

3.6 Implicit Differentiation and Related Rates

Concept Questions page 230

1. a. We differentiate both sides of $F(x, y) = 0$ with respect to x, then solve for dy/dx.

b. The Chain Rule is used to differentiate any expression involving the dependent variable y.

2. $xg(y) + yf(x) = 0$. Differentiating both sides with respect to x gives $xg'(y)\,y' + g(y) + y'f(x) + yf'(x) = 0$, so $\left[xg'(y) + f(x)\right]y' = -\left[g(y) + yf'(x)\right]$, and finally $y' = -\dfrac{g(y) + yf'(x)}{f(x) + xg'(y)}$

3. Suppose x and y are two variables that are related by an equation. Furthermore, suppose x and y are both functions of a third variable t. (Normally t represents time.) Then a related rates problem involves finding dx/dt or dy/dt.

4. See page 228 in the text.

Exercises page 231

1. a. Solving for y in terms of x, we have $y = -\frac{1}{2}x + \frac{5}{2}$. Therefore, $y' = -\frac{1}{2}$.

b. Next, differentiating $x + 2y = 5$ implicitly, we have $1 + 2y' = 0$, or $y' = -\frac{1}{2}$.

2. a. Solving for y in terms of x, we have $y = -\frac{3}{4}x + \frac{3}{2}$. Therefore, $y' = -\frac{3}{4}$.

b. Next, differentiating $3x + 4y = 6$ implicitly, we obtain $3 + 4y' = 0$, or $y' = -\frac{3}{4}$.

3. a. $xy = 1$, $y = \dfrac{1}{x}$, and $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$.

b. $x\dfrac{dy}{dx} + y = 0$, so $x\dfrac{dy}{dx} = -y$ and $\dfrac{dy}{dx} = -\dfrac{y}{x} = \dfrac{-1/x}{x} = -\dfrac{1}{x^2}$.

4. a. Solving for y, we have $y(x-1) = 1$ or $y = (x-1)^{-1}$. Therefore, $y' = -(x-1)^{-2} = -\dfrac{1}{(x-1)^2}$.

b. Next, differentiating $xy - y - 1 = 0$ implicitly, we obtain $y + xy' - y' = 0$, or $y'(x-1) = -y$, so $y' = -\dfrac{y}{x-1} = -\dfrac{1}{(x-1)^2}$.

5. $x^3 - x^2 - xy = 4$.

a. $-xy = 4 - x^3 + x^2$, so $y = -\dfrac{4}{x} + x^2 - x$ and $\dfrac{dy}{dx} = \dfrac{4}{x^2} + 2x - 1$.

b. $x^3 - x^2 - xy = 4$, so $-x\dfrac{dy}{dx} = -3x^2 + 2x + y$, and therefore

$$\frac{dy}{dx} = 3x - 2 - \frac{y}{x} = 3x - 2 - \frac{1}{x}\left(-\frac{4}{x} + x^2 - x\right) = 3x - 2 + \frac{4}{x^2} - x + 1 = \frac{4}{x^2} + 2x - 1.$$

6. $x^2y - x^2 + y - 1 = 0$.

a. $\left(x^2 + 1\right)y = 1 + x^2$, or $y = \dfrac{1 + x^2}{1 + x^2} = 1$. Therefore, $\dfrac{dy}{dx} = 0$.

b. Differentiating implicitly, $x^2y' + 2xy - 2x + y' = 0$, so $\left(x^2 + 1\right)y' = 2x(1 - y)$, and thus $y' = \dfrac{2x(1-y)}{x^2+1}$.

But from part (a), we know that $y = 1$, so $y' = \dfrac{2x(1-1)}{x^2+1} = 0$.

7. a. $\dfrac{x}{y} - x^2 = 1$ is equivalent to $\dfrac{x}{y} = x^2 + 1$, or $y = \dfrac{x}{x^2+1}$. Therefore, $y' = \dfrac{\left(x^2+1\right) - x\,(2x)}{\left(x^2+1\right)^2} = \dfrac{1-x^2}{\left(x^2+1\right)^2}$.

b. Next, differentiating the equation $x - x^2y = y$ implicitly, we obtain $1 - 2xy - x^2y' = y'$, $y'\left(1+x^2\right) = 1 - 2xy$, and thus $y' = \dfrac{1-2xy}{\left(1+x^2\right)}$. This may also be written in the form $-2y^2 + \dfrac{y}{x}$. To show that this is equivalent to the results obtained earlier, use the earlier value of y to get $y' = \dfrac{1 - 2x\left(\dfrac{x}{x^2+1}\right)}{1+x^2} = \dfrac{x^2+1-2x^2}{\left(1+x^2\right)^2} = \dfrac{1-x^2}{\left(1+x^2\right)^2}$.

8. a. $\dfrac{y}{x} - 2x^3 = 4$ is equivalent to $y = 2x^4 + 4x$. Therefore, $y' = 8x^3 + 4$.

b. Next, differentiating the equation $y - 2x^4 = 4x$ implicitly, we obtain $y' - 8x^3 = 4$, and so $y' = 8x^3 + 4$, as obtained earlier.

9. $x^2 + y^2 = 16$. Differentiating both sides of the equation implicitly, we obtain $2x + 2yy' = 0$, and so $y' = -x/y$.

10. $2x^2 + y^2 = 16$, $4x + 2y\dfrac{dy}{dx} = 0$ and $\dfrac{dy}{dx} = -\dfrac{2x}{y}$.

11. $x^2 - 2y^2 = 16$. Differentiating implicitly with respect to x, we have $2x - 4y\dfrac{dy}{dx} = 0$, and so $\dfrac{dy}{dx} = \dfrac{x}{2y}$.

12. $x^3 + y^3 + y - 4 = 0$. Differentiating both sides of the equation implicitly, we obtain $3x^2 + 3y^2y' + y' = 0$ or $y'\left(3y^2+1\right) = -3x^2$. Therefore, $y' = -\dfrac{3x^2}{3y^2+1}$.

13. $x^2 - 2xy = 6$. Differentiating both sides of the equation implicitly, we obtain $2x - 2y - 2xy' = 0$ and so $y' = \dfrac{x-y}{x} = 1 - \dfrac{y}{x}$.

14. $x^2 + 5xy + y^2 = 10$. Differentiating both sides of the equation implicitly, we obtain $2x + 5y + 5xy' + 2yy' = 0$, $2x + 5y + y'\,(5x+2y) = 0$, and so $y' = -\dfrac{2x+5y}{5x+2y}$.

15. $x^2y^2 - xy = 8$. Differentiating both sides of the equation implicitly, we obtain $2xy^2 + 2x^2yy' - y - xy' = 0$, $2xy^2 - y + y'\left(2x^2y - x\right) = 0$, and so $y' = \dfrac{y\,(1-2xy)}{x\,(2xy-1)} = -\dfrac{y}{x}$.

16. $x^2y^3 - 2xy^2 = 5$. Differentiating both sides of the equation implicitly, we obtain $2xy^3 + 3x^2y^2y' - 2y^2 - 4xyy' = 0$, $2y^2\,(xy-1) + xy\,(3xy-4)\,y' = 0$, and so $y' = \dfrac{2y\,(1-xy)}{x\,(3xy-4)}$.

17. $x^{1/2} + y^{1/2} = 1$. Differentiating implicitly with respect to x, we have $\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\dfrac{dy}{dx} = 0$. Therefore, $\dfrac{dy}{dx} = -\dfrac{x^{-1/2}}{y^{-1/2}} = -\dfrac{\sqrt{y}}{\sqrt{x}}$.

18. $x^{1/3} + y^{1/3} = 1$. Differentiating both sides of the equation implicitly, we obtain $\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}y' = 0$, so $y' = -\dfrac{x^{-2/3}}{y^{-2/3}} = -\dfrac{y^{2/3}}{x^{2/3}} = -\left(\dfrac{y}{x}\right)^{2/3}$.

19. $\sqrt{x+y}=x$. Differentiating both sides of the equation implicitly, we obtain $\frac{1}{2}(x+y)^{-1/2}(1+y')=1$, $1+y'=2(x+y)^{1/2}$, and so $y'=2\sqrt{x+y}-1$.

20. $(2x+3y)^{1/3}=x^2$. Differentiating both sides of the equation implicitly, we obtain $\frac{1}{3}(2x+3y)^{-2/3}(2+3y')=2x$, $2+3y'=6x(2x+3y)^{2/3}$, and so $y'=\frac{2}{3}\left[3x(2x+3y)^{2/3}-1\right]$.

21. $\dfrac{1}{x^2}+\dfrac{1}{y^2}=1$. Differentiating both sides of the equation implicitly, we obtain $-\dfrac{2}{x^3}-\dfrac{2}{y^3}y'=0$, or $y'=-\dfrac{y^3}{x^3}$.

22. $\dfrac{1}{x^3}+\dfrac{1}{y^3}=5$. Differentiating both sides of the equation implicitly, we obtain $-\dfrac{3}{x^4}-\dfrac{3}{y^4}y'=0$, or $y'=-\dfrac{y^4}{x^4}$.

23. $\sqrt{xy}=x+y$. Differentiating both sides of the equation implicitly, we obtain $\frac{1}{2}(xy)^{-1/2}(xy'+y)=1+y'$, so $xy'+y=2\sqrt{xy}(1+y')$, $y'(x-2\sqrt{xy})=2\sqrt{xy}-y$, and so $y'=-\dfrac{2\sqrt{xy}-y}{2\sqrt{xy}-x}=\dfrac{2\sqrt{xy}-y}{x-2\sqrt{xy}}$.

24. $\sqrt{xy}=2x+y^2$. Differentiating both sides of the equation implicitly, we obtain $\frac{1}{2}(xy)^{-1/2}(xy'+y)=2+2yy'$, $xy'+y=4\sqrt{xy}+4\sqrt{xy}yy'$, $y'(x-4y\sqrt{xy})=4\sqrt{xy}-y$, and so $y'=\dfrac{4\sqrt{xy}-y}{x-4y\sqrt{xy}}$.

25. $\dfrac{x+y}{x-y}=3x$, or $x+y=3x^2-3xy$. Differentiating both sides of the equation implicitly, we obtain $1+y'=6x-3xy'-3y$, so $y'+3xy'=6x-3y-1$ and $y'=\dfrac{6x-3y-1}{3x+1}$.

26. $\dfrac{x-y}{2x+3y}=2x$, or $x-y=4x^2+6xy$. Differentiating both sides of the equation implicitly, we have $1-y'=8x+6y+6xy'$, so $y'+6xy'=-8x-6y+1$ and $y'=-\dfrac{8x+6y-1}{6x+1}$.

27. $xy^{3/2}=x^2+y^2$. Differentiating implicitly with respect to x, we obtain $y^{3/2}+x\left(\frac{3}{2}\right)y^{1/2}\dfrac{dy}{dx}=2x+2y\dfrac{dy}{dx}$. Multiply both sides by 2 to get $2y^{3/2}+3xy^{1/2}\dfrac{dy}{dx}=4x+4y\dfrac{dy}{dx}$. Then $(3xy^{1/2}-4y)\dfrac{dy}{dx}=4x-2y^{3/2}$, so $\dfrac{dy}{dx}=\dfrac{2(2x-y^{3/2})}{3xy^{1/2}-4y}$.

28. $x^2y^{1/2}=x+2y^3$. Differentiating implicitly with respect to x, we have $2xy^{1/2}+\frac{1}{2}x^2y^{-1/2}y'=1+6y^2y'$, so $4xy+x^2y'=2y^{1/2}+12y^{5/2}y'$, $y'(x^2-12y^{5/2})=-4xy+2y^{1/2}$, and $y'=\dfrac{2\sqrt{y}-4xy}{x^2-12y^{5/2}}$.

29. $(x+y)^3+x^3+y^3=0$. Differentiating implicitly with respect to x, we obtain $3(x+y)^2\left(1+\dfrac{dy}{dx}\right)+3x^2+3y^2\dfrac{dy}{dx}=0$, $(x+y)^2+(x+y)^2\dfrac{dy}{dx}+x^2+y^2\dfrac{dy}{dx}=0$, $\left[(x+y)^2+y^2\right]\dfrac{dy}{dx}=-\left[(x+y)^2+x^2\right]$, and thus $\dfrac{dy}{dx}=-\dfrac{2x^2+2xy+y^2}{x^2+2xy+2y^2}$.

30. $\left(x+y^2\right)^{10}=x^2+25$. Differentiating both sides of this equation with respect to x, we obtain $10\left(x+y^2\right)^9\left(1+2yy'\right)=2x$, so $1+2yy'=\dfrac{2x}{10\left(x+y^2\right)^9}$, $2yy'=\dfrac{2x}{10\left(x+y^2\right)^9}-1$, and $y'=\dfrac{x-5\left(x+y^2\right)^9}{10y\left(x+y^2\right)^9}$.

31. $4x^2+9y^2=36$. Differentiating the equation implicitly, we obtain $8x+18yy'=0$. At the point $(0,2)$, we have $0+36y'=0$, and the slope of the tangent line is 0. Therefore, an equation of the tangent line is $y=2$.

32. $y^2-x^2=16$. Differentiating both sides of this equation implicitly, we obtain $2yy'-2x=0$. At the point $\left(2,2\sqrt{5}\right)$, we have $4\sqrt{5}y'-4=0$, or $y'=m=\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}$. Using the point-slope form of an equation of a line, we have $y=\frac{\sqrt{5}}{5}x+\frac{8\sqrt{5}}{5}$.

33. $x^2y^3-y^2+xy-1=0$. Differentiating implicitly with respect to x, we have $2xy^3+3x^2y^2\dfrac{dy}{dx}-2y\dfrac{dy}{dx}+y+x\dfrac{dy}{dx}=0$. At $(1,1)$, $2+3\dfrac{dy}{dx}-2\dfrac{dy}{dx}+1+\dfrac{dy}{dx}=0$, and so $2\dfrac{dy}{dx}=-3$ and $\dfrac{dy}{dx}=-\dfrac{3}{2}$. Using the point-slope form of an equation of a line, we have $y-1=-\frac{3}{2}(x-1)$, and the equation of the tangent line to the graph of the function f at $(1,1)$ is $y=-\frac{3}{2}x+\frac{5}{2}$.

34. $(x-y-1)^3=x$. Differentiating both sides of the given equation implicitly, we obtain $3(x-y-1)^2\left(1-y'\right)=1$. At the point $(1,-1)$, $3(1+1-1)^2\left(1-y'\right)=1$ or $y'=\frac{2}{3}$. Using the point-slope form of an equation of a line, we have $y+1=\frac{2}{3}(x-1)$ or $y=\frac{2}{3}x-\frac{5}{3}$.

35. $xy=1$. Differentiating implicitly, we have $xy'+y=0$, or $y'=-\dfrac{y}{x}$. Differentiating implicitly once again, we have $xy''+y'+y'=0$. Therefore, $y''=-\dfrac{2y'}{x}=\dfrac{2\left(\frac{y}{x}\right)}{x}=\dfrac{2y}{x^2}$.

36. $x^3+y^3=28$. Differentiating implicitly, we have $3x^2+3y^2y'=0$. Differentiating again, we have $6x+3y^2y''+6y\left(y'\right)^2=0$. Thus, $y''=-\dfrac{2y\left(y'\right)^2+2x}{y^2}$. But $\dfrac{dy}{dx}=-\dfrac{x^2}{y^2}$, and therefore,

$$y''=-\frac{2y\left(\frac{x^4}{y^4}\right)+2x}{y^2}=-\frac{2\left(\frac{x^4}{y^3}+x\right)}{y^2}=-\frac{2x\left(x^3+y^3\right)}{y^5}.$$

37. $y^2-xy=8$. Differentiating implicitly we have $2yy'-y-xy'=0$, and so $y'=\dfrac{y}{2y-x}$. Differentiating implicitly again, we have $2\left(y'\right)^2+2yy''-y'-y'-xy''=0$, so $y''=\dfrac{2y'-2\left(y'\right)^2}{2y-x}=\dfrac{2y'\left(1-y'\right)}{2y-x}$. Then

$$y''=\frac{2\left(\frac{y}{2y-x}\right)\left(1-\frac{y}{2y-x}\right)}{2y-x}=\frac{2y\,(2y-x-y)}{(2y-x)^3}=\frac{2y\,(y-x)}{(2y-x)^3}.$$

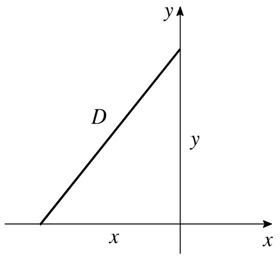
38. Differentiating $x^{1/3}+y^{1/3}=1$ implicitly, we have $\frac{1}{3}x^{-2/3}+\frac{1}{3}y^{-2/3}y'=0$ and $y'=-\frac{y^{2/3}}{x^{2/3}}$. Differentiating implicitly once again, we have

$$y''=-\frac{x^{2/3}\left(\frac{2}{3}\right)y^{-1/3}y'-y^{2/3}\left(\frac{2}{3}\right)x^{-1/3}}{x^{4/3}}=\frac{-\frac{2}{3}x^{2/3}y^{-1/3}\left(-\frac{y^{2/3}}{x^{2/3}}\right)+\frac{2}{3}y^{2/3}x^{-1/3}}{x^{4/3}}=\frac{2}{3}\left(\frac{y^{1/3}+y^{2/3}x^{-1/3}}{x^{4/3}}\right)$$
$$=\frac{2y^{1/3}\left(x^{1/3}+y^{1/3}\right)}{3x^{4/3}x^{1/3}}=\frac{2y^{1/3}}{3x^{5/3}}.$$

39. a. Differentiating the given equation with respect to t, we obtain $\frac{dV}{dt}=\pi r^2\frac{dh}{dt}+2\pi rh\frac{dr}{dt}=\pi r\left(r\frac{dh}{dt}+2h\frac{dr}{dt}\right)$.

b. Substituting $r=2$, $h=6$, $\frac{dr}{dt}=0.1$, and $\frac{dh}{dt}=0.3$ into the expression for $\frac{dV}{dt}$, we obtain$\frac{dV}{dt}=\pi(2)[2(0.3)+2(6)(0.1)]=3.6\pi$, and so the volume is increasing at the rate of 3.6π in^3/sec.

40. Let $(x,0)$ and $(0,y)$ denote the position of the two cars. Then $D^2=x^2+y^2$. Differentiating with respect to t, we obtain $2D\frac{dD}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}$, so $D\frac{dD}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt}$. When $t=4$, $x=-20$, and $y=28$, $\frac{dx}{dt}=-9$ and $\frac{dy}{dt}=11$. Therefore, $\left(\sqrt{(-20)^2+(28)^2}\right)\frac{dD}{dt}=(-20)(-9)+(28)(11)=488$, and so $\frac{dD}{dt}=\frac{488}{\sqrt{1184}}=14.18$ ft/sec. Thus, the distance is changing at the rate of 14.18 ft/sec.



41. We are given $\frac{dp}{dt}=2$ and wish to find $\frac{dx}{dt}$ when $x=9$ and $p=63$. Differentiating the equation $p+x^2=144$ with respect to t, we obtain $\frac{dp}{dt}+2x\frac{dx}{dt}=0$. When $x=9$, $p=63$, and $\frac{dp}{dt}=2$, we have $2+2(9)\frac{dx}{dt}=0$, and so and $\frac{dx}{dt}=-\frac{1}{9}\approx-0.111$. Thus, the quantity demanded is decreasing at the rate of approximately 111 tires per week.

42. $p=\frac{1}{2}x^2+48$. Differentiating implicitly, we have $\frac{dp}{dt}-x\frac{dx}{dt}=0$, so $-x\frac{dx}{dt}=-\frac{dp}{dt}$, and thus $\frac{dx}{dt}=\frac{dp/dt}{x}$. When $x=6$, $p=66$, and $\frac{dp}{dt}=-3$, we have $\frac{dx}{dt}=-\frac{3}{6}=-\frac{1}{2}$, or $\left(-\frac{1}{2}\right)(1000)=-500$ tires/week.

43. $100x^2+9p^2=3600$. Differentiating the given equation implicitly with respect to t, we have $200x\frac{dx}{dt}+18p\frac{dp}{dt}=0$. Next, when $p=14$, the given equation yields $100x^2+9(14)^2=3600$, so $100x^2=1836$, or $x\approx4.2849$. When $p=14$, $\frac{dp}{dt}=-0.15$, and $x\approx4.2849$, we have $200(4.2849)\frac{dx}{dt}+18(14)(-0.15)=0$, and so $\frac{dx}{dt}\approx0.0441$. Thus, the quantity demanded is increasing at the rate of approximately 44 headphones per week.

44. $625p^2-x^2=100$. Differentiating the given equation implicitly with respect to t, we have $1250p\frac{dp}{dt}-2x\frac{dx}{dt}=0$. To find p when $x=25$, we solve the equation $625p^2-625=100$, obtaining $p=\sqrt{\frac{725}{625}}\approx1.0770$. Therefore, $1250(1.077)(-0.02)-2(25)\frac{dx}{dt}=0$, and so $\frac{dx}{dt}=-0.5385$. We conclude that the supply is falling at the rate of 539 dozen eggs per week.

45. Differentiating $625p^2-x^2=100$ implicitly, we have $1250p\frac{dp}{dt}-2x\frac{dx}{dt}=0$. When $p=1.0770$, $x=25$, and $\frac{dx}{dt}=-1$, we find that $1250(1.077)\frac{dp}{dt}-2(25)(-1)=0$, and so $\frac{dp}{dt}=-\frac{50}{1250(1.077)}=-0.037$. We conclude that the price is decreasing at the rate of 3.7 cents per carton.

46. $p = -0.01x^2 - 0.1x + 6$. Differentiating the given equation with respect to p, we obtain $1 = -0.02x\frac{dx}{dp} - 0.1\frac{dx}{dp} = -(0.02x + 0.1)\frac{dx}{dp}$. When $x = 10$, we have $1 = -[0.02(10) + 0.1]\frac{dx}{dp}$, so $\frac{dx}{dp} = -\frac{1}{0.3} = -\frac{10}{3}$. Also, for this value of x, $p = -0.01(100) - 0.1(10) + 6 = 4$. Therefore, for these values of x and p, $E(p) = -\dfrac{pf'(p)}{f(p)} = -\dfrac{p\frac{dx}{dp}}{f(p)} = -\dfrac{4\left(-\frac{10}{3}\right)}{10} = \dfrac{4}{3} > 1$, and so the demand is elastic.

47. $p = -0.01x^2 - 0.2x + 8$. Differentiating the given equation implicitly with respect to p, we have $1 = -0.02x\dfrac{dx}{dp} - 0.2\dfrac{dx}{dp} = -[0.02x + 0.2]\dfrac{dx}{dp}$, so $\dfrac{dx}{dp} = -\dfrac{1}{0.02x + 0.2}$. When $x = 15$, $p = -0.01(15)^2 - 0.2(15) + 8 = 2.75$, and so and $\dfrac{dx}{dp} = -\dfrac{1}{0.02(15) + 0.2} = -2$. Therefore, $E(p) = -\dfrac{pf'(p)}{f(p)} = -\dfrac{(2.75)(-2)}{15} \approx 0.37 < 1$, and the demand is inelastic.

48. a. The required output is $Q(16, 81) = 5\left(16^{1/4}\right)\left(81^{3/4}\right) = 270$, or \$270,000.

b. Rewriting $5x^{1/4}y^{3/4} = 270$ as $x^{1/4}y^{3/4} = 54$ and differentiating implicitly, we have $\frac{1}{4}x^{-3/4}y^{3/4} + x^{1/4}\left(\frac{3}{4}y^{-1/4}\dfrac{dy}{dx}\right) = 0$, so $\dfrac{dy}{dx} = -\frac{1}{4}x^{-3/4}y^{3/4}\left(\frac{4}{3}x^{-1/4}y^{1/4}\right) = -\dfrac{y}{3x}$. If $x = 16$ and $y = 81$, then $\dfrac{dy}{dx} = -\dfrac{81}{3 \cdot 16} = -1.6875$. Thus, to keep the output constant at \$270,000, the amount spent on capital should decrease by \$1687.50 per \$1000 in labor spending. The MRTS is \$1687.50 per thousand dollars.

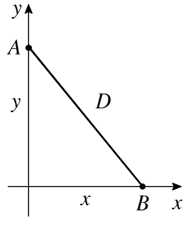
49. a. The required output is $Q(32, 243) = 20\left(32^{3/5}\right)\left(243^{2/5}\right) = 1440$, or \$1440 billion.

b. Differentiating $20x^{3/5}y^{2/5} = 1440$ implicitly with respect to x, we have $20\left(\frac{3}{5}x^{-2/5}y^{2/5}\right) + 20\left(x^{3/5}\right)\left(\frac{2}{5}y^{-3/5}\dfrac{dy}{dx}\right) = 0$, so $\dfrac{dy}{dx} = -\frac{3}{5}x^{-2/5}y^{2/5}\left(\frac{5}{2}x^{-3/5}y^{3/5}\right) = -\dfrac{3y}{2x}$. If $x = 32$ and $y = 243$, then $\dfrac{dy}{dx} = -\dfrac{3 \cdot 243}{2 \cdot 32} \approx -11.39$, so the amount spent on capital should decrease by approximately \$11.4 billion. The MRTS is \$11.4 billion per billion dollars.

50. $V = x^3$, so $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$. When $x = 5$ and $\frac{dx}{dt} = 0.1$, we have $\frac{dV}{dt} = 3(25)(0.1) = 7.5\ \text{in}^3/\text{sec}$.

51. $A = \pi r^2$. Differentiating with respect to t, we obtain $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$. When the radius of the circle is 60 ft and increasing at the rate of $\frac{1}{2}$ ft/sec, $\frac{dA}{dt} = 2\pi(60)\left(\frac{1}{2}\right) = 60\pi\ \text{ft}^2/\text{sec}$. Thus, the area is increasing at a rate of approximately $188.5\ \text{ft}^2/\text{sec}$.

52. Let D denote the distance between the two ships, x the distance that ship A traveled north, and y the distance that ship B traveled east. Then $D^2 = x^2 + y^2$. Differentiating implicitly, we have $2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$, so $D\frac{dD}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$. At 1 p.m., $x = 12$ and $y = 15$, so $\sqrt{144 + 225}\frac{dD}{dt} = (12)(12) + (15)(15)$. Thus, $\frac{dD}{dt} = \frac{369}{\sqrt{369}} \approx 19.21$ ft/sec.

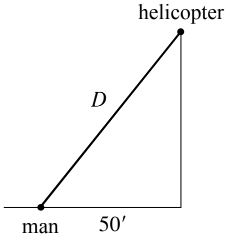


53. $A = \pi r^2$, so $r = \left(\frac{A}{\pi}\right)^{1/2}$. Differentiating with respect to t, we obtain $\frac{dr}{dt} = \frac{1}{2}\left(\frac{A}{\pi}\right)^{-1/2}\frac{dA}{dt}$. When the area of the spill is 1600π ft^2 and increasing at the rate of 80π ft^2/sec, $\frac{dr}{dt} = \frac{1}{2}\left(\frac{1600\pi}{\pi}\right)^{-1/2}(80\pi) = \pi$ ft/sec. Thus, the radius is increasing at the rate of approximately 3.14 ft/sec.

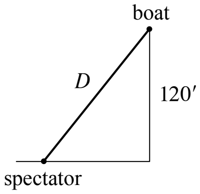
54. Let D denote the distance between the two cars, x the distance traveled by the car heading east, and y the distance traveled by the car heading north. Then $D^2 = x^2 + y^2$. Differentiating with respect to t, we have $2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$, so $D\frac{dD}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$. Notice also that $\frac{dx}{dt} = 2t + 1$ and $\frac{dy}{dt} = 2t + 3$. When $t = 5$, $x = 5^2 + 5 = 30$, $y = 5^2 + 3(5) = 40$, $\frac{dx}{dt} = 2(5) + 1 = 11$, and $\frac{dy}{dt} = 2(5) + 3 = 13$, so $\frac{dD}{dt} = \frac{(30)(11) + (40)(13)}{\sqrt{900 + 1600}} = 17$ ft/sec.

55. Let $(x, 0)$ and $(0, y)$ denote the position of the two cars at time t. Then $y = t^2 + 2t$. Now $D^2 = x^2 + y^2$ so $2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$ and thus $D\frac{dD}{dt} = x\frac{dx}{dt} + \left(t^2 + 2t\right)(2t + 2)$. When $t = 4$, we have $x = -20$, $\frac{dx}{dt} = -9$, and $y = 24$, so $\sqrt{(-20)^2 + (24)^2}\frac{dD}{dt} = (-20)(-9) + (24)(10)$, and therefore $\frac{dD}{dt} = \frac{420}{\sqrt{976}} \approx 13.44$. That is, the distance is changing at approximately 13.44 ft/sec.

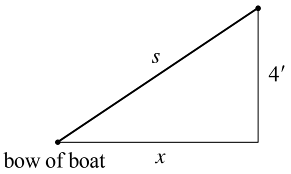
56. $D^2 = x^2 + (50)^2 = x^2 + 2500$. Differentiating implicitly with respect to t, we have $2D\frac{dD}{dt} = 2x\frac{dx}{dt}$, so $\frac{dD}{dt} = \frac{x\frac{dx}{dt}}{D}$. When $x = 120$ and $\frac{dx}{dt} = 44$, $\frac{dD}{dt} = \frac{(120)(44)}{\sqrt{(120)^2 + (50)^2}} \approx 40.6$, and so the distance between the helicopter and the man is increasing at the rate of 40.6 ft/sec.



57. Referring to the diagram, we see that $D^2 = 120^2 + x^2$. Differentiating this last equation with respect to t, we have $2D\frac{dD}{dt} = 2x\frac{dx}{dt}$, and so $\frac{dD}{dt} = \frac{x\frac{dx}{dt}}{D}$. When $x = 50$ and $\frac{dx}{dt} = 20$, $D = \sqrt{120^2 + 50^2} = 130$ and $\frac{dD}{dt} = \frac{(20)(50)}{130} \approx 7.69$, or 7.69 ft/sec.



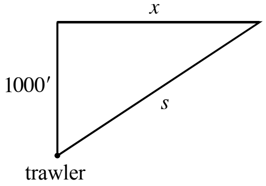
58. By the Pythagorean Theorem, $s^2 = x^2 + 4^2 = x^2 + 16$. We want to find $\frac{dx}{dt}$ when $x = 25$, given that $\frac{ds}{dt} = -3$. Differentiating both sides of the equation with respect to t yields $2s\frac{ds}{dt} = 2x\frac{dx}{dt}$, or $\frac{dx}{dt} = \frac{s\frac{ds}{dt}}{x}$. Now when $x = 25$, $s^2 = 25^2 + 16 = 641$ and $s = \sqrt{641}$. Therefore, when $x = 25$, we have $\frac{dx}{dt} = \frac{\sqrt{641}(-3)}{25} \approx -3.04$; that is, the boat is approaching the dock at the rate of approximately 3.04 ft/sec.



59. Let V and S denote its volume and surface area. Then we are given that $\frac{dV}{dt} = kS$, where k is the constant of proportionality. But from $V = \frac{4}{3}\pi r^3$, we find, upon differentiating both sides with respect to t, that $\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2 \frac{dr}{dt} = kS = k\left(4\pi r^2\right)$. Therefore, $\frac{dr}{dt} = k$ a constant.

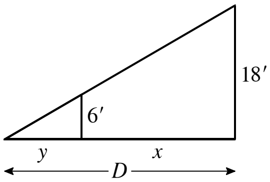
60. Let V denote the volume of the soap bubble and r its radius. Then, we are given $\frac{dV}{dt} = 8$. Differentiating the formula $V = \frac{4}{3}\pi r^3$ with respect to t, we find $\frac{dV}{dt} = \left(\frac{4}{3}\right)\left(3\pi r^2 \frac{dr}{dt}\right) = 4\pi r^2 \frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}$. When $r = 10$, we have $\dfrac{dr}{dt} = \dfrac{8}{4\pi\left(10^2\right)} \approx 0.0064$. Thus, the radius is increasing at the rate of approximately 0.0064 cm/sec. From $s = 4\pi r^2$, we find $\frac{ds}{dt} = 4\pi\,(2r)\,\frac{dr}{dt} = 8\pi r \frac{dr}{dt}$. Therefore, when $r = 10$, we have $\frac{ds}{dt} = 8\pi\,(10)\,(0.0064) \approx 1.6$. Thus, the surface area is increasing at the rate of approximately 1.6 cm^2/sec.

61. We are given that $\frac{dx}{dt} = 264$. Using the Pythagorean Theorem, $s^2 = x^2 + 1000^2 = x^2 + 1{,}000{,}000$. We want to find $\dfrac{ds}{dt}$ when $s = 1500$. Differentiating both sides of the equation with respect to t, we have $2s\dfrac{ds}{dt} = 2x\dfrac{dx}{dt}$ and so $\dfrac{ds}{dt} = \dfrac{x\frac{dx}{dt}}{s}$. When $s = 1500$, we have $1500^2 = x^2 + 1{,}000{,}000$, or $x = \sqrt{1{,}250{,}000}$. Therefore, $\dfrac{ds}{dt} = \dfrac{\sqrt{1{,}250{,}000}\cdot(264)}{1500} \approx 196.8$, that is, the aircraft is receding from the trawler at the speed of approximately 196.8 ft/sec.



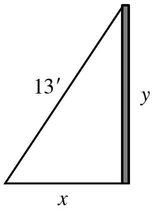
62. The volume V of the water in the pot is $V = \pi r^2 h = \pi\,(16)\,h = 16\pi h$. Differentiating with respect to t, we obtain $\frac{dV}{dt} = 16\pi\frac{dh}{dt}$. Therefore, with $\frac{dh}{dt} = 0.4$, we find $\frac{dV}{dt} = 16\pi\,(0.4) \approx 20.1$; that is, water is being poured into the pot at the rate of approximately 20.1 cm^3/sec.

63. $\dfrac{y}{6} = \dfrac{y+x}{18}$, $18y = 6\,(y+x)$, so $3y = y + x$, $2y = x$, and $y = \frac{1}{2}x$. Thus, $D = y + x = \frac{3}{2}x$. Differentiating implicitly, we have $\frac{dD}{dt} = \frac{3}{2}\cdot\frac{dx}{dt}$, and when $\frac{dx}{dt} = 6$, $\frac{dD}{dt} = \frac{3}{2}\,(6) = 9$, or 9 ft/sec.



64. Differentiating $x^2 + y^2 = 400$ with respect to t gives $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. When $x = 12$, we have $144 + y^2 = 400$, or $y = \sqrt{256} = 16$. Therefore, with $x = 12$, $\frac{dx}{dt} = 5$, and $y = 16$, we find $2\,(12)\,(5) + 2\,(16)\,\frac{dy}{dt} = 0$, or $\frac{dy}{dt} = -3.75$. Thus, the top of the ladder is sliding down the wall at the rate of 3.75 ft/sec.

65. Differentiating $x^2 + y^2 = 13^2 = 169$ with respect to t gives $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. When $x = 12$, we have $144 + y^2 = 169$, or $y = 5$. Therefore, with $x = 12$, $y = 5$, and $\frac{dx}{dt} = 8$, we find $2\,(12)\,(8) + 2\,(5)\,\frac{dy}{dt} = 0$, or $\frac{dy}{dt} = -19.2$. Thus, the top of the ladder is sliding down the wall at the rate of 19.2 ft/sec.



66. Differentiating the equation $2h^{1/2} + \frac{1}{25}t - 2\sqrt{20} = 0$ with respect to t gives $2\left(\frac{1}{2}h^{-1/2}\right)\frac{dh}{dt} + \frac{1}{25} = 0$, or $\frac{dh}{dt} = -\frac{\sqrt{h}}{25}$. Therefore, with $h = 8$, we have $\frac{dh}{dt} = \frac{\sqrt{8}}{25} \approx -0.113$. Thus, the height of the water is decreasing at the rate of approximately 0.11 ft/sec.

67. $P^5V^7 = C$, so $V^7 = CP^{-5}$ and $7V^6\dfrac{dV}{dt} = -5CP^{-6}\dfrac{dP}{dt}$. Therefore, $\dfrac{dV}{dt} = -\dfrac{5C}{7P^6V^6}\dfrac{dP}{dt} = -\dfrac{5P^5V^7}{7P^6V^6}\dfrac{dP}{dt} = -\dfrac{5}{7}\dfrac{V}{P}\dfrac{dP}{dt}$. When $V = 4$ L, $P = 100$ kPa, and $\dfrac{dP}{dt} = -5\,\dfrac{\text{kPa}}{\text{sec}}$, we have $\dfrac{dV}{dt} = -\dfrac{5}{7}\cdot\dfrac{4}{100}(-5) = \dfrac{1}{7}\left(\dfrac{\text{L}}{\text{kPa}}\cdot\dfrac{\text{kPa}}{\text{s}}\right) = \dfrac{1}{7}\,\dfrac{\text{L}}{\text{s}}$.

68. When $v = 2.92 \times 10^8$ and $\dfrac{dv}{dt} = a = 2.42 \times 10^5$, $\dfrac{dm}{dt} = \dfrac{\left(9.11 \times 10^{-31}\right)\left(2.92 \times 10^8\right)\left(2.42 \times 10^5\right)}{\left(2.98 \times 10^8\right)^2\left[1 - \left(\frac{2.92\times10^8}{2.98\times10^8}\right)^2\right]^{3/2}} \approx 9.1 \times 10^{-32}$, so the mass is increasing at the rate of approximately 9.1×10^{-32} kg/sec.

69. False. There are no real numbers x and y such that $x^2 + y^2 = -1$.

70. True. If $-1 \le x < 0$, then $y^2 = \left(\sqrt{1-x^2}\right)^2 = 1 - x^2$, so $x^2 + y^2 = 1$. If $0 \le x \le 1$, then $y^2 = \left(-\sqrt{1-x^2}\right)^2 = 1 - x^2$, so $x^2 + y^2 = 1$.

71. True. Differentiating both sides of the equation with respect to x, we have $\frac{d}{dx}\left[f(x)\,g(y)\right] = \frac{d}{dx}(0)$, so $f(x)\,g'(y)\dfrac{dy}{dx} + f'(x)\,g(y) = 0$, and therefore $\frac{dy}{dx} = -\frac{f'(x)g(y)}{f(x)g'(y)}$, provided $f(x) \neq 0$ and $g'(y) \neq 0$.

72. True. Differentiating both sides of the equation with respect to x, $\frac{d}{dx}\left[f(x) + g(y)\right] = \frac{d}{dx}(0)$, so $f'(x) + g'(y)\frac{dy}{dx} = 0$, and therefore $\frac{dy}{dx} = -\frac{f'(x)}{g'(y)}$.

73. True. If $y = f(x)$, then $\Delta y = f(x + \Delta x) - f(x) \approx f'(x)\,\Delta x$, from which it follows that $f(x + \Delta x) \approx f(x) + f'(x)\,\Delta x$.

74. True. Let $y = f(x) = x^{1/3}$. Then $y' = f'(x) = \frac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$. At $x = a$, $\Delta y = f(a + \Delta x) - f(a) \approx f'(a)\,\Delta x$, so $f(a + \Delta x) \approx f(a) + f'(a)\,\Delta x = a^{1/3} + \dfrac{\Delta x}{3a^{2/3}}$. Letting $\Delta x = h$, we have $(a + h)^{1/3} = f(a + h) \approx a^{1/3} + \dfrac{h}{3a^{2/3}}$.

3.7 Differentials

Concept Questions page 240

1. The differential of x is dx. The differential of y is $dy = f'(x)\,dx$.

2. a. $A = \Delta x$, $B = \Delta y$, and $C = dy$.

b. $f'(x) = \dfrac{dy}{\Delta x}$.

c. From part (b), we see that $dy = f'(x)\,\Delta x$. Because $B \approx C$, $\Delta y \approx f'(x)\,\Delta x = f'(x)\,dx$.

3. Because $\Delta P = P(t_0 + \Delta t) - P(t_0) \approx P'(t_0)\,\Delta t$, we see that $P'(t_0)\,\Delta t$ is an approximation of the change in the population from time t_0 to time $t_0 + \Delta t$.

4. $P(t) = P(t_0 + \Delta t) \approx P(t_0) + P'(t_0)\,\Delta t$.

Exercises page 240

1. $f(x) = 2x^2$ and $dy = 4x\,dx$.

2. $f(x) = 3x^2 + 1$ and $dy = 6x\,dx$.

3. $f(x) = x^3 - x$ and $dy = \left(3x^2 - 1\right)dx$.

4. $f(x) = 2x^3 + x$ and $dy = \left(6x^2 + 1\right)dx$.

5. $f(x) = \sqrt{x+1} = (x+1)^{1/2}$ and $dy = \frac{1}{2}(x+1)^{-1/2}\,dx = \dfrac{dx}{2\sqrt{x+1}}$.

6. $f(x) = 3x^{-1/2}$ and $dy = -\dfrac{3}{2x^{3/2}}\,dx$.

7. $f(t) = 2x^{3/2} + x^{1/2}$ and $dy = \left(3x^{1/2} + \frac{1}{2}x^{-1/2}\right)dx = \frac{1}{2}x^{-1/2}(6x+1)\,dx = \dfrac{6x+1}{2\sqrt{x}}\,dx$.

8. $f(x) = 3x^{5/6} + 7x^{2/3}$ and $dy = \left(\frac{5}{2}x^{-1/6} + \frac{14}{3}x^{-1/3}\right)dx$.

9. $f(x) = x + \dfrac{2}{x}$ and $dy = \left(1 - \dfrac{2}{x^2}\right)dx = \dfrac{x^2 - 2}{x^2}dx$.

10. $f(x) = \dfrac{3}{x-1}$ and $dy = -\dfrac{3}{(x-1)^2}dx$.

11. $f(x) = \dfrac{x-1}{x^2+1}$ and $dy = \dfrac{x^2 + 1 - (x-1)\,2x}{\left(x^2+1\right)^2}\,dx = \dfrac{-x^2 + 2x + 1}{\left(x^2+1\right)^2}\,dx$.

12. $f(x) = \dfrac{2x^2+1}{x+1}$ and $dy = \dfrac{(x+1)(4x) - \left(2x^2+1\right)}{(x+1)^2}\,dx = \dfrac{2x^2 + 4x - 1}{(x+1)^2}\,dx$.

13. $f(x) = \sqrt{3x^2 - x} = \left(3x^2 - x\right)^{1/2}$ and $dy = \frac{1}{2}\left(3x^2 - x\right)^{-1/2}(6x - 1)\,dx = \dfrac{6x-1}{2\sqrt{3x^2 - x}}\,dx$.

14. $f(x) = \left(2x^2 + 3\right)^{1/3}$ and $dy = \frac{1}{3}\left(2x^2+3\right)^{-2/3}(4x)\,dx = \dfrac{4x}{3\left(2x^2+3\right)^{2/3}}\,dx$.

15. $f(x) = x^2 - 1$.

a. $dy = 2x\,dx$.

b. $dy \approx 2(1)(0.02) = 0.04$.

c. $\Delta y = \left[(1.02)^2 - 1\right] - (1 - 1) = 0.0404$.

16. $f(x) = 3x^2 - 2x + 6$

a. $dy = (6x - 2)\,dx$.

b. $dy \approx 10(-0.03) = -0.3$.

c. $\Delta y = \left[3(1.97)^2 - 2(1.97) + 6\right] - \left[3(2)^2 - 2(2) + 6\right] = -0.2973$.

17. $f(x) = \dfrac{1}{x}$.

a. $dy = -\dfrac{dx}{x^2}$.

b. $dy \approx -0.05$.

c. $\Delta y = \frac{1}{-0.95} - \frac{1}{-1} \approx -0.05263$.

18. $f(x) = \sqrt{2x+1} = (2x+1)^{1/2}$.

a. $dy = \frac{1}{2}(2x+1)^{-1/2}(2)\,dx = \dfrac{dx}{\sqrt{2x+1}}$.

b. $dy \approx \frac{0.1}{\sqrt{9}} \approx 0.03333$.

c. $\Delta y = [2(4.1) + 1]^{1/2} - [2(4) + 1]^{1/2} \approx 0.03315$.

19. $y = \sqrt{x}$ and $dy = \dfrac{dx}{2\sqrt{x}}$. Therefore, $\sqrt{10} \approx 3 + \dfrac{1}{2 \cdot \sqrt{9}} \approx 3.167$.

20. $y = \sqrt{x}$ and $dy = \dfrac{dx}{2\sqrt{x}}$. Therefore, $\sqrt{17} \approx 4 + \dfrac{1}{2 \cdot 4} = 4.125$.

21. $y = \sqrt{x}$ and $dy = \dfrac{dx}{2\sqrt{x}}$. Therefore, $\sqrt{49.5} \approx 7 + \dfrac{0.5}{2 \cdot 7} \approx 7.0357$.

22. $y = \sqrt{x}$ and $dy = \dfrac{dx}{2\sqrt{x}}$. Therefore, $\sqrt{99.7} \approx 10 - \dfrac{0.3}{2 \cdot 10} = 9.985$.

23. $y = x^{1/3}$ and $dy = \frac{1}{3}x^{-2/3}\,dx$. Therefore, $\sqrt[3]{7.8} \approx 2 - \dfrac{0.2}{3 \cdot 4} \approx 1.983$.

24. $y = x^{1/4}$ and $dy = \frac{1}{4}x^{-3/4}\,dx$. Therefore, $\sqrt[4]{81.6} \approx 3 + \dfrac{0.6}{4 \cdot 27} \approx 3.0056$.

25. $y = \sqrt{x}$ and $dy = \dfrac{dx}{2\sqrt{x}}$. Therefore, $\sqrt{0.089} = \frac{1}{10}\sqrt{8.9} \approx \frac{1}{10}\left(3 - \frac{0.1}{2 \cdot 3}\right) \approx 0.298$.

26. $y = \sqrt[3]{x}$ and $dy = \dfrac{dx}{3x^{2/3}}$. Therefore, $\sqrt[3]{0.00096} = \dfrac{1}{100}\sqrt[3]{960} \approx \dfrac{1}{100}\left[10 - \dfrac{40}{3(100)}\right] \approx 0.0987$.

27. $y = f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$. Therefore, $\dfrac{dy}{dx} = \dfrac{1}{2}x^{-1/2} - \dfrac{1}{2}x^{-3/2}$, so $dy = \left(\dfrac{1}{2x^{1/2}} - \dfrac{1}{2x^{3/2}}\right)dx$. Letting $x = 4$ and $dx = 0.02$, we find $\sqrt{4.02} + \dfrac{1}{\sqrt{4.02}} - f(4) = f(4.02) - f(4) = \Delta y \approx dy$, so

$$\sqrt{4.02} + \frac{1}{\sqrt{4.02}} \approx f(4) + dy \approx 2 + \frac{1}{2} + \left(\frac{1}{2 \cdot 2} - \frac{1}{16}\right)(0.02) = 2.50375.$$

28. Let $y = f(x) = \dfrac{2x}{x^2+1}$. Then $\dfrac{dy}{dx} = \dfrac{(x^2+1)(2) - 2x(2x)}{(x^2+1)^2} = \dfrac{2(1-x^2)}{(x^2+1)^2}$ and $dy = \dfrac{2(1-x^2)}{(x^2+1)^2}\,dx$.

Letting $x = 5$ and $dx = -0.02$, we find $f(5) - f(4.98) = \dfrac{2(5)}{5^2+1} - \dfrac{2(4.98)}{(4.98)^2+1} = \Delta y \approx dy$, so

$$\frac{2(4.98)}{(4.98)^2+1} \approx \frac{10}{26} - \frac{2(1-5^2)}{(5^2+1)^2}(-0.02) \approx 0.3832.$$

29. The volume of the cube is given by $V = x^3$. Then $dV = 3x^2\,dx$, and when $x = 12$ and $dx = 0.02$, $dV = 3(144)(\pm 0.02) = \pm\, 8.64$. The possible error that might occur in calculating the volume is ± 8.64 cm^3.

30. The area of the cube of side x cm is $S = 6x^2$. Thus, the amount of paint required is approximately $\Delta S = 6(x + \Delta x)^2 - 6x^2 \approx dS = 12x\,dx$. With $x = 30$ and $dx = \Delta x = 0.05$, $\Delta S \approx 12(30)(0.05) = 18$, or approximately 18 cm^3.

31. The volume of the hemisphere is given by $V = \frac{2}{3}\pi r^3$. The amount of rust-proofer needed is $\Delta V = \frac{2}{3}\pi(r + \Delta r)^3 - \frac{2}{3}\pi r^3 \approx dV = \frac{2}{3}(3\pi r^2)\,dr$. Thus, with $r = 60$ and $dr = \dfrac{1}{12}(0.01)$, we have $\Delta V \approx 2\pi(60^2)\left(\frac{1}{12}\right)(0.01) \approx 18.85$. So we need approximately 18.85 ft^3 of rust-proofer.

32. The volume of the tumor is given by $V = \frac{4}{3}\pi r^3$. Then $dV = 4\pi r^2\,dr$. When $r = 1.1$ and $dr = 0.005$, $dV = 4\pi(1.1)^2(\pm 0.005) = \pm 0.076$ cm^3.

33. $dR = \dfrac{d}{dr}\left(k\ell r^{-4}\right)dr = -4k\ell r^{-5}\,dr$. With $\dfrac{dr}{r} = 0.1$, we find $\dfrac{dR}{R} = -\dfrac{4k\ell r^{-5}}{k\ell r^{-4}}\,dr = -4\dfrac{dr}{r} = -4(0.1) = -0.4$. In other words, the resistance will drop by 40%.

34. $f(x) = 640x^{1/5}$ and $df = 128x^{-4/5}\,dx$. When $x = 243$ and $dx = 5$, we have $df = 128(243)^{-4/5}(5) = 128\left(\frac{5}{81}\right) \approx 7.9$, or approximately \$7.9 billion.

35. $f(n) = 4n\sqrt{n-4} = 4n(n-4)^{1/2}$, so $df = 4\left[(n-4)^{1/2} + \frac{1}{2}n(n-4)^{-1/2}\right]dn$. When $n = 85$ and $dn = 5$, $df = 4\left(9 + \dfrac{85}{2 \cdot 9}\right)5 \approx 274$ seconds.

36. $P(x) = -\frac{1}{8}x^2 + 7x + 30$ and $dP = \left(-\frac{1}{4}x + 7\right)dx$. To estimate the increase in profits when the amount spent on advertising each quarter is increased from \$24,000 to \$26,000, we set $x = 24$ and $dx = 2$ and compute $dP = \left(-\frac{24}{4} + 7\right)(2) = 2$, or \$2000.

37. $N(r) = \dfrac{7}{1 + 0.02r^2}$ and $dN = -\dfrac{0.28r}{(1 + 0.02r^2)^2}\,dr$. To estimate the decrease in the number of housing starts when the mortgage rate is increased from 6% to 6.5%, we set $r = 6$ and $dr = 0.5$ and compute $dN = -\dfrac{(0.28)(6)(0.5)}{(1.72)^2} \approx -0.283937$, or 283,937 fewer housing starts.

38. $s(x) = 0.3\sqrt{x} + 10$ and $s' = \dfrac{0.15}{\sqrt{x}}\,dx$. To estimate the change in price when the quantity supplied is increased from 10,000 units to 10,500 units, we compute $ds = \dfrac{(0.15)\,500}{100} = 0.75$, or 75 cents.

39. $p = \dfrac{30}{0.02x^2 + 1}$ and $dp = -\dfrac{1.2x}{\left(0.02x^2 + 1\right)^2}\,dx$. To estimate the change in the price p when the quantity demanded changed from 5000 to 5500 units per week (that is, x changes from 5 to 5.5), we compute $dp = \dfrac{(-1.2)\,(5)\,(0.5)}{[0.02\,(25) + 1]^2} \approx -1.33$, a decrease of \$1.33.

40. $S = kW^{2/3}$ and $dS = \dfrac{0.2}{3W^{1/3}}\,dW$. To determine the percentage error in the calculation of the surface area of a horse that weighs 300 kg when the maximum error in measurement is 0.6 kg and $k = 0.1$, we compute $\dfrac{dS}{S} = \dfrac{0.2}{3W^{1/3}}\,dW \cdot \dfrac{1}{0.1W^{2/3}} = \dfrac{2}{3W}\,dW = \dfrac{2\,(0.6)}{3\,(300)} \approx 0.00133$, or 0.133%.

41. $P(x) = -0.000032x^3 + 6x - 100$ and $dP = \left(-0.000096x^2 + 6\right)dx$. To determine the error in the estimate of Trappee's profits corresponding to a maximum error in the forecast of 15 percent [that is, $dx = \pm 0.15\,(200)$], we compute $dP = \left[(-0.000096)\,(200)^2 + 6\right](\pm 30) = (2.16)\,(30) = \pm 64.80$, or \$64,800.

42. $p = \dfrac{55}{2x^2 + 1}$ and $dp = -\dfrac{220x}{\left(2x^2 + 1\right)^2}\,dx$. To find the error corresponding to a possible error of 15% in a forecast of 1.8 billion bushels, we compute $dp = -\dfrac{(220)\,(1.8)\,(\pm 0.27)}{\left[2\,(1.8)^2 + 1\right]^2} \approx \pm 1.91$, or approximately \$1.91/bushel.

43. The approximate change in the quantity demanded is given by $\Delta x \approx dx = f'(p)\,\Delta p = \dfrac{d}{dp}\,(144 - p)^{1/2}\,\Delta p = -\dfrac{1}{2} \cdot \dfrac{1}{\sqrt{144 - p}} \cdot \Delta p$. When $\Delta p = 110 - 108 = 2$, we find $\Delta x = -\dfrac{1}{2} \cdot \dfrac{1}{\sqrt{144 - 108}}\,(2) = -\tfrac{1}{6} \approx -0.1667$. Thus, the quantity demanded decreases by approximately 167 tires/week.

44. The change is given by

$$\Delta A \approx dA = A'(t)\,dt = 136\frac{d}{dt}\left\{\left[1 + 0.25\,(t - 4.5)^2\right]^{-1} + 28\right\}\Delta t$$

$$= 136\left[1 + 0.25\,(t - 4.5)^2\right]^{-2}(0.25)\,(2)\,(t - 4.5)\,\Delta t = \frac{68\,(t - 4.5)}{\left[1 + 0.25\,(t - 4.5)^2\right]^2}\,\Delta t.$$

When $t = 8$ and $\Delta t = 8.05 - 8 = 0.05$, we find $A \approx 0.7210$, so the change in the amount of nitrogen dioxide is approximately 0.72 PSI.

45. $N(x) = \dfrac{500(400+20x)^{1/2}}{(5+0.2x)^2}$ and

$N'(x) = \dfrac{(5+0.2x)^2\, 250(400+20x)^{-1/2}(20) - 500(400+20x)^{1/2}(2)(5+0.2x)(0.2)}{(5+0.2x)^4}\, dx$. To estimate the change in the number of crimes if the level of reinvestment changes from 20 cents to 22 cents per dollar deposited, we compute

$$dN = \frac{(5+4)^2(250)(800)^{-1/2}(20) - 500(400+400)^{1/2}(2)(9)(0.2)}{(5+4)^4}(2) \approx \frac{(14318.91 - 50911.69)}{9^4}(2)$$

≈ -11, a decrease of approximately 11 crimes per year.

46. a. $P = \dfrac{20{,}000r}{1-\left(1+\frac{r}{12}\right)^{-360}}$ and

$$dP = \frac{\left[1-\left(1+\frac{r}{12}\right)^{-360}\right]20{,}000 - 20{,}000r(360)\left(1+\frac{r}{12}\right)^{-361}\left(\frac{1}{12}\right)}{\left[1-\left(1+\frac{r}{12}\right)^{-360}\right]^2}dr$$
$$= \frac{20{,}000\left\{\left[1-\left(1+\frac{r}{12}\right)^{-360}\right] - 30r\left(1+\frac{r}{12}\right)^{-361}\right\}}{\left[1-\left(1+\frac{r}{12}\right)^{-360}\right]^2}dr$$

b. When $r = 0.05$, $dP \approx \dfrac{20{,}000(0.776173404 - 0.334346782)}{(0.776173404)^2} \approx 14{,}667.77912\, dr$. When the interest rate increases from 5% to 5.2% per year, $dP = 14{,}667.77912(0.002) \approx 29.34$, or approximately \$29.34. When the interest rate increases from 5% to 5.3% per year, $dP = 14{,}667.77912(0.003) \approx 44.00$, or approximately \$44.00. When the interest rate increases from 5% to 5.4% per year, $dP = 14{,}667.77912(0.004) \approx 58.67$, or approximately \$58.67. When the interest rate increases from 5% to 5.5% per year, $dP = 14{,}667.77912(0.005) \approx 73.34$, or approximately \$73.34.

47. $A = 10{,}000\left(1+\frac{r}{12}\right)^{120}$.

a. $dA = 10{,}000(120)\left(1+\frac{r}{12}\right)^{119}\left(\frac{1}{12}\right)dr = 100{,}000\left(1+\frac{r}{12}\right)^{119} dr$.

b. At 3.1%, it will be worth $100{,}000\left(1+\frac{0.03}{12}\right)^{119}(0.001)$, or approximately \$134.60 more. At 3.2%, it will be worth $100{,}000\left(1+\frac{0.03}{12}\right)^{119}(0.002)$, or approximately \$269.20 more. At 3.3%, it will be worth $100{,}000\left(1+\frac{0.03}{12}\right)^{119}(0.003)$, or approximately \$403.80 more.

48. $S = \dfrac{24{,}000\left[\left(1+\frac{r}{12}\right)^{300} - 1\right]}{r}$

a. $dS = 24{,}000\left[\dfrac{(r)300\left(1+\frac{r}{12}\right)^{299}\left(\frac{1}{12}\right) - \left(1+\frac{r}{12}\right)^{300} + 1}{r^2}\right]dr$.

b. With $r = 0.04$, we find $dS = 14{,}864{,}762.53\, dr$. Therefore, if John's account earned 4.1%, it would be worth $dS = 14{,}864{,}762.53(0.001) \approx \$14{,}864.76$ more; if it earned 4.2%, it would be worth $dS = 14{,}864{,}762.53(0.002) \approx \$29{,}729.53$ more, and if it earned 4.3%, it would be worth $dS = 14{,}864{,}762.53(0.003) \approx \$44{,}594.29$ more.

49. True. $dy = f'(x)\,dx = \frac{d}{dx}(ax+b)\,dx = a\,dx$. On the other hand, $\Delta y = f(x+\Delta x) - f(x) = [a(x+\Delta x)+b] - (ax+b) = a\,\Delta x = a\,dx$.

50. True. The percentage change in A is approximately $\dfrac{100\left[f(x+\Delta x)-f(x)\right]}{f(x)} \approx \dfrac{100 f'(x)\,dx}{f(x)}$.

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1. $dy = f'(3)\,dx = 757.87(0.01) \approx 7.5787$.

2. $dy = f'(2)\,dx = -0.125639152666(-0.04) \approx -0.0050256$.

3. $dy = f'(1)\,dx = 1.04067285926(0.03) \approx 0.031220$.

4. $dy = f'(2)(-0.02) \approx 9.66379267622(-0.02) = -0.19328$.

5. $dy = f'(4)(0.1) \approx -0.198761598(0.1) = -0.01988$.

6. $dy = f'(3)(-0.05) \approx 12.3113248654(-0.05) = -0.6155662$.

7. If the interest rate changes from 5% to 5.3% per year, the monthly payment will increase by $dP = f'(0.05)(0.003) \approx 44.00$, or approximately \$44.00 per month. If the rate changes from 5% to 5.4% per year, the payment will increase by \$58.67 per month, and if it changes from 5% to 5.5% per year, the payment will increase by \$73.34 per month.

8. $A = \pi r^2$, so $dA = 2\pi r\,dr$. The area of the ring is approximately $dA = 2\pi(53{,}200)(15)$, or 5,013,982 km^2.

9. $dx = f'(40)(2) \approx -0.625$. That is, the quantity demanded will decrease by 625 watches per week.

10. $T'(22{,}000) = 0.0000570472$, so $\Delta T \approx T'(22{,}000)\,\Delta d \approx -0.0285236$. The period changes by $(-0.0285236)(24) \approx -0.6845664$, a decrease of approximately 0.69 hours.

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1. a. 0 **b.** nx^{n-1} **c.** $cf'(x)$. **d.** $f'(x) \pm g'(x)$

2. a. $f(x)g'(x) + g(x)f'(x)$ **b.** $\dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

3. a. $g'(f(x))f'(x)$ **b.** $n[f(x)]^{n-1}f'(x)$

4. Marginal cost, marginal revenue, marginal profit, marginal average cost

5. a. $-\dfrac{pf'(p)}{f(p)}$ **b.** Elastic, unitary, inelastic

6. Both sides, dy/dx **7.** $y, dy/dt, a$ **8.** $-\dfrac{f(t)f'(t)}{g(t)}, -\dfrac{f(t)g'(t)}{g(t)}$

9. a. $x_2 - x_1$ **b.** $f(x + \Delta x) - f(x)$ **10.** $\Delta x, \Delta x, x, f'(x)\,dx$

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1. $f'(x) = \frac{d}{dx}\left(3x^5 - 2x^4 + 3x^2 - 2x + 1\right) = 15x^4 - 8x^3 + 6x - 2.$

2. $f'(x) = \frac{d}{dx}\left(4x^6 + 2x^4 + 3x^2 - 2\right) = 24x^5 + 8x^3 + 6x.$

3. $g'(x) = \frac{d}{dx}\left(-2x^{-3} + 3x^{-1} + 2\right) = 6x^{-4} - 3x^{-2}.$

4. $f'(t) = \frac{d}{dt}\left(2t^2 - 3t^3 - t^{-1/2}\right) = 4t - 9t^2 + \frac{1}{2}t^{-3/2}.$

5. $g'(t) = \frac{d}{dt}\left(2t^{-1/2} + 4t^{-3/2} + 2\right) = -t^{-3/2} - 6t^{-5/2}.$

6. $h'(x) = \frac{d}{dx}\left(x^2 + \frac{2}{x}\right) = 2x - \frac{2}{x^2}.$

7. $f'(t) = \frac{d}{dt}\left(t + 2t^{-1} + 3t^{-2}\right) = 1 - 2t^{-2} - 6t^{-3} = 1 - \frac{2}{t^2} - \frac{6}{t^3}.$

8. $g'(s) = \frac{d}{ds}\left(2s^2 - 4s^{-1} + 2s^{-1/2}\right) = 4s + 4s^{-2} - s^{-3/2} = 4s + \frac{4}{s^2} - \frac{1}{s^{3/2}}.$

9. $h'(x) = \frac{d}{dx}\left(x^2 - 2x^{-3/2}\right) = 2x + 3x^{-5/2} = 2x + \frac{3}{x^{5/2}}.$

10. $f(x) = \frac{x+1}{2x-1}$, so $f'(x) = \frac{(2x-1)(1) - (x+1)(2)}{(2x-1)^2} = -\frac{3}{(2x-1)^2}.$

11. $g(t) = \frac{t^2}{2t^2+1}$, so $g'(t) = \frac{\left(2t^2+1\right)\frac{d}{dt}\left(t^2\right) - t^2\frac{d}{dt}\left(2t^2+1\right)}{\left(2t^2+1\right)^2} = \frac{\left(2t^2+1\right)(2t) - t^2(4t)}{\left(2t^2+1\right)^2} = \frac{2t}{\left(2t^2+1\right)^2}.$

12. $h(t) = \frac{t^{1/2}}{t^{1/2}+1}$, so $h'(t) = \frac{\left(t^{1/2}+1\right)\frac{1}{2}t^{-1/2} - t^{1/2}\left(\frac{1}{2}t^{-1/2}\right)}{\left(t^{1/2}+1\right)^2} = \frac{1}{2\sqrt{t}\left(\sqrt{t}+1\right)^2}.$

13. $f(x) = \frac{\sqrt{x}-1}{\sqrt{x}+1} = \frac{x^{1/2}-1}{x^{1/2}+1}$, so

$$f'(x) = \frac{\left(x^{1/2}+1\right)\left(\frac{1}{2}x^{-1/2}\right) - \left(x^{1/2}-1\right)\left(\frac{1}{2}x^{-1/2}\right)}{\left(x^{1/2}+1\right)^2} = \frac{\frac{1}{2} + \frac{1}{2}x^{-1/2} - \frac{1}{2} + \frac{1}{2}x^{-1/2}}{\left(x^{1/2}+1\right)^2} = \frac{x^{-1/2}}{\left(x^{1/2}+1\right)^2}$$

$$= \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)^2}.$$

14. $f(t) = \dfrac{t}{2t^2+1}$, so $f'(t) = \dfrac{(2t^2+1)(1) - t(4t)}{(2t^2+1)^2} = \dfrac{1-2t^2}{(2t^2+1)^2}$.

15. $f(x) = \dfrac{x^2(x^2+1)}{x^2-1}$, so

$$\begin{aligned} f'(x) &= \frac{(x^2-1)\dfrac{d}{dx}(x^4+x^2) - (x^4+x^2)\dfrac{d}{dx}(x^2-1)}{(x^2-1)^2} = \frac{(x^2-1)(4x^3+2x) - (x^4+x^2)(2x)}{(x^2-1)^2} \\ &= \frac{4x^5+2x^3-4x^3-2x-2x^5-2x^3}{(x^2-1)^2} = \frac{2x^5-4x^3-2x}{(x^2-1)^2} = \frac{2x(x^4-2x^2-1)}{(x^2-1)^2}. \end{aligned}$$

16. $f(x) = (2x^2+x)^3$, so $f'(x) = 3(2x^2+x)^2 \dfrac{d}{dx}(2x^2+x) = 3(4x+1)(2x^2+x)^2$.

17. $f(x) = (3x^3-2)^8$, so $f'(x) = 8(3x^3-2)^7(9x^2) = 72x^2(3x^3-2)^7$.

18. $h(x) = (\sqrt{x}+2)^5$, so $h'(x) = 5(x^{1/2}+2)^4 \dfrac{d}{dx}x^{1/2} = 5(x^{1/2}+2)^4 \cdot \frac{1}{2}x^{-1/2} = \dfrac{5(\sqrt{x}+2)^4}{2\sqrt{x}}$.

19. $f'(t) = \dfrac{d}{dt}(2t^2+1)^{1/2} = \frac{1}{2}(2t^2+1)^{-1/2}\dfrac{d}{dt}(2t^2+1) = \frac{1}{2}(2t^2+1)^{-1/2}(4t) = \dfrac{2t}{\sqrt{2t^2+1}}$.

20. $g(t) = \sqrt[3]{1-2t^3} = (1-2t^3)^{1/3}$, so $g'(t) = \frac{1}{3}(1-2t^3)^{-2/3}(-6t^2) = -2t^2(1-2t^3)^{-2/3}$.

21. $s(t) \;=\; (3t^2-2t+5)^{-2}$, so

$$s'(t) = -2(3t^2-2t+5)^{-3}(6t-2) = -4(3t^2-2t+5)^{-3}(3t-1) = -\frac{4(3t-1)}{(3t^2-2t+5)^3}.$$

22. $f(x) \;=\; (2x^3-3x^2+1)^{-3/2}$, so

$$f'(x) = -\tfrac{3}{2}(2x^3-3x^2+1)^{-5/2}(6x^2-6x) = -9x(x-1)(2x^3-3x^2+1)^{-5/2}.$$

23. $h(x) \;=\; \left(x+\dfrac{1}{x}\right)^2 \;=\; (x+x^{-1})^2$, so

$$h'(x) = 2(x+x^{-1})(1-x^{-2}) = 2\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right) = 2\left(\frac{x^2+1}{x}\right)\left(\frac{x^2-1}{x^2}\right) = \frac{2(x^2+1)(x^2-1)}{x^3}.$$

24. $h(x) = \dfrac{1+x}{(2x^2+1)^2}$, so

$$h'(x) = \frac{(2x^2+1)^2(1) - (1+x)2(2x^2+1)(4x)}{(2x^2+1)^4} = \frac{(2x^2+1)\left[(2x^2+1) - 8x - 8x^2\right]}{(2x^2+1)^4} = -\frac{6x^2+8x-1}{(2x^2+1)^3}.$$

25. $h(t) = (t^2+t)^4(2t^2)$, so

$$\begin{aligned} h'(t) &= (t^2+t)^4\frac{d}{dt}(2t^2) + 2t^2\frac{d}{dt}(t^2+t)^4 = (t^2+t)^4(4t) + 2t^2 \cdot 4(t^2+t)^3(2t+1) \\ &= 4t(t^2+t)^3\left[(t^2+t) + 4t^2 + 2t\right] = 4t^2(5t+3)(t^2+t)^3. \end{aligned}$$

26. $f(x) = (2x+1)^3 \left(x^2+x\right)^2$, so

$$f'(x) = (2x+1)^3 \cdot 2\left(x^2+x\right)(2x+1) + \left(x^2+x\right)^2 3(2x+1)^2(2)$$
$$= 2(2x+1)^2\left(x^2+x\right)\left[(2x+1)^2 + 3\left(x^2+x\right)\right] = 2(2x+1)^2\left(x^2+x\right)\left(7x^2+7x+1\right).$$

27. $g(x) = x^{1/2}\left(x^2-1\right)^3$, so

$$g'(x) = \frac{d}{dx}\left[x^{1/2}\left(x^2-1\right)^3\right] = x^{1/2}\cdot 3\left(x^2-1\right)^2(2x) + \left(x^2-1\right)^3\cdot \tfrac{1}{2}x^{-1/2}$$
$$= \tfrac{1}{2}x^{-1/2}\left(x^2-1\right)^2\left[12x^2+\left(x^2-1\right)\right] = \frac{\left(13x^2-1\right)\left(x^2-1\right)^2}{2\sqrt{x}}.$$

28. $f(x) = \dfrac{x}{\left(x^3+2\right)^{1/2}}$, so

$$f'(x) = \frac{\left(x^3+2\right)^{1/2}(1) - x\cdot\frac{1}{2}\left(x^3+2\right)^{-1/2}\cdot 3x^2}{x^3+2} = \frac{\frac{1}{2}\left(x^3+2\right)^{-1/2}\left[2\left(x^3+2\right)-3x^3\right]}{x^3+2} = \frac{4-x^3}{2\left(x^3+2\right)^{3/2}}.$$

29. $h(x) = \dfrac{(3x+2)^{1/2}}{4x-3}$, so

$$h'(x) = \frac{(4x-3)\frac{1}{2}(3x+2)^{-1/2}(3) - (3x+2)^{1/2}(4)}{(4x-3)^2} = \frac{\frac{1}{2}(3x+2)^{-1/2}\left[3(4x-3)-8(3x+2)\right]}{(4x-3)^2}$$
$$= -\frac{12x+25}{2\sqrt{3x+2}\,(4x-3)^2}.$$

30. $f(t) = \dfrac{(2t+1)^{1/2}}{(t+1)^3}$, so

$$f'(t) = \frac{(t+1)^3\frac{1}{2}(2t+1)^{-1/2}(2) - (2t+1)^{1/2}\cdot 3(t+1)^2(1)}{(t+1)^6}$$
$$= \frac{(2t+1)^{-1/2}(t+1)^2\left[(t+1)-3(2t+1)\right]}{(t+1)^6} = -\frac{5t+2}{\sqrt{2t+1}\,(t+1)^4}.$$

31. $f(x) = 2x^4-3x^3+2x^2+x+4$, so $f'(x) = \dfrac{d}{dx}\left(2x^4-3x^3+2x^2+x+4\right) = 8x^3-9x^2+4x+1$ and

$$f''(x) = \frac{d}{dx}\left(8x^3-9x^2+4x+1\right) = 24x^2-18x+4 = 2\left(12x^2-9x+2\right).$$

32. $g(x) = x^{1/2}+x^{-1/2}$, so $g'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$ and $g''(x) = -\frac{1}{4}x^{-3/2} + \frac{3}{4}x^{-5/2} = -\dfrac{1}{4x^{3/2}} + \dfrac{3}{4x^{5/2}}$.

33. $h(t) = \dfrac{t}{t^2+4}$, so $h'(t) = \dfrac{\left(t^2+4\right)(1) - t(2t)}{\left(t^2+4\right)^2} = \dfrac{4-t^2}{\left(t^2+4\right)^2}$ and

$$h''(t) = \frac{\left(t^2+4\right)^2(-2t) - \left(4-t^2\right)2\left(t^2+4\right)(2t)}{\left(t^2+4\right)^4} = \frac{-2t\left(t^2+4\right)\left[\left(t^2+4\right)+2\left(4-t^2\right)\right]}{\left(t^2+4\right)^4} = \frac{2t\left(t^2-12\right)}{\left(t^2+4\right)^3}.$$

34. $f(x) = \left(x^3+x+1\right)^2$, so

$$f'(x) = 2\left(x^3+x+1\right)\left(3x^2+1\right) = 2\left(3x^5+3x^3+3x^2+x^3+x+1\right) = 2\left(3x^5+4x^3+3x^2+x+1\right)$$

and $f''(x) = 2\left(15x^4+12x^2+6x+1\right)$.

35. $f'(x) = \frac{d}{dx}\left(2x^2+1\right)^{1/2} = \frac{1}{2}\left(2x^2+1\right)^{-1/2}(4x) = 2x\left(2x^2+1\right)^{-1/2}$, so

$$f''(x) = 2\left(2x^2+1\right)^{-1/2} + 2x\cdot\left(-\tfrac{1}{2}\right)\left(2x^2+1\right)^{-3/2}(4x) = 2\left(2x^2+1\right)^{-3/2}\left[\left(2x^2+1\right)-2x^2\right] = \frac{2}{\left(2x^2+1\right)^{3/2}}.$$

36. $f(t) = t\left(t^2+1\right)^3$, so

$$f'(t) = \left(t^2+1\right)^3 + t\cdot 3\left(t^2+1\right)^2(2t) = \left(t^2+1\right)^2\left[\left(t^2+1\right)+6t^2\right] = \left(t^2+1\right)^2\left(7t^2+1\right) \text{ and}$$

$$\begin{aligned} f''(t) &= \left(t^2+1\right)^2(14t) + \left(7t^2+1\right)(2)\left(t^2+1\right)(2t) = 2t\left(t^2+1\right)\left[7\left(t^2+1\right)+2\left(7t^2+1\right)\right] \\ &= 6t\left(t^2+1\right)\left(7t^2+3\right). \end{aligned}$$

37. $6x^2 - 3y^2 = 9$. Differentiating this equation implicitly, we have $12x - 6y\frac{dy}{dx} = 0$ and $-6y\frac{dy}{dx} = -12x$. Therefore, $\frac{dy}{dx} = \frac{-12x}{-6y} = \frac{2x}{y}$.

38. $2x^3 - 3xy = 4$. Differentiating this equation implicitly, we have $6x^2 - 3y - 3x\frac{dy}{dx} = 0$, so $-3x\frac{dy}{dx} = -6x^2 + 3y$. Thus, $\frac{dy}{dx} = \frac{2x^2-y}{x}$.

39. $y^3 + 3x^2 = 3y$. Differentiating this equation implicitly, we have $3y^2y' + 6x = 3y'$, $3y^2y' - 3y' = -6x$, and $y'\left(3y^2-3\right) = -6x$. Therefore, $y' = -\frac{6x}{3\left(y^2-1\right)} = -\frac{2x}{y^2-1}$.

40. $x^2 + 2x^2y^2 + y^2 = 10$. Differentiating this equation implicitly, we have $2x + 4xy^2 + 2x^2\left(2yy'\right) + 2yy' = 0$, $2yy'\left(2x^2+1\right) = -2x\left(1+2y^2\right)$, and thus $y' = -\frac{x\left(1+2y^2\right)}{y\left(2x^2+1\right)}$.

41. $x^2 - 4xy - y^2 = 12$. Differentiating this equation implicitly, we have $2x - 4xy' - 4y - 2yy' = 0$ and $y'(-4x-2y) = -2x + 4y$. Therefore, $y' = \frac{-2(x-2y)}{-2(2x+y)} = \frac{x-2y}{2x+y}$.

42. $3x^2y - 4xy + x - 2y = 6$. Differentiating this equation implicitly, we have $6xy + 3x^2y' - 4y - 4xy' + 1 - 2y' = 0$, $y'\left(3x^2-4x-2\right) = 4y - 6xy - 1$, and thus $y' = \frac{4y-6xy-1}{3x^2-4x-2}$.

43. $f(x) = x^2 + \frac{1}{x^2}$, so $df = f'(x)\,dx = \left(2x - 2x^{-3}\right)dx = \left(2x - \frac{2}{x^3}\right)dx = \frac{2\left(x^4-1\right)}{x^3}\,dx$.

44. $f(x) = \frac{1}{\sqrt{x^3+1}}$, so $df = f'(x)\,dx = \frac{d}{dx}\left(x^3+1\right)^{-1/2}\,dx = -\frac{1}{2}\left(x^3+1\right)^{-3/2}\left(3x^2\right)dx = -\frac{3x^2}{2\left(x^3+1\right)^{3/2}}\,dx$.

45. a. $df = f'(x)\,dx = \frac{d}{dx}\left(2x^2+4\right)^{1/2}\,dx = \frac{1}{2}\left(2x^2+4\right)^{-1/2}(4x) = \frac{2x}{\sqrt{2x^2+4}}\,dx$.

b. Setting $x = 4$ and $dx = 0.1$, we find $\Delta f \approx df = \frac{2(4)(0.1)}{\sqrt{2(16)+4}} = \frac{0.8}{6} = \frac{8}{60} = \frac{2}{15}$.

c. $\Delta f = f(4.1) - f(4) = \sqrt{2(4.1)^2+4} - \sqrt{2(16)+4} \approx 0.1335$. From part (b), $\Delta f \approx \frac{2}{15} \approx 0.1333$.

46. Take $y = f(x) = x^{1/3}$ and $x = 27$. Then $\Delta x = dx = 26.8 - 27 = -0.2$, so $\Delta y \approx dy = f'(x)\,\Delta x = \frac{1}{3}x^{-2/3}\Big|_{x=27} \cdot (-0.2) = \frac{1}{3(9)}(-0.2) = -\frac{2}{270} = -\frac{1}{135}$. Therefore, $\sqrt[3]{26.8} - \sqrt[3]{27} = \Delta y = -\frac{1}{135}$, so $\sqrt[3]{26.8} = \sqrt[3]{27} - \frac{1}{135} = 3 - \frac{1}{135} \approx 2.9926$.

47. $f(x) = 2x^3 - 3x^2 - 16x + 3$ and $f'(x) = 6x^2 - 6x - 16$.

a. To find the point(s) on the graph of f where the slope of the tangent line is equal to -4, we solve $6x^2 - 6x - 16 = -4$, obtaining $6x^2 - 6x - 12 = 0$, $6\left(x^2 - x - 2\right) = 0$, and $6(x-2)(x+1) = 0$. Thus, $x = 2$ or $x = -1$. Now $f(2) = 2(2)^3 - 3(2)^2 - 16(2) + 3 = -25$ and $f(-1) = 2(-1)^3 - 3(-1)^2 - 16(-1) + 3 = 14$, so the points are $(2, -25)$ and $(-1, 14)$.

b. Using the point-slope form of the equation of a line, we find that the equation of the tangent line at $(2, -25)$ is $y - (-25) = -4(x-2)$, $y + 25 = -4x + 8$, or $y = -4x - 17$, and the equation of the tangent line at $(-1, 14)$ is $y - 14 = -4(x+1)$, or $y = -4x + 10$.

48. $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 4x + 1$, so $f'(x) = x^2 + x - 4$.

a. Set $x^2 + x - 4 = -2$, so $x^2 + x - 2 = (x+2)(x-1) = 0$. Therefore, $x = -2$ or 1. The points are $\left(-2, \frac{25}{3}\right)$ and $\left(1, -\frac{13}{6}\right)$.

b. Equations are $y - \frac{25}{3} = -2(x+2)$, or $y = -2x + \frac{13}{3}$; and $y + \frac{13}{6} = -2(x-1)$, or $y = -2x - \frac{1}{6}$.

49. $y = \left(4 - x^2\right)^{1/2}$, so $y' = \frac{1}{2}\left(4 - x^2\right)^{-1/2}(-2x) = -\dfrac{x}{\sqrt{4 - x^2}}$. The slope of the tangent line is obtained by letting $x = 1$, giving $m = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$. Therefore, an equation of the tangent line at $x = 1$ is $y - \sqrt{3} = -\frac{\sqrt{3}}{3}(x-1)$, or $y = -\frac{\sqrt{3}}{3}x + \frac{4\sqrt{3}}{3}$.

50. $y = x(x+1)^5$, so $y' = (x+1)^5 + x \cdot 5(x+1)^4(1) = (x+1)^4[(x+1) + 5x] = (6x+1)(x+1)^4$. The slope of the tangent line is obtained by letting $x = 1$. Then $m = (6+1)(2)^4 = 112$. An equation of the tangent line is $y - 32 = 112(x-1)$, or $y = 112x - 80$.

51. $f(x) = (2x-1)^{-1}$, so $f'(x) = -2(2x-1)^{-2}$, $f''(x) = 8(2x-1)^{-3} = \dfrac{8}{(2x-1)^3}$, and $f'''(x) = -48(2x-1)^4 = -\dfrac{48}{(2x-1)^4}$. Because $(2x-1)^4 = 0$ when $x = \frac{1}{2}$, we see that the domain of f''' is $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$.

52. a. $S(0) = 3.1$, or \$3.1 billion. $S(5) = 0.14(5)^2 + 0.68(5) + 3.1 = 10$, or \$10 billion.

b. $S'(t) = 0.28t + 0.68$, so $S'(0) = 0.28(0) + 0.68 = 0.68$, or \$0.68 billion, and $S'(5) = 0.28(5) + 0.68 = 2.08$, or \$2.08 billion/yr.

53. a. The number of UK digital viewers in 2015 is projected to be $N(t) = 65.71(5)^{0.085} \approx 75.3$, or 75.3 million.

b. $N'(t) = 65.71(0.085)\,t^{-0.915}$, so $N'(5) = 65.71(0.085)(5)^{-0.915} \approx 1.28$. Thus, the number of viewers is expected to be increasing at the rate of approximately 1.3 million per year.

54. $P(t) = 0.01484t^2 + 0.446t + 15$.

a. $P(0) = 15$, or 15%, and $P(22) = 0.01484(22)^2 + 0.446(22) + 15 \approx 31.99$, or approximately 31.99%.

b. $P'(t) = 2(0.01484)t + 0.446 = 0.02968t + 0.446$, so $P'(2) = 0.02968(2) + 0.446 = 0.50536$, or approximately 0.51%/yr, and $P'(20) = 0.02968(20) + 0.446 = 1.0396$, or approximately 1.04%/yr.

55. a. The number of cameras that will be shipped after 2 years is given by $N(2) = 6(2^2) + 200(2) + 4\sqrt{2} + 20{,}000 \approx 20{,}429.7$, or approximately 20,430 cameras.

b. The rate of change in the number of cameras shipped after 2 years is given by $N'(2) = \left(12t + 200 + 2t^{-1/2}\right)\big|_2 = 12(2) + 200 + \frac{2}{\sqrt{2}} \approx 225.4$, or approximately 225 cameras/yr.

56. a. The GDP in 2013 is given by $f(3) = 0.1(3)^3 + 0.5(3)^2 + 2(3) + 20 = 33.2$, or \$33.2 billion.

b. The rate of change of the GDP in 2013 is given by $f'(3) = \left(0.3t^2 + t + 2\right)\big|_{t=3} = 0.3(3)^2 + 3 + 2 = 7.7$, or \$7.7 billion/yr.

57. a. The population after 3 years is given by $P(3) = 30 - \dfrac{20}{2(3) + 3} \approx 27.7778$, or approximately 27,778. The current population is $P(0) = 30 - \frac{20}{3} \approx 23.333$, or approximately 23,333. So the population will have changed by $27{,}778 - 23{,}333 = 4445$; that is, it would have increased by 4445.

b. $P'(t) = \dfrac{d}{dt}\left[30 - 20(2t + 3)^{-1}\right] = \dfrac{40}{(2t + 3)^2}$, so the rate of change after 3 years is $P'(3) = \dfrac{40}{[2(3) + 3]^2} \approx 0.4938$; that is, it will be increasing at the rate of approximately 494 people/yr.

58. a. The number of copies sold after 12 weeks is given by $N(12) = [4 + 5(12)]^{5/3} = 1024$, or 1,024,000.

b. The rate of change after 12 weeks is given by $N'(12) = \frac{5}{3}(4 + 5t)^{2/3}(5)\Big|_{t=12} \approx 133.33$, or approximately 133,000 copies/wk.

59. $N(x) = 1000(1 + 2x)^{1/2}$, so $N'(x) = 1000\left(\frac{1}{2}\right)(1 + 2x)^{-1/2}(2) = \dfrac{1000}{\sqrt{1 + 2x}}$. The rate of increase at the end of the twelfth week is $N'(12) = \frac{1000}{\sqrt{25}} = 200$, or 200 subscribers/week.

60. $f(t) = 31.88(1 + t)^{-0.45}$, so $f'(t) = 31.88(-0.45)(1 + t)^{-1.45} = -14.346(1 + t)^{-1.45}$. It is changing at the rate of $f'(2) \approx -2.917$; that is, decreasing at the rate of approximately 2.9 cents/minute/yr. The average price per minute at the beginning of 2000 was $f(2) = 31.88(1 + 2)^{-0.45}$, or approximately 19.45 cents/minute.

61. He can expect to live $f(100) = 46.9[1 + 1.09(100)]^{0.1} \approx 75.0433$, or approximately 75.04 years. $f'(t) = 46.9(0.1)(1 + 1.09t)^{-0.9}(1.09) = 5.1121(1 + 1.09t)^{-0.9}$, so the required rate of change is $f'(100) = 5.1121[1 + 1.09(100)]^{-0.9} \approx 0.074$, or approximately 0.07 yr/yr.

62. $C(x) = 2500 + 2.2x$.

a. The marginal cost is $C'(x) = 2.2$. The marginal cost when $x = 1000$ is $C'(1000) = 2.2$. The marginal cost when $x = 2000$ is $C'(2000) = 2.2$.

b. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{2500 + 2.2x}{x} = 2.2 + \dfrac{2500}{x}$, so $\overline{C}'(x) = -\dfrac{2500}{x^2}$.

c. $\lim_{x\to\infty} \overline{C}(x) = \lim_{x\to\infty} \left(2.2 + \frac{2500}{x}\right) = 2.2.$

63. $p'(x) = \frac{d}{dt}\left[\frac{1}{10}x^{3/2} + 10\right] = \frac{3}{20}x^{1/2} = \frac{3}{20}\sqrt{x}$, so $p'(40) = \frac{3}{20}\sqrt{40} \approx 0.9487$, or \$0.9487. When the number of units is 40,000, the price will increase \$0.9487 for each 1000 radios demanded.

64. $p'(x) = \frac{d}{dx}\left[20\left(-x^2+100\right)^{1/2}\right] = 10\left(-x^2+100\right)^{-1/2}(-2x) = \frac{-20x}{\sqrt{-x^2+100}}$, so $p'(6) = -\frac{20(6)}{\sqrt{-36+100}} = -15$. When the number of units is 6000, the price drops by \$15 for each thousand units demanded.

65. a. The actual cost incurred in the manufacturing of the 301st MP3 player is

$$\begin{aligned} C(301) - C(300) &= \left[0.0001(301)^3 - 0.02(301)^2 + 24(301) + 2000\right] \\ &\qquad - \left[0.0001(300)^3 - 0.02(300)^2 + 24(300) + 2000\right] \\ &\approx 39.07, \text{ or approximately \$39.07.} \end{aligned}$$

b. The marginal cost is $C'(300) = \left(0.0003x^2 - 0.04x + 24\right)\big|_{x=300} \approx 39$, or approximately \$39.

66. a. $R(x) = px = (-0.02x + 600)x = -0.02x^2 + 600x.$

b. $R'(x) = -0.04x + 600.$

c. $R'(10{,}000) = -0.04(10{,}000) + 600 = 200$. This says that the sale of the 10,001st phone will bring a revenue of \$200.

67. a. $R(x) = px = (2000 - 0.04x)x = 2000x - 0.04x^2$, so

$$\begin{aligned} P(x) = R(x) - C(x) &= \left(2000x - 0.04x^2\right) - \left(0.000002x^3 - 0.02x^2 + 1000x + 120{,}000\right) \\ &= -0.000002x^3 - 0.02x^2 + 1000x - 120{,}000. \end{aligned}$$

Therefore,

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{0.000002x^3 - 0.02x^2 + 1000x + 120{,}000}{x} = 0.000002x^2 - 0.02x + 1000 + \frac{120{,}000}{x}.$$

b. $C'(x) = \frac{d}{dx}\left(0.000002x^3 - 0.02x^2 + 1000x + 120{,}000\right) = 0.000006x^2 - 0.04x + 1000,$

$R'(x) = \frac{d}{dx}\left(2000x - 0.04x^2\right) = 2000 - 0.08x,$

$P'(x) = \frac{d}{dx}\left(-0.000002x^3 - 0.02x^2 + 1000x - 120{,}000\right) = -0.000006x^2 - 0.04x + 1000$, and

$\overline{C}'(x) = \frac{d}{dx}\left(0.000002x^2 - 0.02x + 1000 + 120{,}000x^{-1}\right) = 0.000004x - 0.02 - 120{,}000x^{-2}.$

c. $C'(3000) = 0.000006(3000)^2 - 0.04(3000) + 1000 = 934$, $R'(3000) = 2000 - 0.08(3000) = 1760$, and $P'(3000) = -0.000006(3000)^2 - 0.04(3000) + 1000 = 826.$

d. $\overline{C}'(5000) = 0.000004(5000) - 0.02 - 120{,}000(5000)^{-2} = -0.0048$, and $\overline{C}'(8000) = 0.000004(8000) - 0.02 - 120{,}000(8000)^{-2} \approx 0.0101$. At a production level of 5000 machines, the average cost of each additional unit is decreasing at a rate of 0.48 cents. At a production level of 8000 machines, the average cost of each additional unit is increasing at a rate of approximately 1 cent per unit.

68. **a.** $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{80x + 150{,}000}{x} = 80 + \dfrac{150{,}000}{x}$.

b. $\overline{C}'(x) = -\dfrac{150{,}000}{x^2}$.

c. $\lim_{x\to\infty} \overline{C}(x) = \lim_{x\to\infty} \left(80 + \dfrac{150{,}000}{x}\right) = 80$. If the production level is very high, then the unit cost approaches \$80/unit.

69. $x = f(p) = -\frac{5}{2}p + 30$, so $f'(p) = -\frac{5}{2}$ and $E(p) = -\dfrac{pf'(p)}{f(p)} = -\dfrac{p\left(-\frac{5}{2}\right)}{-\frac{5}{2}p + 30} = \dfrac{p}{12 - p}$.

a. $E(3) = \frac{3}{9} = \frac{1}{3}$, so demand is inelastic.

b. $E(6) = \frac{6}{12-6} = 1$, so demand is unitary.

c. $E(9) = \frac{9}{12-9} = 3$, so demand is elastic.

70. $x = \dfrac{25}{\sqrt{p}} - 1$, so $f'(p) = -\dfrac{25}{2p^{3/2}}$ and $E(p) = -\dfrac{p\left(-\frac{25}{2p^{3/2}}\right)}{\frac{25}{p^{1/2}} - 1} = \dfrac{\frac{25}{2p^{1/2}}}{\frac{25-p^{1/2}}{p^{1/2}}} = \dfrac{25}{2\left(25 - p^{1/2}\right)}$. If $E(p) = 1$, then $2\left(25 - p^{1/2}\right) = 25$, so $25 - p^{1/2} = \frac{25}{2}$, $p^{1/2} = \frac{25}{2}$, and $p = \frac{625}{4}$. $E(p) > 1$ and demand is elastic if $p > 156.25$, $E(p) = 1$ and demand is unitary if $p = 156.25$, and $E(p) < 1$ and demand is inelastic if $p < 156.25$.

71. $x = 100 - 0.01p^2$, so $f'(p) = -0.02p$ and $E(p) = -\dfrac{p(-0.02p)}{100 - 0.01p^2} = \dfrac{p^2}{5000 - \frac{1}{2}p^2}$.

a. $E(40) = \dfrac{1600}{5000 - \frac{1}{2}(1600)} = \dfrac{1600}{4200} = \dfrac{8}{21} < 1$ and so demand is inelastic.

b. Because demand is inelastic, raising the unit price slightly causes revenue to increase.

72. **a.** $p = 9\sqrt[3]{1000 - x}$, so $\sqrt[3]{1000 - x} = \dfrac{p}{9}$, $1000 - x = \dfrac{p^3}{729}$, and $x = 1000 - \dfrac{p^3}{729}$. Therefore,

$$x = f(p) = \frac{729{,}000 - p^3}{729} \text{ and } f'(p) = -\frac{3p^2}{729} = -\frac{p^2}{243}. \text{ Then } E(p) = -\frac{p\left(-\frac{p^2}{243}\right)}{\frac{729{,}000-p^3}{729}} = \frac{3p^3}{729{,}000 - p^3}.$$

$$E(60) = \frac{3(60)^3}{729{,}000 - 60^3} = \frac{648{,}000}{513{,}000} = \frac{648}{513} > 1, \text{ and so demand is elastic.}$$

b. From part (a), we see that raising the price slightly causes revenue to decrease.

73. $G'(t) = \dfrac{d}{dt}\left(-0.3t^3 + 1.2t^2 + 500\right) = -0.9t^2 + 2.4t$, so $G'(2) = -0.9(4) + 2.4(2) = 1.2$. Thus, the GDP is growing at the rate of \$1.2 billion/year. $G''(2) = (-1.8t + 2.4)|_{t=2} = -1.2$, so the rate of rate of change of the GDP is decreasing at the rate of \$1.2 billion/yr/yr.

74. $v = \frac{d}{dt}\left[t\left(2t^2+1\right)^{1/2}\right] = \left(2t^2+1\right)^{1/2} + t\left(\frac{1}{2}\right)\left(2t^2+1\right)^{-1/2}(4t) = \left(2t^2+1\right)^{1/2} + \frac{2t^2}{\left(2t^2+1\right)^{1/2}}$ and

$$a = \frac{d}{dt}\left[\left(2t^2+1\right)^{1/2} + 2t^2\left(2t^2+1\right)^{-1/2}\right]$$
$$= \tfrac{1}{2}\left(2t^2+1\right)^{-1/2}(4t) + 4t\left(2t^2+1\right)^{-1/2} + \left(2t^2\right)\left(-\tfrac{1}{2}\right)\left(2t^2+1\right)^{-3/2}(4t) = \frac{6t}{\left(2t^2+1\right)^{1/2}} - \frac{4t^3}{\left(2t^2+1\right)^{3/2}}.$$

Thus, the velocity after 2 seconds is $v(2) = 9^{1/2} + \frac{2(4)}{9^{1/2}} = \frac{17}{3}$ ft/sec and the acceleration after 2 seconds is $a(2) = \frac{12}{9^{1/2}} - \frac{4(8)}{9^{3/2}} = \frac{76}{27}$ ft/sec^2.

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1. $f(x) = 2x^3 - 3x^{1/3} + 5x^{-2/3}$, so $f'(x) = 2\left(3x^2\right) - 3\left(\frac{1}{3}x^{-2/3}\right) + 5\left(-\frac{2}{3}x^{-5/3}\right) = 6x^2 - x^{-2/3} - \frac{10}{3}x^{-5/3}$.

2. $$g'(x) = \frac{d}{dx}\left[x\left(2x^2-1\right)^{1/2}\right] = \left(2x^2-1\right)^{1/2} + x\left(\tfrac{1}{2}\right)\left(2x^2-1\right)^{-1/2}\frac{d}{dx}\left(2x^2-1\right)$$
$$= \left(2x^2-1\right)^{1/2} + \tfrac{1}{2}x\left(2x^2-1\right)^{-1/2}(4x) = \left(2x^2-1\right)^{-1/2}\left[\left(2x^2-1\right)+2x^2\right] = \frac{4x^2-1}{\sqrt{2x^2-1}}.$$

3. $y = f(x) = \frac{2x+1}{x^2+x+1}$, so

$$\frac{dy}{dx} = \frac{\left(x^2+x+1\right)(2) - (2x+1)(2x+1)}{\left(x^2+x+1\right)^2} = \frac{2x^2+2x+2-\left(4x^2+4x+1\right)}{\left(x^2+x+1\right)^2} = -\frac{2x^2+2x-1}{\left(x^2+x+1\right)^2}.$$

4. $f(x) = \frac{1}{\sqrt{x+1}} = (x+1)^{-1/2}$, so $f'(x) = \frac{d}{dx}(x+1)^{-1/2} = -\frac{1}{2}(x+1)^{-3/2} = -\frac{1}{2(x+1)^{3/2}}$.

Thus, $f''(x) = -\frac{1}{2}\left(-\frac{3}{2}\right)(x+1)^{-5/2} = \frac{3}{4}(x+1)^{-5/2} = \frac{3}{4(x+1)^{5/2}}$ and

$f'''(x) = \frac{3}{4}\left(-\frac{5}{2}\right)(x+1)^{-7/2} = -\frac{15}{8}(x+1)^{-7/2} = -\frac{15}{8(x+1)^{7/2}}$.

5. $xy^2 - x^2y + x^3 = 4$. Differentiating both sides of the equation implicitly with respect to x gives $y^2 + x\left(2yy'\right) - 2xy - x^2y' + 3x^2 = 0$, so $\left(2xy - x^2\right)y' + \left(y^2 - 2xy + 3x^2\right) = 0$ and

$$y' = \frac{-y^2+2xy-3x^2}{2xy-x^2} = \frac{-y^2+2xy-3x^2}{x(2y-x)}.$$

6. a. $y = x\sqrt{x^2+5}$, so $dy = \frac{d}{dx}\left[x\left(x^2+5\right)^{1/2}\right]dx = \left[x\left(\frac{1}{2}\right)\left(x^2+5\right)^{-1/2}(2x)\right]dx + \left[\left(x^2+5\right)^{1/2}(1)\right]dx$

$$= \left(x^2+5\right)^{-1/2}\left[\left(x^2+5\right)+x^2\right]dx = \frac{2x^2+5}{\sqrt{x^2+5}}dx.$$

b. Here $dx = \Delta x = 2.01 - 2 = 0.01$. Therefore, $\Delta y \approx dy = \frac{2(4)+5}{\sqrt{4+5}}(0.01) = \frac{0.13}{3} \approx 0.043$.

CHAPTER 3 Explore & Discuss

Page 179

1. $R'(x) = p(x) + xp'(x)$. This says that the rate of change of the revenue (marginal revenue) is equal to the sum of the unit price of the product plus the product of the number of units sold and the rate of change of the unit price.

2. If $p(x)$ is a constant, say p, then $R'(x) = p$. In other words, the marginal revenue is equal to the unit price. This is expected because if the unit price is constant, then the revenue realized in selling one more unit (the marginal revenue) is p.

Page 192

1. The required expression is $\dfrac{dP}{dx} = g'(x)$.

2. The required expression is $\dfrac{dx}{dt} = f'(t)$.

3. $P = g(x) = g(f(x))$. Using the Chain Rule, we have $\dfrac{dP}{dt} = g'(f(x))f'(x)$.

Page 193

1. $\dfrac{dP}{dt}$ measures the rate of change of the population P with respect to the temperature of the medium.

2. $\dfrac{dT}{dt}$ measures the rate of change of the temperature of the medium with respect to time.

3. $\dfrac{dP}{dt} = \dfrac{dP}{dT} \cdot \dfrac{dT}{dt} = f'(T)g'(t)$ measures the rate of change of the population with respect to time.

4. $(f \circ g)(t) = f(g(t)) = P$ gives the population of bacteria at any time t.

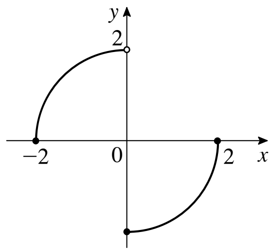
5. $f'(g(t))g'(t) = \dfrac{dP}{dt}$ (by the Chain Rule), and this gives the rate of change of the population with respect to time (see part (c)).

Page 223

1. Thinking of x as a function of y and differentiating the given equation with respect to the independent variable y, we obtain $\dfrac{d}{dy}\left(y^3 - y + 2x^3 - x\right) = \dfrac{d}{dy}(8)$. Thus, $\dfrac{d}{dy}\left(y^3\right) - \dfrac{d}{dy}(y) + \dfrac{d}{dy}\left(2x^3\right) + \dfrac{d}{dy}(-x) = 0$, so $3y^2 - 1 + 2\left(3x^2\dfrac{dx}{dy}\right) - \dfrac{dx}{dy} = 0$, $\left(6x^2 - 1\right)\dfrac{dx}{dy} = 1 - 3y^2$, and $\dfrac{dx}{dy} = \dfrac{1 - 3y^2}{6x^2 - 1}$.

Page 224

1.



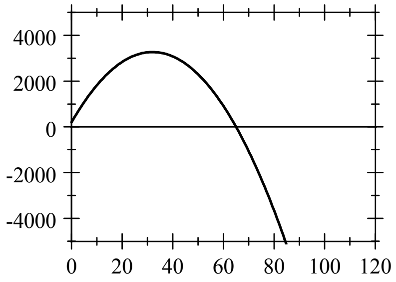
2. If $-2 \le x < 0$, then $h(x) = \left(4 - x^2\right)^{1/2}$, so $h'(x) = \frac{1}{2}\left(4 - x^2\right)^{-1/2}(-2x) = -\frac{x}{\sqrt{4-x^2}} = -\frac{x}{y}$. If $0 \le x \le 2$, then $h(x) = -\left(4 - x^2\right)^{1/2}$, and $h'(x) = \frac{1}{2}\left(4 - x^2\right)^{-1/2}(-2x) = -\frac{x}{\sqrt{4-x^2}} = -\frac{x}{y}$ for $0 \le x < 2$.

3. At the point $\left(1, -\sqrt{3}\right)$, $h'(x) = -\frac{x}{y} = \frac{1}{\sqrt{3}}$, so $y + \sqrt{3} = \frac{\sqrt{3}}{3}(x - 1)$, $y + \sqrt{3} = \frac{\sqrt{3}}{3}x - \frac{\sqrt{3}}{3}$, and so $y = \frac{\sqrt{3}}{3}x - \frac{4\sqrt{3}}{3}$.

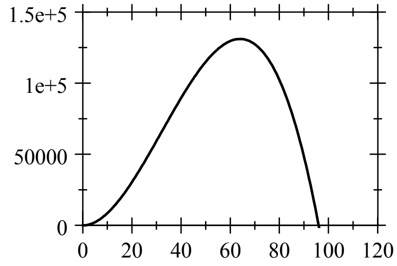
CHAPTER 3 Exploring with Technology

Page 168

1.



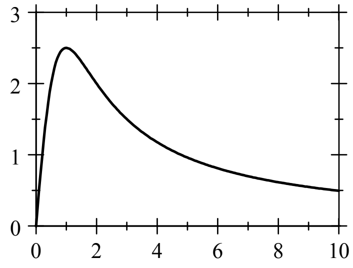
2.



3. We see that $f(t) = 0$ when $t = 98$. We conclude that the rocket returns to Earth 98 seconds later.

Page 179

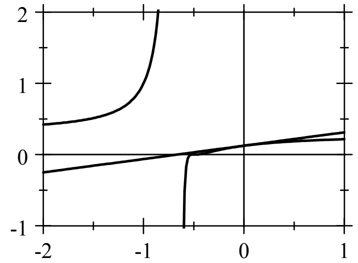
1.



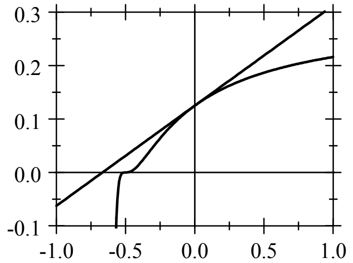
2. The highest point on the graph of S is $(1, 2.5)$, and this tells us that the sales of the laser disc reach a maximum of \$2.5 million one year after its release.

Page 191

1.



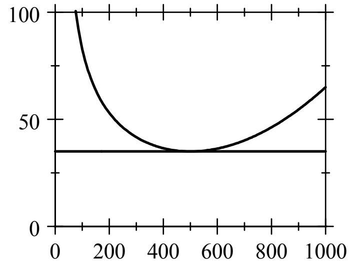
2.



3. The slope of the tangent line is 0.1875, as expected.

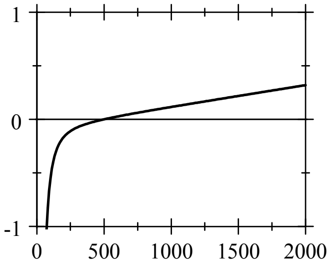
Page 203

1, 2.



The slope is zero. Yes, the point of tangency is the lowest point on the graph of $\overline{C}$.

3.



Yes. At the lowest point on the graph of $\overline{C}$, the derivative of $\overline{C}$ must be zero.

4 APPLICATIONS OF THE DERIVATIVE

4.1 Applications of the First Derivative

Concept Questions page 263

1. **a.** f is increasing on I if whenever x_1 and x_2 are in I with $x_1 < x_2$, then $f(x_1) < f(x_2)$.

   b. f is decreasing on I if whenever x_1 and x_2 are in I with $x_1 < x_2$, then $f(x_1) > f(x_2)$.

2. Find all numbers such that $f'(x)$ does not exist or $f'(x)$ is not defined. Using test numbers if necessary, draw the sign diagram for f'. On a subinterval where $f'(x) < 0$, f is decreasing; on a subinterval where $f'(x) > 0$, f is increasing.

3. **a.** f has a relative maximum at $x = a$ if there is an open interval I containing a such that $f(x) \leq f(a)$ for all x in I.

   b. f has a relative minimum at $x = a$ if there is an open interval I containing a such that $f(x) \geq f(a)$ for all x in I.

4. **a.** A critical number of f is a number c in the domain of f such that $f'(c) = 0$ or f' does not exist at c.

   b. If f has a relative extremum at c, then c must be a critical number of f.

5. See page 260 of the text.

Exercises page 264

1. f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

2. f is decreasing on $(-\infty, -1)$, constant on $(-1, 1)$, and increasing on $(1, \infty)$.

3. f is increasing on $(-\infty, -1)$ and $(1, \infty)$, and decreasing on $(-1, 1)$.

4. f is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 0)$ and $(0, 1)$.

5. f is increasing on $(0, 2)$ and decreasing on $(-\infty, 0)$ and $(2, \infty)$.

6. f is increasing on $(-1, 0)$ and $(1, \infty)$ and decreasing on $(-\infty, -1)$ and $(0, 1)$.

7. f is decreasing on $(-\infty, -1)$ and $(1, \infty)$ and increasing on $(-1, 1)$.

8. f is increasing on $(-\infty, -1)$ and $(-1, \infty)$.

9. Increasing on $(20.2, 20.6)$ and $(21.7, 21.8)$, constant on $(19.6, 20.2)$ and $(20.6, 21.1)$, and decreasing on $(21.1, 21.7)$ and $(21.8, 22.7)$.

10. f is increasing on the interval $(0, 6)$, constant on the intervals $(6, 9)$ and $(14, 15)$, and decreasing on the interval $(9, 14)$. Beginning at 5 A.M., the amount of power generated increases until 11 A.M. ($t = 6$), when the power generated reaches its peak (100% capacity). The solar panel continues to generate at 100% capacity until 2 P.M. ($t = 9$), at which point the output begins to drop, reaching its lowest point at 7 P.M. From 7 P.M. until 8 A.M., the power generated remains constant at its lowest level.

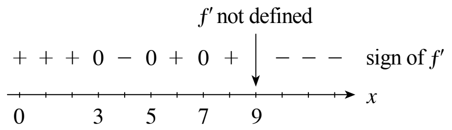
11. **a.** f is decreasing on $(0, 4)$. **b.** f is constant on $(4, 12)$. **c.** f is increasing on $(12, 24)$.

12. **a.** Positive **b.** Positive **c.** Zero **d.** Zero

e. Negative **f.** Negative **g.** Positive

13. **a.** 3, 5, and 7 are critical numbers because $f'(3) = f'(5) = f'(7) = 0$ and 9 is a critical number because $f'(9)$ is not defined.

b.

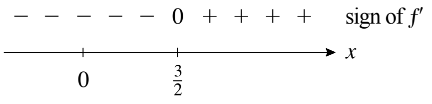


c. f has relative maxima at $(3, 3)$ and $(9, 6)$ and a relative minimum at $(5, 1)$.

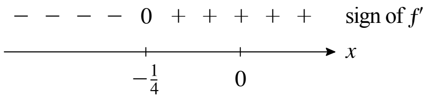
14. $f(x) = 4 - 5x$, so $f'(x) = -5$. Therefore, f is decreasing everywhere; that is, f is decreasing on $(-\infty, \infty)$.

15. $f(x) = 3x + 5$, so $f'(x) = 3 > 0$ for all x. Thus, f is increasing on $(-\infty, \infty)$.

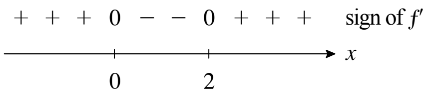
16. $f(x) = x^2 - 3x$, so $f'(x) = 2x - 3$. f' is continuous everywhere and is equal to zero when $x = \frac{3}{2}$. From the sign diagram, we see that f is decreasing on $\left(-\infty, \frac{3}{2}\right)$ and increasing on $\left(\frac{3}{2}, \infty\right)$.



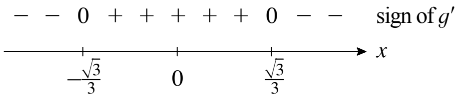
17. $f(x) = 2x^2 + x + 1$, so $f'(x) = 4x + 1 = 0$ if $x = -\frac{1}{4}$. From the sign diagram of f', we see that f is decreasing on $\left(-\infty, -\frac{1}{4}\right)$ and increasing on $\left(-\frac{1}{4}, \infty\right)$.



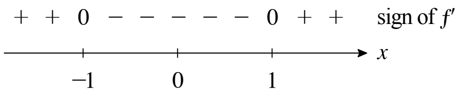
18. $f(x) = x^3 - 3x^2$, so $f'(x) = 3x^2 - 6x = 3x(x - 2) = 0$ if $x = 0$ or 2. From the sign diagram of f', we see that f is increasing on $(-\infty, 0)$ and $(2, \infty)$ and decreasing on $(0, 2)$.



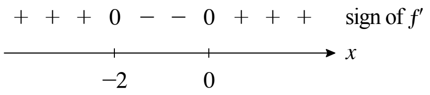
19. $g(x) = x - x^3$, so $g'(x) = 1 - 3x^2$ is continuous everywhere and is equal to zero when $1 - 3x^2 = 0$, or $x = \pm\frac{\sqrt{3}}{3}$. From the sign diagram, we see that f is decreasing on $\left(-\infty, -\frac{\sqrt{3}}{3}\right)$ and $\left(\frac{\sqrt{3}}{3}, \infty\right)$ and increasing on $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.



20. $f(x) = x^3 - 3x + 4$, so $f'(x) = 3x^2 - 3$. f' is continuous everywhere and is equal to zero when $x = \pm 1$. From the sign diagram, we see that f is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$.

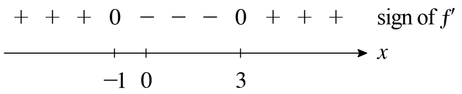


21. $g(x) = x^3 + 3x^2 + 1$, so $g'(x) = 3x^2 + 6x = 3x(x + 2)$. From the sign diagram, we see that g is increasing on $(-\infty, -2)$ and $(0, \infty)$ and decreasing on $(-2, 0)$.

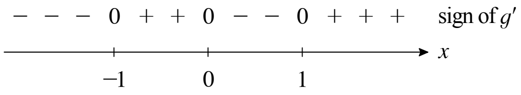


22. $f(x) = \frac{1}{3}x^3 - 3x^2 + 9x + 20$, so $f'(x) = x^2 - 6x + 9 = (x - 3)^2 > 0$ for all x except $x = 3$, at which point $f'(3) = 0$. Therefore, f is increasing on $(-\infty, \infty)$.

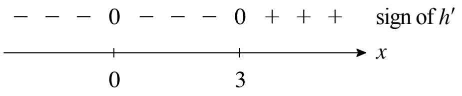
23. $f(x) = \frac{2}{3}x^3 - 2x^2 - 6x - 2$, so $f'(x) = 2x^2 - 4x - 6 = 2\left(x^2 - 2x - 3\right) = 2(x - 3)(x + 1) = 0$ if $x = -1$ or 3. From the sign diagram of f', we see that f is increasing on $(-\infty, -1)$ and $(3, \infty)$ and decreasing on $(-1, 3)$.



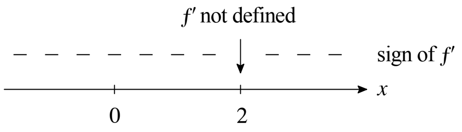
24. $g(x) = x^4 - 2x^2 + 4$, so $g'(x) = 4x^3 - 4x = 4x\left(x^2 - 1\right)$ is continuous everywhere and is equal to zero when $x = 0$, 1, or -1. From the sign diagram, we see that g is decreasing on $(-\infty, -1)$ and $(0, 1)$ and increasing on $(-1, 0)$ and $(1, \infty)$.



25. $h(x) = x^4 - 4x^3 + 10$, so $h'(x) = 4x^3 - 12x^2 = 4x^2(x - 3) = 0$ if $x = 0$ or 3. From the sign diagram of h', we see that h is increasing on $(3, \infty)$ and decreasing on $(-\infty, 3)$.

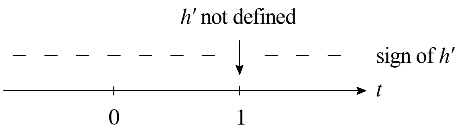


26. $f(x) = \dfrac{1}{x - 2} = (x - 2)^{-1}$, so $f'(x) = -1(x - 2)^{-2}(1) = -\dfrac{1}{(x - 2)^2}$ is discontinuous at $x = 2$ and is continuous and nonzero everywhere else. From the sign diagram, we see that f is decreasing on $(-\infty, 2)$ and $(2, \infty)$.



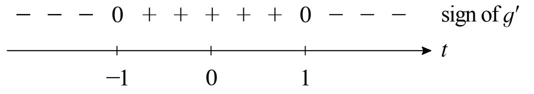
27. $h(x) = \dfrac{1}{2x + 3}$, so $h'(x) = \dfrac{-2}{(2x + 3)^2}$ and we see that h' is not defined at $x = -\frac{3}{2}$. But $h'(x) < 0$ for all x except $x = -\frac{3}{2}$. Therefore, h is decreasing on $\left(-\infty, -\frac{3}{2}\right)$ and $\left(-\frac{3}{2}, \infty\right)$.

28. $h(t) = \dfrac{t}{t - 1}$, so $h'(t) = \dfrac{(t - 1)(1) - t(1)}{(t - 1)^2} = -\dfrac{1}{(t - 1)^2}$. From the sign diagram, we see that $h'(t) < 0$ whenever it is defined. We conclude that h is decreasing on $(-\infty, 1)$ and $(1, \infty)$.



29. $g(t) = \dfrac{2t}{t^2+1}$, so $g'(t) = \dfrac{\left(t^2+1\right)(2) - (2t)(2t)}{\left(t^2+1\right)^2} = \dfrac{2t^2+2-4t^2}{\left(t^2+1\right)^2} = -\dfrac{2\left(t^2-1\right)}{\left(t^2+1\right)^2}$. Next, $g'(t) = 0$ if $t = \pm 1$.

From the sign diagram of g', we see that g is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1)$ and $(1, \infty)$.



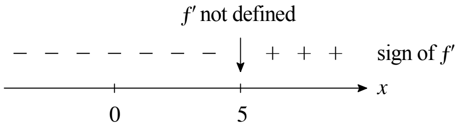
30. $f(x) = x^{3/5}$, so $f'(x) = \frac{3}{5}x^{-2/5} = \dfrac{3}{5x^{2/5}}$. Observe that $f'(x)$ is not defined at $x = 0$, but is positive everywhere else. Therefore, f is increasing on $(-\infty, \infty)$.

31. $f(x) = x^{2/3} + 5$, so $f'(x) = \frac{2}{3}x^{-1/3} = \dfrac{2}{3x^{1/3}}$, and so f' is not defined at $x = 0$. Now $f'(x) < 0$ if $x < 0$ and $f'(x) > 0$ if $x > 0$, and so f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

32. $f(x) = \sqrt{x+1}$, so $f'(x) = \dfrac{d}{dx}(x+1)^{1/2} = \frac{1}{2}(x+1)^{-1/2} = \dfrac{1}{2\sqrt{x+1}}$, and we see that $f'(x) > 0$ if $x > -1$. Therefore, f is increasing on $(-1, \infty)$.

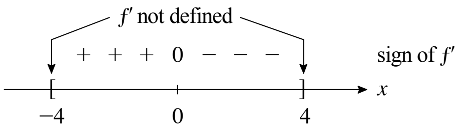
33. $f(x) = (x-5)^{2/3}$, so

$f'(x) = \frac{2}{3}(x-5)^{-1/3} = \dfrac{2}{3(x-5)^{1/3}}$. From the sign diagram, we see that f is decreasing on $(-\infty, 5)$ and increasing on $(5, \infty)$.



34. $f(x) = \sqrt{16-x^2} = \left(16-x^2\right)^{1/2}$, so

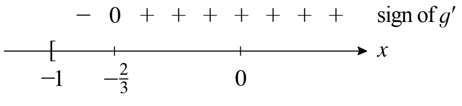
$f'(x) = \frac{1}{2}\left(16-x^2\right)^{-1/2}(-2x) = -\dfrac{x}{\sqrt{16-x^2}}$. Because the domain of f is $[-4, 4]$, we consider the sign diagram for f' on this interval. We see that f is increasing on $(-4, 0)$ and decreasing on $(0, 4)$.



35. $g(x) = x(x+1)^{1/2}$, so

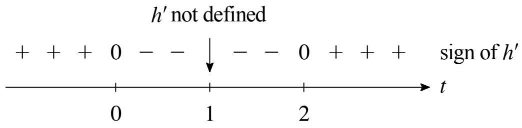
$g'(x) = (x+1)^{1/2} + x\left(\frac{1}{2}\right)(x+1)^{-1/2} = (x+1)^{-1/2}\left(x+1+\frac{1}{2}x\right) = (x+1)^{-1/2}\left(\frac{3}{2}x+1\right) = \dfrac{3x+2}{2\sqrt{x+1}}$.

Thus, g' is continuous on $(-1, \infty)$ and has a zero at $x = -\frac{2}{3}$. From the sign diagram, we see that g is decreasing on $\left(-1, -\frac{2}{3}\right)$ and increasing on $\left(-\frac{2}{3}, \infty\right)$.



36. $f'(x) = \dfrac{d}{dx}\left(x^{-1} - x\right) = -1 - \dfrac{1}{x^2} = -\dfrac{x^2+1}{x^2} < 0$ for all $x \neq 0$. Therefore, f is decreasing on $(-\infty, 0)$ and $(0, \infty)$.

37. $h(x) = \dfrac{x^2}{x-1}$, so $h'(x) = \dfrac{(x-1)(2x) - x^2}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$. Thus, h' is continuous everywhere except at $x = 1$ and has zeros at $x = 0$ and $x = 2$. From the sign diagram, we see that h is increasing on $(-\infty, 0)$ and $(2, \infty)$ and decreasing on $(0, 1)$ and $(1, 2)$.



38. f has a relative maximum of $f(0) = 1$ and relative minima of $f(-1) = 0$ and $f(1) = 0$.

39. f has a relative maximum of $f(0) = 1$ and relative minima of $f(-1) = 0$ and $f(1) = 0$.

40. f has a relative maximum at $(0, 0)$ and a relative minimum at $(4, -32)$.

41. f has a relative maximum of $f(-1) = 2$ and a relative minimum of $f(1) = -2$.

42. f has a relative minimum at $(-1, 0)$.

43. f has a relative maximum of $f(1) = 3$ and a relative minimum of $f(2) = 2$.

44. f has a relative minimum at $(0, 2)$.

45. f has a relative maximum at $\left(-3, -\frac{9}{2}\right)$ and a relative minimum at $\left(3, \frac{9}{2}\right)$.

46. f is increasing on the interval (a, c), where $f'(x) > 0$; f is decreasing on (c, d), where $f'(x) < 0$; and f is increasing once again on (d, b), where $f'(x) > 0$.

47. f is decreasing on the interval (a, c), where $f'(x) < 0$; f is increasing on (c, d), where $f'(x) > 0$; f is constant on (d, e), where $f'(x) = 0$; and finally f is decreasing on (e, b), where $f'(x) < 0$.

48. The car is moving in the positive direction from $t = a$ to $t = c$, where $v(t) > 0$; it stops at $t = c$, where $v(t) = 0$; then it starts moving in the negative direction from $t = c$ to $t = d$, where $v(t) < 0$. It stops again at $t = d$, where $v(t) = 0$, before moving in the positive direction once again.

49. The profit is increasing at a level of production between 0 units and c units, corresponding to the interval $(0, c)$ on which $P'(t) > 0$. The profit is neither increasing nor decreasing when the level of production is c units; this corresponds to the number c at which $P'(t) = 0$. Finally, the profit is decreasing when the level of production is between c units and b units.

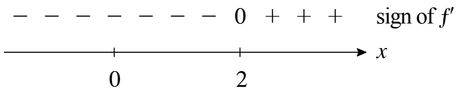
50. c

51. a

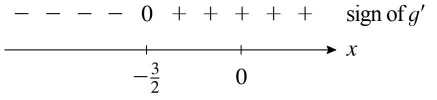
52. b

53. d

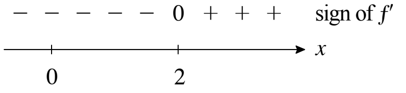
54. $f(x) = x^2 - 4x$, so $f'(x) = 2x - 4 = 2(x-2)$ has a critical point at $x = 2$. From the sign diagram, we see that $f(2) = -4$ is a relative minimum by the First Derivative Test.



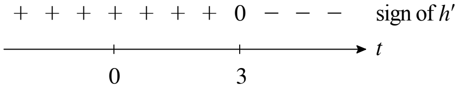
55. $g(x) = x^2 + 3x + 8$, so $g'(x) = 2x + 3$ has a critical point at $x = -\frac{3}{2}$. From the sign diagram, we see that $g\left(-\frac{3}{2}\right) = \frac{23}{4}$ is a relative minimum by the First Derivative Test.



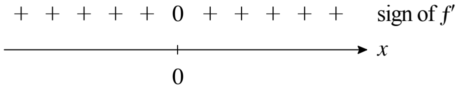
56. $f(x) = \frac{1}{2}x^2 - 2x + 4$, so $f'(x) = x - 2$, giving the critical number $x = 2$. From the sign diagram, we see that $f(2) = 2$ is a relative minimum.



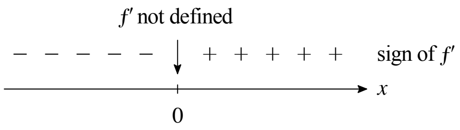
57. $h(t) = -t^2 + 6t + 6$, so $h'(t) = -2t + 6 = -2(t - 3) = 0$ if $t = 3$, a critical number. The sign diagram and the First Derivative Test imply that h has a relative maximum at 3 with value $h(3) = -9 + 18 + 6 = 15$.



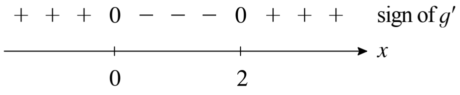
58. $f(x) = x^{5/3}$, so $f'(x) = \frac{5}{3}x^{2/3}$, $x = 0$ as the critical number of f. From the sign diagram, we see that f' does not change sign as we move across $x = 0$, and conclude that f has no relative extremum.



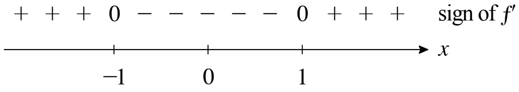
59. $f(x) = x^{2/3} + 2$, so $f'(x) = \frac{2}{3}x^{-1/3} = \dfrac{2}{3x^{1/3}}$ is discontinuous at $x = 0$, a critical number. From the sign diagram and the First Derivative Test, we see that f has a relative minimum at $(0, 2)$.



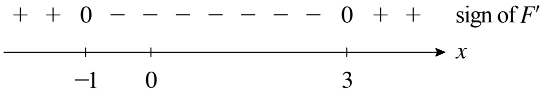
60. $g(x) = x^3 - 3x^2 + 5$, so $g'(x) = 3x^2 - 6x = 3x(x - 2) = 0$ if $x = 0$ or 2. From the sign diagram, we see that the critical number $x = 0$ gives a relative maximum, whereas $x = 2$ gives a relative minimum. The values are $g(0) = 5$ and $g(2) = 8 - 12 + 5 = 1$.



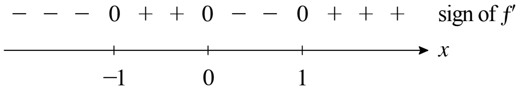
61. $f(x) = x^3 - 3x + 6$. Setting $f'(x) = 3x^2 - 3 = 3\left(x^2 - 1\right) = 3(x + 1)(x - 1) = 0$ gives $x = -1$ and $x = 1$ as critical numbers. The sign diagram of f' shows that $(-1, 8)$ is a relative maximum and $(1, 4)$ is a relative minimum.



62. $F(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$, so $F'(x) = x^2 - 2x - 3 = (x - 3)(x + 1) = 0$ gives $x = -1$ and $x = 3$ as critical numbers. From the sign diagram, we see that $x = -1$ gives a relative maximum and $x = 3$ gives a relative minimum. The values are $F(-1) = -\frac{1}{3} - 1 + 3 + 4 = \frac{17}{3}$ and $F(3) = 9 - 9 - 9 + 4 = -5$.



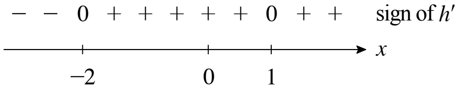
63. $f(x) = \frac{1}{2}x^4 - x^2$, so $f'(x) = 2x^3 - 2x = 2x\left(x^2 - 1\right) = 2x(x+1)(x-1)$ is continuous everywhere and has zeros at $x = -1$, 0, and 1, the critical numbers of f. Using the First Derivative Test and the sign diagram of f', we see that $f(-1) = -\frac{1}{2}$ and $f(1) = -\frac{1}{2}$ are relative minima of f and $f(0) = 0$ is a relative maximum of f.



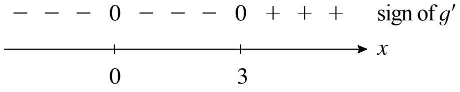
64. $h(x) = \frac{1}{2}x^4 - 3x^2 + 4x - 8$, so

$$h'(x) = 2x^3 - 6x + 4 = 2\left(x^3 - 3x + 2\right) = 2(x-1)\left(x^2 + x - 2\right)$$
$$= 2(x-1)(x-1)(x+2) = 2(x-1)^2(x+2).$$

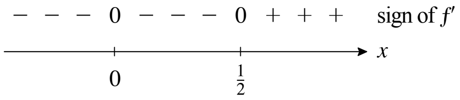
We see that $h'(x) = 0$ at $x = -2$ and $x = 1$; both are critical numbers of h. From the sign diagram of h', we see that h has a relative minimum at $(-2, -20)$.



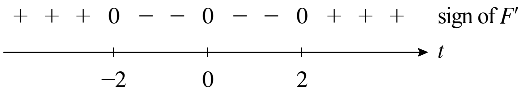
65. $g(x) = x^4 - 4x^3 + 12$. Setting $g'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0$ gives $x = 0$ and $x = 3$ as critical numbers. From the sign diagram, we see that $x = 3$ gives a relative minimum. Its value is $g(3) = 3^4 - 4(3)^3 + 12 = -15$.



66. $f(x) = 3x^4 - 2x^3 + 4$, so $f'(x) = 12x^3 - 6x^2 = 6x^2(2x-1) = 0$ if $x = 0$ or $\frac{1}{2}$. The sign diagram of f' shows that f has a relative minimum at $\left(\frac{1}{2}, \frac{63}{16}\right)$.



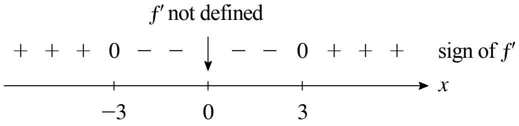
67. $F(t) = 3t^5 - 20t^3 + 20$. Setting $F'(t) = 15t^4 - 60t^2 = 15t^2\left(t^2 - 4\right) = 15t^2(t+2)(t-2) = 0$ gives $t = -2$, 0, and 2 as critical numbers. From the sign diagram, we see that $t = -2$ gives a relative maximum and $t = 2$ gives a relative minimum. The values are $F(-2) = 3(-32) - 20(-8) + 20 = 84$ and $F(2) = 3(32) - 20(8) + 20 = -44$.



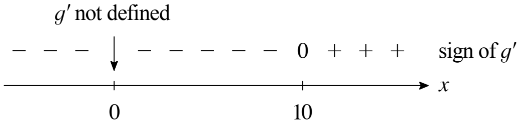
68. $h(x) = \dfrac{x}{x+1}$, so $h'(x) = \dfrac{(x+1)(1) - x(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}$. Because $x = -1$ is not in the domain of h, we see that $x = -1$ is not a critical number of h and conclude that h has no extremum.

69. $g'(x) = \dfrac{d}{dx}\left(1 + \dfrac{1}{x}\right) = -\dfrac{1}{x^2}$. Observe that g' is nonzero for all values of x. Furthermore, g' is undefined at $x = 0$, but $x = 0$ is not in the domain of g. Therefore, g has no critical number and hence no relative extremum.

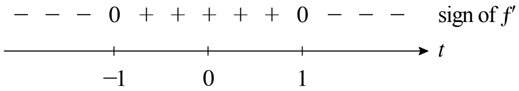
70. $f(x) = x + \dfrac{9}{x} + 2$, so $f'(x) = 1 - \dfrac{9}{x^2} = \dfrac{x^2 - 9}{x^2} = \dfrac{(x+3)(x-3)}{x^2} = 0$ gives $x = -3$ and $x = 3$ as critical numbers. From the sign diagram, we see that $(-3, -4)$ is a relative maximum and $(3, 8)$ is a relative minimum.



71. $g(x) = 2x^2 + \dfrac{4000}{x} + 10$, so $g'(x) = 4x - \dfrac{4000}{x^2} = \dfrac{4(x^3 - 1000)}{x^2}$. The only critical number of g is $x = 10$; $x = 0$ is not a critical number of g since $g(x)$ is not defined there. Using the sign diagram of g' and the First Derivative Test, we conclude that $(10, 610)$ is a relative minimum of g.



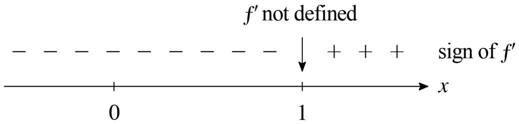
72. $f(x) = \dfrac{x}{1+x^2}$, so $f'(x) = \dfrac{(1+x^2)(1) - x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2} = \dfrac{(1-x)(1+x)}{(1+x^2)^2} = 0$ if $x = \pm 1$, and these are critical numbers of f. From the sign diagram of f', we see that f has a relative minimum at $\left(-1, -\frac{1}{2}\right)$ and a relative maximum at $\left(1, \frac{1}{2}\right)$.



73. $g(x) = \dfrac{x}{x^2 - 1}$, so $g'(x) = \dfrac{(x^2-1) - x(2x)}{(x^2-1)^2} = -\dfrac{1+x^2}{(x^2-1)^2}$ is never zero. Furthermore, $x \pm 1$ are not critical numbers since they are not in the domain of g. Therefore, g has no relative extremum.

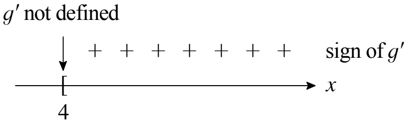
74. $f(x) = (x-1)^{2/3}$, so

$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \dfrac{2}{3(x-1)^{1/3}}$. $f'(x)$ is discontinuous at $x = 1$. From the sign diagram for f', we conclude that $f(1) = 0$ is a relative minimum.

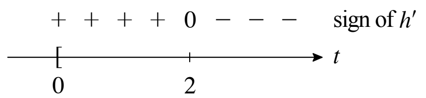


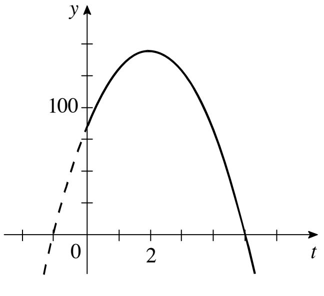
75. $g(x) = x\sqrt{x-4} = x(x-4)^{1/2}$, so

$g'(x) = (x-4)^{1/2} + x\left(\frac{1}{2}\right)(x-4)^{-1/2} = \frac{1}{2}(x-4)^{-1/2}[2(x-4) + x] = \dfrac{3x-8}{2\sqrt{x-4}}$ is continuous everywhere except at $x = 4$ and has a zero at $x = \frac{8}{3}$. But both $\frac{8}{3}$ and 4 lie outside the interval $(4, \infty)$, so there is no critical number, and accordingly g has no relative minimum.



76. $h(t) = -16t^2 + 64t + 80$, so $h'(t) = -32t + 64 = -32(t - 2)$. The sign diagram shows us that the stone is rising in the time interval $(0, 2)$ and falling when $t > 2$. It hits the ground when $h(t) = -16t^2 + 64t + 80 = 0$, that is, $t^2 - 4t - 5 = (t - 5)(t + 1) = 0$ or $t = 5$. (We reject the root $t = -1$.)





77. $N(t) = 1.1375t^2 + 0.25t + 4.6$, so $N'(t) = 2.275t + 0.25$. Because $N'(t) > 0$ for all t in the interval under consideration, we see that N is increasing on the interval $(0, 4)$. We conclude that the percentage of U.S. homes and businesses with digital meters was always increasing between 2008 and 2012.

78. $P(x) = -0.001x^2 + 8x - 5000$, so $P'(x) = -0.002x + 8 = 0$ if $x = 4000$. Observe that $P'(x) > 0$ if $x < 4000$ and $P'(x) < 0$ if $x > 4000$. So P is increasing on $(0, 4000)$ and decreasing on $(4000, \infty)$.

79. **a.** $f'(t) = \frac{d}{dt}\left(0.469t^2 + 0.758t + 0.44\right) = 0.938t + 0.758$. Because $f'(t) > 0$ for t in $(0, 10)$, we conclude that f is increasing on that interval.

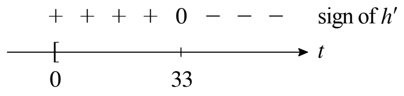
b. The result of part (a) tells us that sales in managed services grew from 1999 through 2009.

80. **a.** The number of Hispanic voters in the 2000 election was approximately $N(0) = 6$ (million); the number in the 2004 election was approximately $N(4) \approx 7.6$ (million); the number in the 2008 election was approximately $N(8) \approx 9.35$ (million); the number in the 2012 election was approximately $N(12) \approx 11.25$ (million); and the number in the 2016 election is projected to be $N(16) \approx 13.3$ (million).

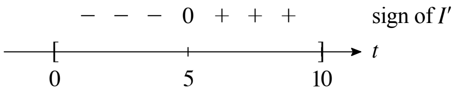
b. $N'(t) = \frac{d}{dt}\left(0.0046875t^2 + 0.38125t + 6\right) = 0.009375t + 0.38125$.

c. $N'(4) = 0.009375(4) + 0.38125 = 0.41875$, $N'(8) = 0.45625$, and $N'(12) = 0.49375$. Because $N'(12) > N'(8) > N'(4)$, we see that the growth of Hispanic voters in recent elections has been increasing steadily.

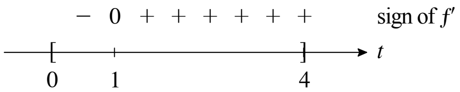
81. $h(t) = -\frac{1}{3}t^3 + 16t^2 + 33t + 10$, so $h'(t) = -t^2 + 32t + 33 = -(t + 1)(t - 33)$. The sign diagram for h' shows that the rocket is ascending on the time interval $(0, 33)$ and descending on $(33, T)$ for some positive number T.



82. $I(t) = \frac{1}{3}t^3 - \frac{5}{2}t^2 + 80$, so $I'(t) = t^2 - 5t = t(t - 5) = 0$ if $t = 0$ or 5. From the sign diagram, we see that I is decreasing on $(0, 5)$ and increasing on $(5, 10)$. After declining from 2004 through 2009, the index begins to increase after 2009.



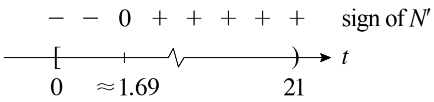
83. $f(t) = 20t - 40\sqrt{t} + 50 = 20t - 40t^{1/2} + 50$, so $f'(t) = 20 - 40\left(\frac{1}{2}t^{-1/2}\right) = 20\left(1 - \frac{1}{\sqrt{t}}\right) = \frac{20\left(\sqrt{t} - 1\right)}{\sqrt{t}}$. Thus, f' is continuous on $(0, 4)$ and is equal to zero at $t = 1$. From the sign diagram, we see that f is decreasing on $(0, 1)$ and increasing on $(1, 4)$. We conclude that the average speed decreases from 6 a.m. to 7 a.m. and then picks up from 7 a.m. to 10 a.m.



84. $\overline{C}(x) = -0.0001x + 2 + \frac{2000}{x}$, so $\overline{C}'(x) = -0.0001 - \frac{2000}{x^2} < 0$ for all values of x, and therefore $\overline{C}$ is always decreasing.

85. $D'(t) = \frac{d}{dt}\left(0.0032t^3 - 0.0698t^2 + 0.6048t + 3.22\right) = 0.0096t^2 - 0.1396t + 0.6048$. The discriminant of the quadratic equation $D'(t) = 0$ is $b^2 - 4ac = (-0.1396)^2 - 4(0.0096)(0.6048) = -0.00373616$. Since this is negative, we conclude that the function D has no zeros on $(-\infty, \infty)$ and, in particular, on $(0, 21)$. Also, $D'(0) = 0.6048 > 0$. These two results tell us that $D'(t) > 0$ on $(0, 21)$. Thus, the U.S. public debt outstanding was increasing throughout the period under consideration.

86. $N(t) = -0.062t^3 + 0.617t^2 - 1.557t + 7.7$. The number of tourists visiting Hawaii in 2007 was approximately $N(0) = 7.7$ (million). $N'(t) = -0.186t^2 + 1.234t - 1.557$, so using the quadratic formula with $N'(t) = 0$ gives $t = \frac{-1.234 \pm \sqrt{(1.234)^2 - 4(-0.186)(-1.557)}}{2(-0.186)} \approx -4.94$ or 1.69. So on the interval $(0, 21)$, the only critical number of N is approximately 1.69. From the sign diagram, we see that N has a relative minimum at $t \approx 1.69$. Thus, tourism reached a low point at some time short of 2009, then recovered to approximately $N(4) \approx 7.4$ (million visitors) in 2011.



87. a. $f'(t) = \frac{d}{dt}\left(-0.05t^3 + 0.56t^2 + 5.47t + 7.5\right) = -0.15t^2 + 1.12t + 5.47$. Setting $f'(t) = 0$ gives $-0.15t^2 + 1.12t + 5.47 = 0$. Using the quadratic formula, we find $t = \frac{-1.12 \pm \sqrt{(1.12)^2 - 4(-0.15)(5.47)}}{-0.3}$; that is, $t = -3.37$ or 10.83. Because f' is continuous, the only critical numbers of f are $t \approx -3.4$ and $t \approx 10.8$, both of which lie outside the interval of interest. Nevertheless, this result can be used to tell us that f' does not change sign in the interval $(-3.4, 10.8)$. Using $t = 0$ as the test number, we see that $f'(0) = 5.47 > 0$ and so we see that f is increasing on $(-3.4, 10.8)$ and, in particular, in the interval $(0, 10)$. Thus, we conclude that f is increasing on $(0, 10)$.

b. The result of part (a) tells us that sales in the web hosting industry will be increasing from 1999 through 2009.

88. $C'(x) = \frac{d}{dx}\left(1910.5x^{-1.72} + 42.9\right) = -\frac{3286.06}{x^{2.72}} < 0$ for all $x > 0$, and therefore for $5 \le x \le 10$. This says that the average cost of driving a 2012 medium-sized sedan decreases as the number of miles driven increases.

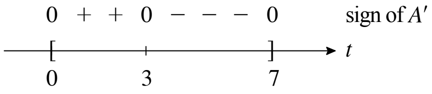
89. $f'(t) = \frac{d}{dt}\left[90.7\left(0.01t^2+0.01t+1\right)^{-1}\right] = -90.7\left(0.01t^2+0.01t+1\right)^{-2}(0.02t+0.01)$

$$= -\frac{90.7(0.02t+0.01)}{\left(0.01t^2+0.01t+1\right)^2}.$$

We see that $f'(t)$ is negative for $t \geq 0$, and thus for $0 \leq t \leq 4$, showing that the volume of first class mail deliveries has been declining throughout the period under consideration.

90. $A(t) = 0.03t^3(t-7)^4 + 60.2$, so

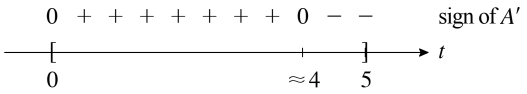
$A'(t) = 0.09t^2(t-7)^4 + 0.03t^3(4)(t-7)^3 = 0.03t^2(t-7)^3[3(t-7)+4t] = 0.21t^2(t-3)(t-7)^3$.



From the sign diagram of A', we see that A is increasing on $(0,3)$ and decreasing on $(3,7)$. This says that the pollution is increasing from 7 a.m. to 10 a.m. and decreasing from 10 a.m. to 2 p.m.

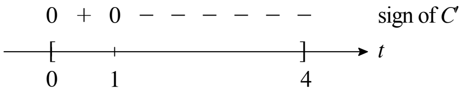
91. $A(t) = -96.6t^4 + 403.6t^3 + 660.9t^2 + 250$, so

$A'(t) = -386.4t^3 + 1210.8t^2 + 1321.8t = t\left(-386.4t^2 + 1210.8t + 1321.8\right)$. Solving $A'(t) = 0$, we find $t = 0$ and $t = \dfrac{-1210.8 \pm \sqrt{(1210.8)^2 - 4(-386.4)(1321.8)}}{-2(386.4)} = \dfrac{-1210.8 \pm 1873.2}{-2(386.4)} \approx 4$. Because t lies in the interval $(0,5)$, we see that the continuous function A' has zeros at $t = 0$ and $t = 4$.



From the sign diagram, we see that A is increasing on $(0,4)$ and decreasing on $(4,5)$. We conclude that the cash in the Central Provident Trust Funds will be increasing from 2005 to 2045 and decreasing from 2045 to 2055.

92. $C(t) = \dfrac{t^2}{2t^3+1}$, so $C'(t) = \dfrac{\left(2t^3+1\right)(2t) - t^2\left(6t^2\right)}{\left(2t^3+1\right)^2} = \dfrac{2t-2t^4}{\left(2t^3+1\right)^2} = \dfrac{2t\left(1-t^3\right)}{\left(2t^3+1\right)^2}$. From the sign diagram of C', we see that the drug concentration is increasing on $(0,1)$ and decreasing on $(1,4)$.



93. **a.** In 2005, the percentage was $f(0) = \dfrac{5.3\sqrt{0}-300}{\sqrt{0}-10} = 30$ (%). In 2015, it will be

$$f(10) = \frac{5.3\sqrt{10}-300}{\sqrt{10}-10} \approx 41.4\ (\%).$$

b. $f'(t) = \dfrac{\left(t^{1/2}-10\right)(5.3)\left(\frac{1}{2}t^{-1/2}\right) - \left(5.3t^{1/2}-300\right)\left(\frac{1}{2}t^{-1/2}\right)}{\left(t^{1/2}-10\right)^2} = \dfrac{247}{2\sqrt{t}\left(\sqrt{t}-10\right)^2} > 0$. Thus, f is increasing on $(0,10)$, indicating that the percentage of small and lower-midsize vehicles is increasing from 2005 through 2015.

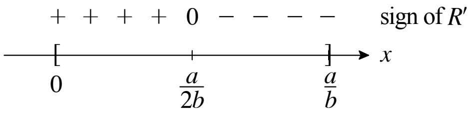
94. $A(t) = \dfrac{136}{1+0.25(t-4.5)^2} + 28$, so

$$A'(t) = 136\frac{d}{dt}\left[\frac{1}{1+0.25(t-4.5)^2} + 28\right] = -\frac{136}{\left[1+0.25(t-4.5)^2\right]^2}\cdot 2(0.25)(t-4.5) = -\frac{68(t-4.5)}{\left[1+0.25(t-4.5)^2\right]^2}.$$

Observe that $A'(t) > 0$ if $t < 4.5$ and $A'(t) < 0$ if $t > 4.5$, so the pollution is increasing from 7 A.M. to 11:30 A.M. and decreasing from 11:30 A.M. to 6 P.M.

95. **a.** $R(x) = px = (a - bx)x = ax - bx^2$.

b. $R'(x) = a - 2bx$. Solving $R'(x) = a - bx = 0$ gives $x = \frac{a}{2b}$. From the sign diagram for R', we see that the revenue is increasing when the level of production is between $x = 0$ and $x = \frac{a}{2b}$, revenue is stable at a production level of $x = \frac{a}{2b}$, and revenue is decreasing when the production level is between $x = \frac{a}{2b}$ and $x = \frac{a}{b}$.

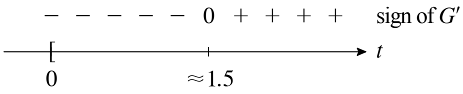


c. From the results of part (b), we conclude that a production level of $x = \frac{a}{2b}$ would yield a maximum revenue of $R\left(\frac{a}{2b}\right) = \frac{a^2}{4b}$.

96. **a.** $G(t) = (D - S)(t) = D(t) - S(t) = \left(0.0007t^2 + 0.0265t + 2\right) - \left(-0.0014t^2 + 0.0326t + 1.9\right)$

$= 0.0021t^2 - 0.0061t + 0.1$.

b. $G'(t) = 0.0042t - 0.0061 = 0$ implies $t \approx 1.45$. From the sign diagram of G', we see that G is decreasing on (0, 1.5) and increasing on (1.5, 15). This shows that the gap between the demand and supply of nurses was increasing from 2000 through the middle of 2001 but starts widening from the middle of 2001 through 2015.



c. The relative minimum of G occurs at $t = 1.5$ and is $f(1.5) \approx 0.0956$. This says that the smallest shortage is approximately 96,000.

97. False. The function $f(x) = \begin{cases} -x + 1 & \text{if } x < 0 \\ -\frac{1}{2}x + 1 & \text{if } x \geq 0 \end{cases}$ is decreasing on $(-1, 1)$, but $f'(0)$ does not exist.

98. True. We will show that $g = -f$ is decreasing on (a, b). Let x_1 and x_2 be any two numbers such that $a < x_1 < x_2 < b$. Since f is increasing on (a, b), we have $f(x_1) < f(x_2)$, so $-f(x_1) > -f(x_2)$; that is, $g(x_1) > g(x_2)$. This says that g is decreasing on (a, b). If f is differentiable, then the result follows from the fact that $g' = -f'$ and so if $f'(x) > 0$ on (a, b), then $g'(x) < 0$ on (a, b).

99. True. Let $a < x_1 < x_2 < b$. Then $f(x_2) > f(x_1)$ and $g(x_2) > g(x_1)$. Therefore, $(f + g)(x_2) = f(x_2) + g(x_2) > f(x_1) + g(x_1) = (f + g)(x_1)$, and so $f + g$ is increasing on (a, b).

100. True. Let $a < x_1 < x_2 < b$. Then $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$. We find

$$\begin{aligned}(fg)(x_2) - (fg)(x_1) &= f(x_2)g(x_2) - f(x_1)g(x_1) \\ &= f(x_2)g(x_2) - f(x_2)g(x_1) + f(x_2)g(x_1) - f(x_1)g(x_1) \\ &= f(x_2)\left[g(x_2) - g(x_1)\right] + g(x_1)\left[f(x_2) - f(x_1)\right] > 0,\end{aligned}$$

so $(fg)(x_2) > (fg)(x_1)$ and fg is increasing on (a, b).

101. False. Consider $f(x) = g(x) = -x$ on $(-1, 1)$. Then both f and g are decreasing on $(-1, 1)$, but $(fg)(x) = f(x)g(x) = (-x)(-x) = x^2$ is not decreasing on $(-1, 1)$.

102. False. Let $f(x) = x^3$. Then $f'(0) = 3x^2\big|_{x=0} = 0$, but f does not have a relative extremum at $x = 0$.

103. True. The derivative of a polynomial function of degree n ($n \geq 2$) is a polynomial function of degree $n - 1$. Since a polynomial of degree $n - 1$ has at most $n - 1$ zeros, and therefore at most $n - 1$ critical numbers, it follows that the original function of degree n has at most $n - 1$ relative extrema.

104. False. consider $f(x) = g(x) = -x$. Both are decreasing on $(0, 1)$, but $h(x) = (g \circ f)(x) = -(-x) = x$ is increasing on that interval.

105. We compute $f'(x) = m$. If $m > 0$, then $f'(x) > 0$ for all x and f is increasing; if $m < 0$, then $f'(x) < 0$ for all x and f is decreasing; if $m = 0$, then $f'(x) = 0$ for all x and f is a constant function.

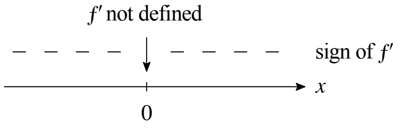
106. We require that $f'(2) = 0$, that is, $f'(2) = (-2x + a)|_{x=2} = -4 + a = 0$, so $a = 4$. Now $f(2) = 7$ implies that $f(2) = -4 + 2a + b = 7$, so $-4 + 2(4) + b = 7$ and $b = 3$. Therefore, $a = 4$ and $b = 3$.

107. We require that $f'(-1) = 0$; that is, $f'(-1) = \left(3ax^2 + 12x + b\right)\big|_{x=-1} = 3a - 12 + b = 0$, and therefore $f'(2) = 0$, or $f'(2) = \left(3ax^2 + 12x + b\right)\big|_{x=2} = 12a + 24 + b = 0$. Solving the system $\begin{cases} 3a + b = 12 \\ 12a + b = -24 \end{cases}$ simultaneously gives $a = -4$ and $b = 24$.

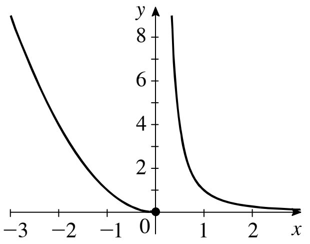
108. **a.** $f'(x) = \begin{cases} -3 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}$ $f'(-1) = -3$ and $f'(1) = 2$, so $f'(x)$ changes sign as we move across $x = 0$.

b. f does not have a relative minimum at $x = 0$ because $f(0) = 4$ but $f(x) < 4$ if x is a little less than 4. This does not contradict the First Derivative Test because f' is not continuous at $x = 0$.

109. **a.** $f'(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ -\dfrac{2}{x^3} & \text{if } x > 0 \end{cases}$ The sign diagram of f' shows that f' does not change sign as we move across $x = 0$.



b. From the graph of f, we see that f has a relative minimum at $x = 0$. This does not contradict the First Derivative Test since f is not a continuous function.



110. **a.** $f'(x) = 3x^2 + 1$, and so $f'(x) > 1$ on the interval $(0, 1)$. Therefore, f is increasing on $(0, 1)$.

b. $f(0) = -1$ and $f(1) = 1 + 1 - 1 = 1$. Thus, the Intermediate Value Theorem guarantees that there is at least one root of $f(x) = 0$ in $(0, 1)$. Because f is increasing on $(0, 1)$, the graph of f can cross the x-axis at only one point in $(0, 1)$, and so $f(x) = 0$ has exactly one root.

Using Technology page 271

1. **a.** f is decreasing on $(-\infty, -0.2934)$ and increasing on $(-0.2934, \infty)$.

b. Relative minimum $f(-0.2934) = -2.5435$.

2. **a.** f is decreasing on $(-\infty, -0.4067)$ and $(0.4563, 3.7421)$ and increasing on $(-0.4067, 0.4563)$ and $(3.7421, \infty)$.

b. Relative maximum $(0.4563, -2.5050)$, relative minima $(-0.4067, -5.3721)$ and $(3.7421, -109.1789)$.

3. **a.** f is increasing on $(-\infty, -1.6144)$ and $(0.2390, \infty)$ and decreasing on $(-1.6144, 0.2390)$.

b. Relative maximum $f(-1.6144) = 26.7991$, relative minimum $f(0.2390) = 1.6733$.

4. **a.** f is increasing on $(-\infty, \infty)$. **b.** None.

5. **a.** f is decreasing on $(-\infty, -1)$ and $(0.33, \infty)$ and increasing on $(-1, 0.33)$.

b. Relative maximum $f(0.33) = 1.11$, relative minimum $f(-1) = -0.63$.

6. **a.** f is increasing on $(-\infty, -0.87)$ and $(0.89, \infty)$ and decreasing on $(-0.87, 0)$ and $(0, 0.89)$.

b. Relative maximum $(-0.87, -2.22)$, relative minimum $(0.89, 0.51)$.

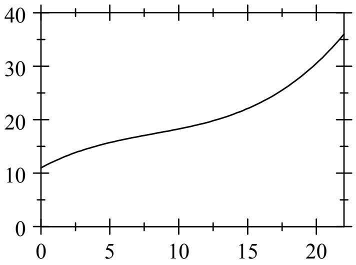
7. **a.** f is decreasing on $(-1, -0.71)$ and increasing on $(-0.71, 1)$.

b. f has a relative minimum at $(-0.71, -1.41)$.

8. **a.** f is decreasing on $(0, 2)$ and $(2, 2.47)$ and increasing on $(2.47, \infty)$.

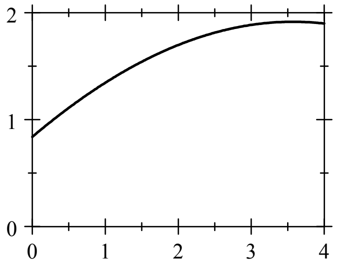
b. Relative minimum $(2.47, 87.01)$.

9. **a.**



b. $f(t) = 0.004545t^3 - 0.1113t^2 + 1.385t + 11$, so $f'(t) = 0.013635t^2 - 0.2226t + 1.385$. The discriminant of $f'(t) = 0$ is $(-0.2226)^2 - 4(0.013635)(1.385) \approx -0.026$ and $f'(0) = 1.385 > 0$, so f is increasing everywhere.

10.

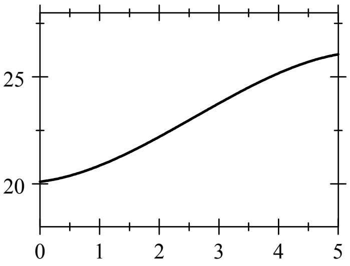


b. $N(3.57) \approx 1.91$, or 1.91 million, around July 2008.

11. The PSI is increasing on the interval $(0, 4.5)$ and decreasing on $(4.5, 11)$. It is highest when $t = 4.5$ (at 11:30 a.m.) and has value 164.

12. a. $y = -0.06204t^3 + 0.48135t^2 + 0.3354t + 20.1$, $0 \le t \le 5$.

b.



c. From the graph, it appears that f is increasing on $(0, 5)$. This tells us that the age at first marriage is increasing from 1960 through 2010.

d. $y' = -0.18612t^2 + 0.9627t + 0.3354$. Setting $f'(t) = 0$ and solving for t gives $t \approx -0.328$ or 5.50. Both numbers lie outside the domain of f, and because $f'(0) = 0.3354 > 0$, we conclude that $f'(t) > 0$ on $(0, 5)$. Thus, f is increasing on $(0, 5)$.

4.2 Applications of the Second Derivative

Concept Questions page 282

1. a. f is concave upward on (a, b) if f' is increasing on (a, b). f is concave downward on (a, b) if f' is decreasing on (a, b).

b. For the procedure for determining where f is concave upward and where f is concave downward, see page 274 of the text.

2. An inflection point of the graph of f is a point on the graph of f where its concavity changes from upward to downward or vice versa. See page 276 of the text for a procedure for finding inflection points.

3. The Second Derivative Test is stated on page 280 of the text. In general, if f'' is easy to compute, then use the Second Derivative Test. However, keep in mind that (1) in order to use this test f'' must exist, (2) the test is inconclusive if $f''(c) = 0$, and (3) the test is inconvenient to use if f'' is difficult to compute.

Exercises page 282

1. f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$. f has an inflection point at $(0, 0)$.

2. f is concave downward on $\left(0, \frac{3}{2}\right)$ and concave upward on $\left(\frac{3}{2}, \infty\right)$. f has an inflection point at $\left(\frac{3}{2}, 2\right)$.

3. f is concave downward on $(-\infty, 0)$ and $(0, \infty)$.

4. f is concave upward on $(-\infty, -4)$ and $(4, \infty)$ and concave downward on $(-4, 4)$.

5. f is concave upward on $(-\infty, 0)$ and $(1, \infty)$ and concave downward on $(0, 1)$. $(0, 0)$ and $(1, -1)$ are inflection points of f.

6. f is concave upward on $(0, 1)$ and $(5, \infty)$ and concave downward on $(1, 5)$.

7. f is concave downward on $(-\infty, -2)$ and $(-2, 2)$ and $(2, \infty)$.

8. f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$. $(0, 1)$ is an inflection point.

9. a. f is concave upward on $(0, 2)$, $(4, 6)$, $(7, 9)$, and $(9, 12)$ and concave downward on $(2, 4)$ and $(6, 7)$.

b. f has inflection points at $\left(2, \frac{5}{2}\right)$, $(4, 2)$, $(6, 2)$, and $(7, 3)$.

10. **a.** $f'(3) = 0$ and so 3 is a critical number of f. Also, $f''(3) < 0$, and so the Second Derivative Test applies and tells us that 3 gives a relative maximum. $f'(5) = 0$ and so 5 is a critical number of f. Also, $f''(5) > 0$, and so 5 gives a relative minimum according to the SDT.

b. The Second Derivative Test is not conclusive when applied to the critical number 7 because $f''(7) = 0$. It cannot be used at the critical number 9 because $f''(9)$ does not exist.

11. (a) **12.** (b) **13.** (b) **14.** (c)

15. **a.** $D_1'(t) > 0$, $D_2'(t) > 0$, $D_1''(t) > 0$, and $D_2''(t) < 0$ on $(0, 12)$.

b. With or without the proposed promotional campaign, the deposits will increase, but with the promotion, the deposits will increase at an increasing rate whereas without the promotion, the deposits will increase at a decreasing rate.

16. **a.** The graph of f is concave upward on $(0, t_1)$, indicating that Car A is accelerating for $0 < t < t_1$. The graph of g is concave downward on $(0, t_1)$, indicating that Car B is decelerating for $0 < t < t_1$.

b. The graph of f is concave downward on (t_1, t_2), indicating that Car A is decelerating for $t_1 < t < t_2$. The graph of g is concave upward on (t_1, t_2), indicating that Car B is accelerating for $t_1 < t < t_2$.

c. f has an inflection point at t_1, so the acceleration of Car A is greatest at $t = t_1$. g has an inflection point at t_1, so the acceleration of Car B is least at t_1.

d. The two cars have the same velocities at $t = 0$, $t = t_1$, and $t = t_2$.

17. (c) **18.** (a) **19.** (d) **20.** (b)

21. **a.** Between 8 a.m. and 10 a.m. the rate of change of the rate of smartphone assembly is increasing; between 10 a.m. and 12 noon, that rate is decreasing.

b. If you look at the tangent lines to the graph of N, you will see that the tangent line at P has the greatest slope. This means that the rate at which the average worker is assembling smartphones is greatest—that is, the worker is most efficient—when $t = 2$, at 10 a.m.

22. **a.** $f'(t)$ is negative on $(0, t_2)$ and positive on (t_2, t_3). $f''(t)$ is negative on $(0, t_1)$ and positive on (t_1, t_3).

b. The graph of f has an inflection point at $(t_1, f(t_1))$.

c. Total deposits were initially decreasing faster and faster. At time t_1, the rate of decrease reached its peak and began to lessen. At time t_2, deposits reached their lowest total and began to rebound.

23. The significance of the inflection point Q is that the restoration process is working at its peak at the time t_0 corresponding to the t-coordinate of Q.

24. **a.** The total change in the index over the period in question is zero.

b. The index was highest in June 2005.

c. The index was increasing at the greatest rate in June 2003 and decreasing at the greatest rate in June 2006.

25. $f(x) = 4x^2 - 12x + 7$, so $f'(x) = 8x - 12$ and $f''(x) = 8$. Thus, $f''(x) > 0$ everywhere, and so f is concave upward everywhere.

26. $g(x) = x^4 + \frac{1}{2}x^2 + 6x + 10$, so $g'(x) = 4x^3 + x + 6$ and $g''(x) = 12x^2 + 1$. We see that $g''(x) \geq 1$ for all values of x, and so g is concave upward everywhere.

27. $f(x) = \dfrac{1}{x^4} = x^{-4}$, so $f'(x) = -\dfrac{4}{x^5}$ and $f''(x) = \dfrac{20}{x^6} > 0$ for all values of x in $(-\infty, 0)$ and $(0, \infty)$, and so f is concave upward on its domain.

28. $g(x) = -\sqrt{4-x^2}$, so $g'(x) = \dfrac{d}{dx}\left[-\left(4-x^2\right)^{1/2}\right] = -\frac{1}{2}\left(4-x^2\right)^{-1/2}(-2x) = x\left(4-x^2\right)^{-1/2}$ and

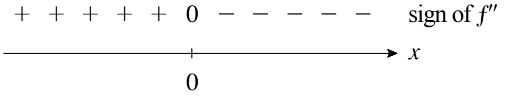
$$g''(x) = \left(4-x^2\right)^{-1/2} + x\left(-\tfrac{1}{2}\right)\left(4-x^2\right)^{-3/2}(-2x) = \left(4-x^2\right)^{-3/2}\left[\left(4-x^2\right)+x^2\right] = \frac{4}{\left(4-x^2\right)^{3/2}} > 0$$

whenever it is defined. Thus, g is concave upward wherever it is defined.

29. $f(x) = 2x^2 - 3x + 4$, so $f'(x) = 4x - 3$ and $f''(x) = 4x > 0$ for all values of x. Thus, f is concave upward on $(-\infty, \infty)$.

30. $g(x) = -x^2 + 3x + 4$, so $g'(x) = -2x + 3$ and $g''(x) = -2 < 0$ for all values of x. Thus, g is concave downward on $(-\infty, \infty)$.

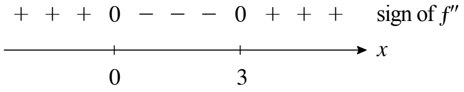
31. $f(x) = 1 - x^3$, so $f'(x) = -3x^2$ and $f''(x) = -6x$. From the sign diagram of f'', we see that f is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$.



32. $g(x) = x^3 - x$, so $g'(x) = 3x^2 - 1$ and $g''(x) = 6x$. Because $g''(x) < 0$ if $x < 0$ and $g''(x) > 0$ if $x > 0$, we see that g is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$.

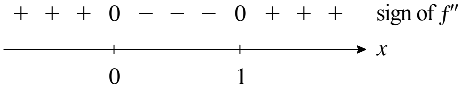
33. $f(x) = x^4 - 6x^3 + 2x + 8$, so $f'(x) = 4x^3 - 18x^2 + 2$ and $f''(x) = 12x^2 - 36x = 12x(x-3)$.

The sign diagram of f'' shows that f is concave upward on $(-\infty, 0)$ and $(3, \infty)$ and concave downward on $(0, 3)$.



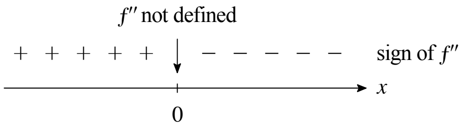
34. $f(x) = 3x^4 - 6x^3 + x - 8$, so $f'(x) = 12x^3 - 18x^2 + 1$ and $f''(x) = 36x^2 - 36x = 36x(x-1)$.

From the sign diagram of f'', we conclude that f is concave upward on $(-\infty, 0)$ and $(1, \infty)$ and concave downward on $(0, 1)$.



35. $f(x) = x^{4/7}$, so $f'(x) = \frac{4}{7}x^{-3/7}$ and $f''(x) = -\frac{12}{49}x^{-10/7} = -\dfrac{12}{49x^{10/7}}$. Observe that $f''(x) < 0$ for all $x \neq 0$, so f is concave downward on $(-\infty, 0)$ and $(0, \infty)$.

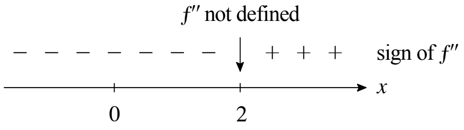
36. $f(x) = x^{1/3}$, so $f'(x) = \frac{1}{3}x^{-2/3}$ and $f''(x) = -\frac{2}{9}x^{-5/3} = -\dfrac{2}{9x^{5/3}}$. From the sign diagram of f'', we see that f is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$.



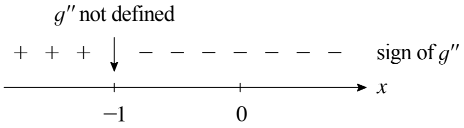
37. $f(x) = (4-x)^{1/2}$, so $f'(x) = \frac{1}{2}(4-x)^{-1/2}(-1) = -\frac{1}{2}(4-x)^{-1/2}$ and $f''(x) = \frac{1}{4}(4-x)^{-3/2}(-1) = -\dfrac{1}{4(4-x)^{3/2}} < 0$ whenever it is defined, so f is concave downward on $(-\infty, 4)$.

38. $g(x)=\sqrt{x-2}=(x-2)^{1/2}$, so $g'(x)=\frac{1}{2}(x-2)^{-1/2}$ and $g''(x)=-\frac{1}{4}(x-2)^{-3/2}=-\dfrac{1}{4(x-2)^{3/2}}$, which is negative for $x>2$. Next, the domain of g is $[2,\infty)$, and we conclude that g is concave downward on $(2,\infty)$.

39. $f'(x)=\dfrac{d}{dx}(x-2)^{-1}=-(x-2)^{-2}$ and $f''(x)=2(x-2)^{-3}=\dfrac{2}{(x-2)^3}$. The sign diagram of f''shows that f is concave downward on $(-\infty,2)$ and concave upward on $(2,\infty)$.



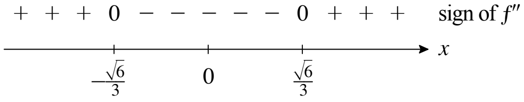
40. $g(x)=\dfrac{x}{x+1}$, so $g'(x)=\dfrac{(x+1)(1)-x(1)}{(x+1)^2}=\dfrac{1}{(x+1)^2}=(x+1)^{-2}$ and $g''(x)=-2(x+1)^{-3}=-\dfrac{2}{(x+1)^3}$. The sign diagram of $g''(x)$ shows that g is concave upward on $(-\infty,-1)$ and concave downward on $(-1,\infty)$.



41. $f'(x)=\dfrac{d}{dx}\left(2+x^2\right)^{-1}=-\left(2+x^2\right)^{-2}(2x)=-2x\left(2+x^2\right)^{-2}$ and

$f''(x)=-2\left(2+x^2\right)^{-2}-2x(-2)\left(2+x^2\right)^{-3}(2x)=2\left(2+x^2\right)^{-3}\left[-\left(2+x^2\right)+4x^2\right]=\dfrac{2\left(3x^2-2\right)}{\left(2+x^2\right)^3}=0$ if

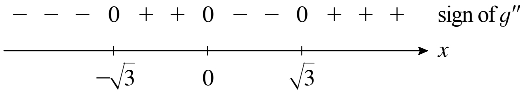
$x=\pm\sqrt{\frac{2}{3}}=\frac{\sqrt{6}}{3}$. From the sign diagram of f'', we see that f is concave upward on $\left(-\infty,-\frac{\sqrt{6}}{3}\right)$ and $\left(\frac{\sqrt{6}}{3},\infty\right)$ and concave downward on $\left(-\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{3}\right)$.



42. $g(x)=\dfrac{x}{1+x^2}$, so $g'(x)=\dfrac{\left(1+x^2\right)(1)-x(2x)}{\left(1+x^2\right)^2}=\dfrac{1-x^2}{\left(1+x^2\right)^2}$ and

$$g''(x)=\frac{\left(1+x^2\right)^2(-2x)-\left(1-x^2\right)2\left(1+x^2\right)(2x)}{\left(1+x^2\right)^4}=\frac{-2x\left(1+x^2\right)\left(1+x^2+2-2x^2\right)}{\left(1+x^2\right)^4}=-\frac{2x\left(3-x^2\right)}{\left(1+x^2\right)^3}.$$

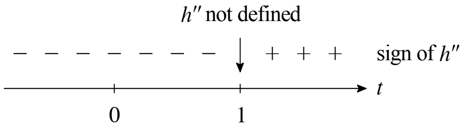
The sign diagram for g'' shows that g is concave downward on $\left(-\infty,-\sqrt{3}\right)$ and $\left(0,\sqrt{3}\right)$ and concave upward on $\left(-\sqrt{3},0\right)$ and $\left(\sqrt{3},\infty\right)$.



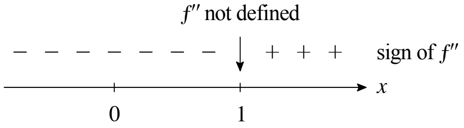
43. $h(t) = \frac{t^2}{t-1}$, so $h'(t) = \frac{(t-1)(2t) - t^2(1)}{(t-1)^2} = \frac{t^2-2t}{(t-1)^2}$ and

$$h''(t) = \frac{(t-1)^2(2t-2) - \left(t^2-2t\right)2(t-1)}{(t-1)^4} = \frac{(t-1)\left(2t^2-4t+2-2t^2+4t\right)}{(t-1)^4} = \frac{2}{(t-1)^3}.$$

The sign diagram of h'' shows that h is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.



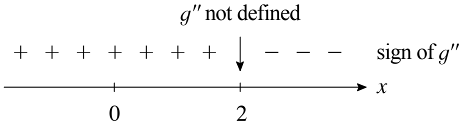
44. $f(x) = \frac{x+1}{x-1}$, so $f'(x) = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} = -\frac{2}{(x-1)^2} = -2(x-1)^{-2}$ and $f''(x) = (-2)(-2)(x-1)^{-3} = \frac{4}{(x-1)^3}$. The sign diagram of f'' shows that f is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.



45. $g(x) = x + \frac{1}{x^2}$, so $g'(x) = 1 - 2x^{-3}$ and $g''(x) = 6x^{-4} = \frac{6}{x^4} > 0$ whenever $x \neq 0$. Therefore, g is concave upward on $(-\infty, 0)$ and $(0, \infty)$.

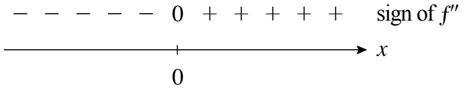
46. $h(r) = -(r-2)^{-2}$, so $h'(r) = 2(r-2)^{-3}$ and $h''(r) = -6(r-2)^{-4} < 0$ for all $r \neq 2$. Thus, h is concave downward on $(-\infty, 2)$ and $(2, \infty)$.

47. $g(t) = (2t-4)^{1/3}$, so $g'(t) = \frac{1}{3}(2t-4)^{-2/3}(2) = \frac{2}{3}(2t-4)^{-2/3}$ and $g''(t) = -\frac{4}{9}(2t-4)^{-5/3} = -\frac{4}{9(2t-4)^{5/3}}$. The sign diagram of g'' shows that g is concave upward on $(-\infty, 2)$ and concave downward on $(2, \infty)$.



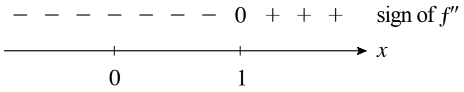
48. $f(x) = (x-2)^{2/3}$, so $f'(x) = \frac{2}{3}(x-2)^{-1/3}$ and $f''(x) = -\frac{2}{9}(x-2)^{-4/3} = -\frac{2}{9(x-2)^{4/3}} < 0$ for all $x \neq 2$. Therefore, f is concave downward on $(-\infty, 2)$ and $(2, \infty)$.

49. $f(x) = x^3 - 2$, so $f'(x) = 3x^2$ and $f''(x) = 6x$. $f''(x)$ is continuous everywhere and has a zero at $x = 0$. From the sign diagram of f'', we conclude that $(0, -2)$ is an inflection point of f.

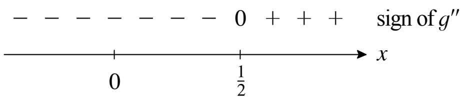


50. $g(x) = x^3 - 6x$, so $g'(x) = 3x^2 - 6$ and $g''(x) = 6x$. Observe that $g''(x) = 0$ if $x = 0$. Because $g''(x) < 0$ if $x < 0$ and $g''(x) > 0$ if $x > 0$, we see that $(0, 0)$ is an inflection point of g.

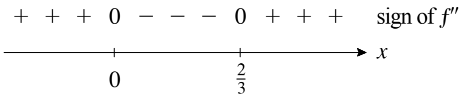
51. $f(x) = 6x^3 - 18x^2 + 12x - 20$, so $f'(x) = 18x^2 - 36x + 12$ and $f''(x) = 36x - 36 = 36(x-1) = 0$ if $x = 1$. The sign diagram of f'' shows that f has an inflection point at $(1, -20)$.



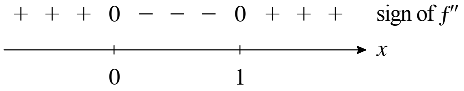
52. $g(x) = 2x^3 - 3x^2 + 18x - 8$, so $g'(x) = 6x^2 - 6x + 18$ and $g''(x) = 12x - 6 = 6(2x - 1)$. From the sign diagram of g'', we conclude that $\left(\frac{1}{2}, \frac{1}{2}\right)$ is an inflection point of g.



53. $f(x) = 3x^4 - 4x^3 + 1$, so $f'(x) = 12x^3 - 12x^2$ and $f''(x) = 36x^2 - 24x = 12x(3x - 2) = 0$ if $x = 0$ or $\frac{2}{3}$. These are candidates for inflection points. The sign diagram of f'' shows that $(0, 1)$ and $\left(\frac{2}{3}, \frac{11}{27}\right)$ are inflection points of f.



54. $f(x) = x^4 - 2x^3 + 6$, so $f'(x) = 4x^3 - 6x^2$ and $f''(x) = 12x^2 - 12x = 12x(x - 1)$. $f''(x)$ is continuous everywhere and has zeros at $x = 0$ and $x = 1$. From the sign diagram of f'', we conclude that $(0, 6)$ and $(1, 5)$ are inflection points of f.



55. $g(t) = t^{1/3}$, so $g'(t) = \frac{1}{3}t^{-2/3}$and $g''(t) = -\frac{2}{9}t^{-5/3} = -\dfrac{2}{9t^{5/3}}$. Observe that $t = 0$ is in the domain of g. Next, since $g''(t) > 0$ if $t < 0$ and $g''(t) < 0$ if $t > 0$, we see that $(0, 0)$ is an inflection point of g.

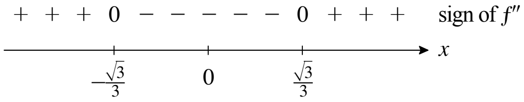
56. $f(x) = x^{1/5}$, so $f'(x) = \frac{1}{5}x^{-4/5}$ and $f''(x) = -\frac{4}{25}x^{-9/5} = -\dfrac{4}{25x^{9/5}}$. Observe that $f''(x) > 0$ if $x < 0$ and $f''(x) < 0$ if $x > 0$. Therefore, $(0, 0)$ is an inflection point.

57. $f(x) = (x - 1)^3 + 2$, so $f'(x) = 3(x - 1)^2$ and $f''(x) = 6(x - 1)$. Observe that $f''(x) < 0$ if $x < 1$ and $f''(x) > 0$ if $x > 1$ and so $(1, 2)$ is an inflection point of f.

58. $f(x) = (x - 2)^{4/3}$, so $f'(x) = \frac{4}{3}(x - 2)^{1/3}$ and $f''(x) = \frac{4}{9}(x - 2)^{-2/3} = \dfrac{4}{9(x - 2)^{2/3}}$. $x = 2$ is a candidate for an inflection point of f, but $f''(x) > 0$ for all values of x except $x = 2$ and so f has no inflection point.

59. $f(x) = \dfrac{2}{1 + x^2} = 2\left(1 + x^2\right)^{-1}$, so $f'(x) = -2\left(1 + x^2\right)^{-2}(2x) = -4x\left(1 + x^2\right)^{-2}$ and

$$f''(x) = -4\left(1 + x^2\right)^{-2} - 4x(-2)\left(1 + x^2\right)^{-3}(2x) = 4\left(1 + x^2\right)^{-3}\left[-\left(1 + x^2\right) + 4x^2\right] = \frac{4\left(3x^2 - 1\right)}{\left(1 + x^2\right)^3},$$ which is continuous everywhere and has zeros at $x = \pm\frac{\sqrt{3}}{3}$. From the sign diagram of f'', we conclude that $\left(-\frac{\sqrt{3}}{3}, \frac{3}{2}\right)$ and $\left(\frac{\sqrt{3}}{3}, \frac{3}{2}\right)$ are inflection points of f.



60. $f(x) = 2 + \dfrac{3}{x}$, so $f'(x) = -\dfrac{3}{x^2}$ and $f''(x) = \dfrac{6}{x^3}$. Now f'' changes sign as we move across $x = 0$, but $x = 0$ is not in the domain of f and so f has no inflection point.

61. $f(x) = -x^2 + 2x + 4$, so $f'(x) = -2x + 2$. The critical number of f is $x = 1$. Because $f''(x) = -2$ and $f''(1) = -2 < 0$, we conclude that $f(1) = 5$ is a relative maximum of f.

62. $g(x) = 2x^2 + 3x + 7$, so $g'(x) = 4x + 3 = 0$ if $x = -\frac{3}{4}$ and this is a critical number of g. Next, $g''(x) = 4$, and so $g''\left(-\frac{3}{4}\right) = 4 > 0$. Thus, $\left(-\frac{3}{4}, \frac{47}{8}\right)$ is a relative minimum of g.

63. $f(x) = 2x^3 + 1$, so $f'(x) = 6x^2 = 0$ if $x = 0$ and this is a critical number of f. Next, $f''(x) = 12x$, and so $f''(0) = 0$. Thus, the Second Derivative Test fails. But the First Derivative Test shows that $(0, 0)$ is not a relative extremum.

64. $g(x) = x^3 - 6x$, so $g'(x) = 3x^2 - 6 = 3\left(x^2 - 2\right) = 0$ implies that $x = \pm\sqrt{2}$ are the critical numbers of g. Next, $g''(x) = 6x$. Because $g''\left(-\sqrt{2}\right) = -6\sqrt{2} < 0$ and $g''\left(\sqrt{2}\right) = 6\sqrt{2} > 0$, we conclude, by the Second Derivative Test, that $\left(-\sqrt{2}, 4\sqrt{2}\right)$ is a relative maximum and $\left(\sqrt{2}, -4\sqrt{2}\right)$ is a relative minimum of g.

65. $f(x) = \frac{1}{3}x^3 - 2x^2 - 5x - 5$, so $f'(x) = x^2 - 4x - 5 = (x - 5)(x + 1)$ and this gives $x = -1$ and $x = 5$ as critical numbers of f. Next, $f''(x) = 2x - 4$. Because $f''(-1) = -6 < 0$, we see that $\left(-1, -\frac{7}{3}\right)$ is a relative maximum of f. Next, $f''(5) = 6 > 0$ and this shows that $\left(5, -\frac{115}{3}\right)$ is a relative minimum of f.

66. $f(x) = 2x^3 + 3x^2 - 12x - 4$, so $f'(x) = 6x^2 + 6x - 12 = 6\left(x^2 + x - 2\right) = 6(x + 2)(x - 1)$. The critical numbers of f are $x = -2$ and $x = 1$. $f''(x) = 12x + 6 = 6(2x + 1)$, so $f''(-2) = 6(-4 + 1) = -18 < 0$ and $f''(1) = 6(2 + 1) = 18 > 0$. Using the Second Derivative Test, we conclude that $f(-2) = 16$ is a relative maximum of f and $f(1) = -11$ is a relative minimum f.

67. $g(t) = t + \dfrac{9}{t}$, so $g'(t) = 1 - \dfrac{9}{t^2} = \dfrac{t^2 - 9}{t^2} = \dfrac{(t + 3)(t - 3)}{t^2}$, showing that $t = \pm 3$ are critical numbers of g. Now, $g''(t) = 18t^{-3} = \dfrac{18}{t^3}$. Because $g''(-3) = -\frac{18}{27} < 0$, the Second Derivative Test implies that g has a relative maximum at $(-3, -6)$. Also, $g''(3) = \frac{18}{27} > 0$ and so g has a relative minimum at $(3, 6)$.

68. $f(t) = 2t + 3t^{-1}$, so $f'(t) = 2 - 3t^{-2}$. Setting $f'(t) = 0$ gives $3t^{-2} = 2$ or $t^2 = \frac{3}{2}$, so $t = \pm\sqrt{\frac{3}{2}}$ are critical numbers of f. Next, we compute $f''(t) = 6/t^3$. Because $f''\left(-\sqrt{\frac{3}{2}}\right) < 0$ and $f''\left(\sqrt{\frac{3}{2}}\right) > 0$, we see that $f\left(-\sqrt{\frac{3}{2}}\right) = -2\sqrt{\frac{3}{2}} - 3\sqrt{\frac{2}{3}}$ is a relative maximum and $f\left(\sqrt{\frac{3}{2}}\right) = 2\sqrt{\frac{3}{2}} + 3\sqrt{\frac{2}{3}}$ is a relative minimum of f.

69. $f(x) = \dfrac{x}{1 - x}$, so $f'(x) = \dfrac{(1 - x)(1) - x(-1)}{(1 - x)^2} = \dfrac{1}{(1 - x)^2}$ is never zero. Thus, there is no critical number and f has no relative extremum.

70. $f(x) = \dfrac{2x}{x^2 + 1}$, so $f'(x) = \dfrac{\left(x^2 + 1\right)(2) - 2x(2x)}{\left(x^2 + 1\right)^2} = \dfrac{2\left(1 - x^2\right)}{\left(x^2 + 1\right)^2} = 0$ if $x = \pm 1$. Thus, $x = \pm 1$ are critical numbers of f. Next,

$$f''(x) = \frac{\left(x^2 + 1\right)^2(-4x) - 2\left(1 - x^2\right)2\left(x^2 + 1\right)(2x)}{\left(x^2 + 1\right)^4} = \frac{2x\left(x^2 + 1\right)\left(-2x^2 - 2 - 4 + 4x^2\right)}{\left(x^2 + 1\right)^4} = \frac{4x\left(x^2 - 3\right)}{\left(x^2 + 1\right)^3}.$$

Because $f''(-1) = \dfrac{-2(-4)}{2^3} = 1 > 0$ and $f''(1) = \dfrac{4(-2)}{2^3} = -1 < 0$, we see that f has a relative minimum at $(-1, -1)$ and a relative maximum at $(1, 1)$.

71. $f(t) = t^2 - \frac{16}{t}$, so $f'(t) = 2t + \frac{16}{t^2} = \frac{2t^3+16}{t^2} = \frac{2\left(t^3+8\right)}{t^2}$. Setting $f'(t) = 0$ gives $t = -2$ as a critical number. Next, we compute $f''(t) = \frac{d}{dt}\left(2t + 16t^{-2}\right) = 2 - 32t^{-3} = 2 - \frac{32}{t^3}$. Because $f''(-2) = 2 - \frac{32}{(-8)} = 6 > 0$, we see that $(-2, 12)$ is a relative minimum.

72. $g(x) = x^2 + \frac{2}{x}$, so $g'(x) = 2x - \frac{2}{x^2}$. Setting $g'(x) = 0$ gives $x^3 = 1$, or $x = 1$. Thus, $x = 1$ is the only critical number of g. Next, $g''(x) = 2 + \frac{4}{x^3}$. Because $g''(1) = 6 > 0$, we conclude that $g(1) = 3$ is a relative minimum of g.

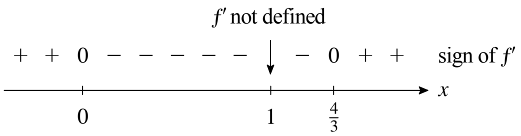
73. $g(s) = \frac{s}{1+s^2}$, so $g'(s) = \frac{\left(1+s^2\right)(1) - s(2s)}{\left(1+s^2\right)^2} = \frac{1-s^2}{\left(1+s^2\right)^2} = 0$ gives $s = -1$ and $s = 1$ as critical numbers of g. Next, we compute $g''(s) = \frac{\left(1+s^2\right)^2(-2s) - \left(1-s^2\right)2\left(1+s^2\right)(2s)}{\left(1+s^2\right)^4} = \frac{2s\left(1+s^2\right)\left(-1-s^2-2+2s^2\right)}{\left(1+s^2\right)^4} = \frac{2s\left(s^2-3\right)}{\left(1+s^2\right)^3}$. Now $g''(-1) = \frac{1}{2} > 0$, and so $g(-1) = -\frac{1}{2}$ is a relative minimum of g. Next, $g''(1) = -\frac{1}{2} < 0$ and so $g(1) = \frac{1}{2}$ is a relative maximum of g.

74. $g'(x) = \frac{d}{dx}\left(1+x^2\right)^{-1} = -\left(1+x^2\right)^{-2}(2x) = -\frac{2x}{\left(1+x^2\right)^2}$. Setting $g'(x) = 0$ gives $x = 0$ as the only critical number. Next, we find $g''(x) = \frac{\left(1+x^2\right)^2(-2) + 2x(2)\left(1+x^2\right)(2x)}{\left(1+x^2\right)^4} = \frac{-2\left(1+x^2\right)\left(1+x^2-4x^2\right)}{\left(1+x^2\right)^4} = -\frac{2\left(1-3x^2\right)}{\left(1+x^2\right)^3}$. Because $g''(0) = -2 < 0$, we see that $(0, 1)$ is a relative maximum.

75. $f(x) = \frac{x^4}{x-1}$, so $f'(x) = \frac{(x-1)\left(4x^3\right) - x^4(1)}{(x-1)^2} = \frac{4x^4-4x^3-x^4}{(x-1)^2} = \frac{3x^4-4x^3}{(x-1)^2} = \frac{x^3(3x-4)}{(x-1)^2}$. Thus, $x = 0$ and $x = \frac{4}{3}$ are critical numbers of f. Next,

$$\begin{aligned} f''(x) &= \frac{(x-1)^2\left(12x^3-12x^2\right) - \left(3x^4-4x^3\right)(2)(x-1)}{(x-1)^4} \\ &= \frac{(x-1)\left(12x^4-12x^3-12x^3+12x^2-6x^4+8x^3\right)}{(x-1)^4} \\ &= \frac{6x^4-16x^3+12x^2}{(x-1)^3} = \frac{2x^2\left(3x^2-8x+6\right)}{(x-1)^3}. \end{aligned}$$

Because $f''\left(\frac{4}{3}\right) > 0$, we see that $f\left(\frac{4}{3}\right) = \frac{256}{27}$ is a relative minimum. Because $f''(0) = 0$, the Second Derivative Test fails. Using the sign diagram for f' and the First Derivative Test, we see that $f(0) = 0$ is a relative maximum.

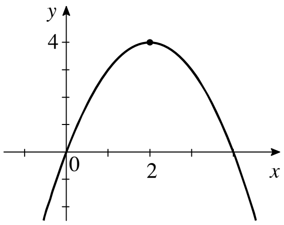


76. $f(x) = \dfrac{x^2}{x^2+1}$, so $f'(x) = \dfrac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} = \dfrac{2x}{(1+x^2)^2}$.

Setting $f'(x) = 0$ gives $x = 0$ as the only critical number of f.

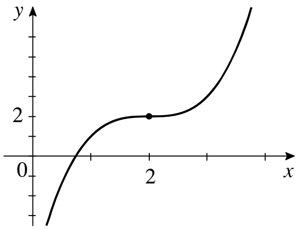
$f''(x) = \dfrac{(1+x^2)^2(2) - 2x(2)(1+x^2)(2x)}{(1+x^2)^4} = \dfrac{2(1+x^2)\left[(1+x^2) - 4x^2\right]}{(1+x^2)^4} = \dfrac{2(1-3x^2)}{(1+x^2)^3}$, so since $f''(0) = 2 > 0$, we see that $(0, 0)$ is a relative minimum.

77.



(Graph is not unique.)

78.



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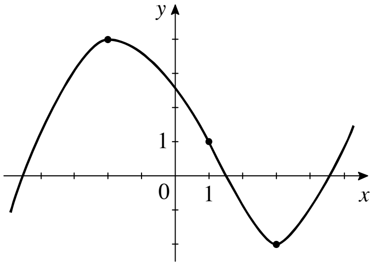

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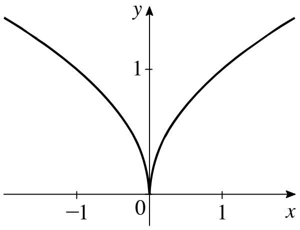

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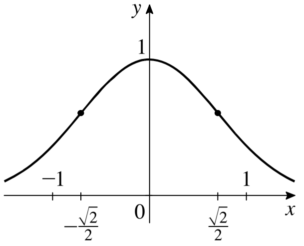
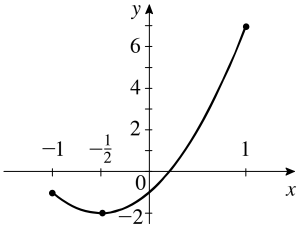


82.



83. **a.** $N'(t)$ is positive because N is increasing on $(0, 12)$.

b. $N''(t) < 0$ on $(0, 6)$ and $N''(t) > 0$ on $(6, 12)$.

c. The rate of growth of the number of help-wanted advertisements was decreasing over the first six months of the year and increasing over the last six months.

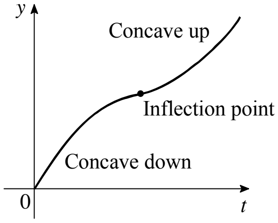
84. **a.** Both $N_1(t)$ and $N_2(t)$ are increasing on $(0, 12)$.

b. $N_1''(t) < 0$ and $N_2''(t) > 0$ on $(0, 12)$.

c. Although the projected number of crimes will increase in either case, a cut in the budget will see an accelerated increase in the number of crimes committed. With the budget intact, the rate of increase of crimes committed will continue to drop.

85. $f(t)$ increases at an increasing rate until the water level reaches the middle of the vase (this corresponds to the inflection point of f). At this point, $f(t)$ is increasing at the fastest rate. Though $f(t)$ still increases until the vase is filled, it does so at a decreasing rate.

86. The behavior of $f(t)$ is just the opposite of that given in the solution to Exercise 85. $f(t)$ increases at a decreasing rate until the water level reaches the middle of the vase (this corresponds to the inflection point of f). After that, $f(t)$ increases until the vase is filled and does so at an increasing rate (see the figure).



87. **a.** $f'(t) = \frac{d}{dt}\left(0.43t^{0.43}\right) = \left(0.43^2\right)t^{-0.57} = \frac{0.1849}{t^{0.57}}$ is positive if $t \geq 1$. This shows that f is increasing for $t \geq 1$, and this implies that the average state cigarette tax was increasing during the period in question.

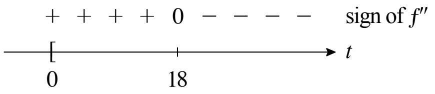
b. $f''(t) = \frac{d}{dt}\left(0.1849t^{-0.57}\right) = (0.1849)(-0.57)t^{-1.57} = -\frac{0.105393}{t^{1.57}}$ is negative if $t \geq 1$. Thus, the rate of the increase of the cigarette tax is decreasing over the period in question.

88. $C'(x) = \frac{d}{dx}\left(1910.5x^{-1.72} + 42.9\right) = -1.72(1910.5)x^{-2.72} = -3286.06x^{-2.72}$, so $C''(x) = -2.72(-3286.06)x^{-2.72} = \frac{8938.0832}{x^{2.72}}$. $C''(x) > 0$ for $x > 0$ and, in particular, for $5 < x < 20$. The per-mile cost of driving a 2012 medium-sized sedan decreases as the number of miles driven increases, but at a decreasing rate.

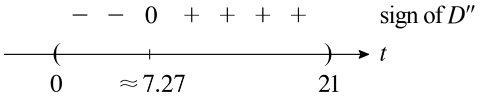
89. **a.** $A'(t) = \frac{d}{dt}\left(0.012414t^2 + 0.7485t + 313.9\right) = 0.024828t + 0.7485 \geq 0.024828(1) + 0.7485 = 0.773328$, so $A'(t)$ is positive for $t \geq 1$. Therefore, A is increasing on $(1, 56)$, showing that the average amount of atmospheric carbon dioxide is increasing from 1958 through 2013.

b. $A''(t) = \frac{d}{dt}A'(t) = \frac{d}{dt}(0.024828t + 0.7485) = 0.024828 > 0$, showing that the rate of increase of the amount of atmospheric carbon dioxide is increasing from 1958 through 2013.

90. $s = f(t) = -t^3 + 54t^2 + 480t + 6$, so the velocity of the rocket is $v = f'(t) = -3t^2 + 108t + 480$ and its acceleration is $a = f''(t) = -6t + 108 = -6(t - 18)$. From the sign diagram, we see that $(18, 20310)$ is an inflection point of f. Our computations reveal that the maximum velocity of the rocket is attained when $t = 18$. The maximum velocity is $f'(18) = -3(18)^2 + 108(18) + 480 = 1452$, or 1452 ft/sec.



91. **a.** $D(t) = 0.0032t^3 - 0.0698t^2 + 0.6048t + 3.22$, $0 \leq t \leq 21$, so $D'(t) = 0.0096t^2 - 0.1396t + 0.6048$ and $D''(t) = 0.0192t - 0.1396$. Setting $D''(t) = 0$ gives $t \approx 7.27$. From the sign diagram for D'', we see that the graph of D is concave downward on approximately $(0, 7.27)$ and concave upward on approximately $(7.27, 21)$.



b. The inflection point of the graph of D is approximately $(7.27, 5.16)$. The U.S. public debt was increasing at a decreasing rate from 1990 through the first quarter of 1997 (approximately), and then continued to increase, but at an increasing rate, from that point onward.

Note: In Exercise 4.1.85, we showed that D is increasing on $(0, 21)$.

92. $f(t) = -0.0004401t^3 + 0.007t^2 + 0.112t + 0.28$, so $f'(t) = -0.0013203t^2 + 0.014t + 0.12$ and $f''(t) = -0.0026406t + 0.014$. Setting $f''(t) = 0$, we find $t \approx 5.302$. We see that $f''(t) > 0$ on approximately $(0, 5.3)$ and $f''(t) < 0$ on approximately $(5.3, 21)$, so $t \approx 5.3$ gives an inflection point of f. $f(5.3) \approx -0.0004401(5.3)^3 + 0.007(5.3)^2 + 0.112(5.3) + 0.28 \approx 1.00$, so the point of inflection is approximately $(5.3, 1)$. We conclude that the death rate from AIDS worldwide was increasing most rapidly around March of 1995. At that point the rate was approximately 1 million deaths per year.

93. a. $f(t) = -0.083t^3 + 0.6t^2 + 0.18t + 20.1$, so at the beginning of 1960, the median age of women at first marriage was $f(0) = 20.1$. In 2000, it was $f(4) = -0.083(4)^3 + 0.6(4)^2 + 0.18(4) + 20.1 \approx 25.1$, and in 2001, it was $f(5) = -0.083(5)^3 + 0.6(5)^2 + 0.18(5) + 20.1 \approx 25.6$.

b. $f'(t) = -0.083(3t^2) + 0.6(2t) + 0.18 = -0.249t^2 + 1.2t + 0.18$, so $f''(t) = -0.249(2)t + 1.2 = 1.2 - 0.498t$. Thus, $f''(t) = 0$ when $t = \dfrac{1.2}{0.498} \approx 2.41$. Therefore, the median age was changing most rapidly approximately 2.41 decades after the beginning of 1960; that is, early in 1984.

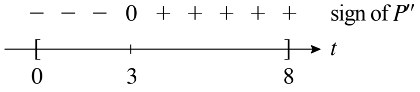
94. a. $R'(x) = \dfrac{d}{dx}(-0.003x^3 + 1.35x^2 + 2x + 8000) = -0.009x^2 + 2.7x + 2$; $R''(x) = -0.018x + 2.7$. Setting $R''(x) = 0$ gives $x = 150$. Because $R''(x) > 0$ if $x < 150$ and $R''(x) < 0$ if $x > 150$, we see that the graph of R is concave upward on $(0, 150)$ and concave downward on $(150, 400)$, so $x = 150$ gives an inflection point of R. Thus, the inflection point is $(150, 28550)$.

b. $R''(140) = 0.18$ and $R''(160) = -0.18$ This shows that at $x = 140$, a slight increase in x (spending) results in increased revenue. At $x = 160$, the opposite conclusion holds. So it would be more beneficial to increase the expenditure when it is \$140,000 than when it is \$160,000.

95. a. $A'(t) = \dfrac{d}{dt}\left[0.92(t+1)^{0.61}\right] = 0.92(0.61)(t+1)^{-0.39} = \dfrac{0.5612}{(t+1)^{0.39}} > 0$ on $(0, 4)$, so A is increasing on $(0, 4)$. This tells us that the spending is increasing over the years in question.

b. $A''(t) = (0.5612)(-0.39)(t+1)^{-1.39} = -\dfrac{0.218868}{(t+1)^{1.39}} < 0$ on $(0, 4)$, so A'' is concave downward on $(0, 4)$. This tells us that the spending is increasing but at a decreasing rate.

96. $P(t) = t^3 - 9t^2 + 40t + 50$, so $P'(t) = 3t^2 - 18t + 40$ and $P''(t) = 6t - 18 = 6(t - 3)$. The sign diagram of P'' shows that $(3, 116)$ is an inflection point. This analysis reveals that after declining the first 3 years, the growth rate of the company's profit is once again rising.



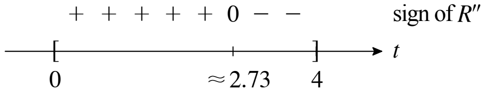
97. a. $P(t) = -0.007333t^3 + 0.91343t^2 + 8.507t + 439$, so $P'(t) = -0.021999t^2 + 1.82686t + 8.507$. Setting $P'(t) = 0$ and using the quadratic formula, we find $t = \dfrac{-1.82686 \pm \sqrt{(1.82686)^2 - 4(-0.021999)(8.507)}}{2(-0.021999)} \approx -4.42$ or 87.46. Both roots lie outside $(0, 31)$, so P has no critical number on that interval. Since $P'(1) \approx 10.31 > 0$, we see that $P'(t) > 0$ on $(0, 31)$. We conclude that the number of people aged 80 and over in Canada was increasing from 1981 through 2011.

b. $P''(t) = -0.043998t + 1.82686$. Setting $P''(t) = 0$ and solving, we find $t \approx 41.52$, which lies outside the interval $(0, 31)$. Because $P''(1) = 1.782862 > 0$, we see that $P''(t) > 0$ for all t in $(0, 31)$. Therefore, P' is increasing on $(0, 31)$. We conclude that the population of Canadians aged 80 and over was increasing at an increasing rate from 1981 through 2011.

98. a. $R(t) = -0.2t^3 + 1.64t^2 + 1.31t + 3.2$, so $R'(t) = -0.6t^2 + 3.28t + 1.31$ and $R''(t) = -1.2t + 3.28$.

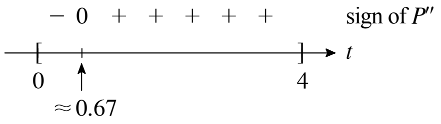
b. Setting $R'(t) = 0$ and solving for t gives $t = \dfrac{-3.28 \pm \sqrt{(3.28)^2 - 4(-0.6)(1.31)}}{2(-0.6)} \approx -0.374$ or 5.840. Both roots lie outside the interval $(0, 4)$. Because $R'(0) = 1.31 > 0$, we conclude that $R'(t) > 0$ for all t in $(0, 4)$.

c. Setting $R''(t) = 0$ gives $-1.2t + 3.28 = 0$, so $t \approx 2.73$. From the sign diagram, we see that R has an inflection point at approximately $(2.73, 14.93)$. This says that between 2004 and 2008, Google's revenue was increasing fastest in late August 2006.



99. a. $P'(t) = \frac{d}{dt}\left(44560t^3 - 89394t^2 + 234633t + 273288\right) = 133680t^2 - 178788 + 234633$. Observe that P' is continuous everywhere and $P'(t) = 0$ has no real solution since the discriminant $b^2 - 4ac = (-178788)^2 - 4(133680)(234633) = -93497808816 < 0$. Because $P'(0) = 234633 > 0$, we may conclude that $P'(t) > 0$ for all t in $(0, 4)$, so the population is always increasing.

b. $P''(t) = 267360t - 178788 = 0$ implies that $t = 0.67$. The sign diagram of P''shows that $t = 0.67$ is an inflection point of the graph of P, so the population was increasing at the slowest pace sometime during August of 1976.



100. a. $D(t) = D_2(t) - D_1(t) = 0.035t^2 + 0.211t + 0.24 - \left(0.0275t^2 + 0.081t + 0.07\right) = 0.0075t^2 + 0.13t + 0.17$, so $D'(t) = 0.015t + 0.13 \geq 0.13$ for $0 < t < 3$, showing that D is increasing on $(0, 3)$. Thus, the projected difference in year t between the deficit without the \$160 million rescue package and the deficit with the rescue package is increasing between 2011 and 2014.

b. $D''(t) = 0.015 > 0$ on $(0, 3)$, and so D is concave upward on $(0, 3)$. This says that the difference referred to in part (a) is increasing at an increasing rate between 2011 and 2014.

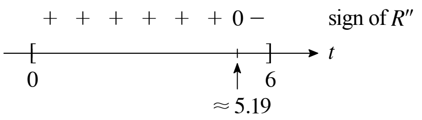
101. $A(t) = 1.0974t^3 - 0.0915t^4$, so $A'(t) = 3.2922t^2 - 0.366t^3$ and $A''(t) = 6.5844t - 1.098t^2$. Setting $A'(t) = 0$, we obtain $t^2(3.2922 - 0.366t) = 0$, and this gives $t = 0$ or $t \approx 8.995 \approx 9$. Using the Second Derivative Test, we find $A''(9) = 6.5844(9) - 1.098(81) = -29.6784 < 0$, and this tells us that $t \approx 9$ gives rise to a relative maximum of A. Our analysis tells us that on that May day, the level of ozone peaked at approximately 4 p.m.

102. a. $N'(t) = \frac{d}{dt}\left(-0.9307t^3 + 74.04t^2 + 46.8667t + 3967\right) = -2.7921t^2 + 148.08t + 46.8667$. $N'$is continuous everywhere and has zeros at $t = \frac{-148.08 \pm \sqrt{(148.08)^2 - 4(-0.9307)(46.8667)}}{2(-2.7921)}$, that is, at $t \approx -0.31$ or 53.35. Both these numbers lie outside the interval of interest. Picking $t = 0$ for a test number, we see that $N'(0) = 46.86667 > 0$ and conclude that N is increasing on $(0, 16)$. This shows that the number of participants is increasing over the years in question.

b. $N''(t) = \frac{d}{dt}\left(-2.7921t^2 + 148.08t + 46.86667\right) = -5.5842t + 148.08 = 0$ if $t \approx 26.518$. Thus, $N''(t)$ does not change sign in the interval $(0, 16)$. Because $N''(0) = 148.08 > 0$, we see that $N'(t)$ is increasing on $(0, 16)$ and the desired conclusion follows.

103. **a.** $R'(t) = \dfrac{d}{dt}\left(0.00731t^4 - 0.174t^3 + 1.528t^2 + 0.48t + 19.3\right) = 0.02924t^3 - 0.522t^2 + 3.056t + 0.48$ and $R''(t) = 0.08772t^2 - 1.044t + 3.056$. Solving the equation $R''(t) = 0$, we obtain $t = \dfrac{1.044 \pm \sqrt{(-1.044)^2 - 4(0.08772)(3.056)}}{2(0.08772)} \approx 5.19$ or 6.71.

From the sign diagram of R'', we see that the inflection points are approximately $(5.19, 43.95)$ and $(6.71, 53.56)$. We see that the dependency ratio will be increasing at the greatest pace around $t = 5.2$, that is, around 2052.



b. The dependency ratio will be $R(5.2) \approx 43.99$, or approximately 44.

104. True. If f' is increasing on (a, b), then $-f'$ is decreasing on (a, b), and so if the graph of f is concave upward on (a, b), the graph of $-f$ must be concave downward on (a, b).

105. False. Let $f(x) = x + \dfrac{1}{x}$ (see Example 2). Then f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$, but f does not have an inflection point at 0.

106. False. If $f(x) = x^2 - 1$, then f is concave up on $(0, 1)$, but $g(x) = \left(x^2 - 1\right)^2 = x^4 - 2x^2 + 1$ and $g''(x) = 12x^2 - 4 < 0$ on $\left(0, \frac{\sqrt{3}}{3}\right)$, so g is not concave up on $(0, 1)$.

107. False. Take $f(x) = x^{1/3}$ on $(-1, 1)$. Then f is defined on $(-1, 1)$ and f has an inflection point at $(0, 0)$, but $f'(x) = \frac{1}{3}x^{-2/3}$ and $f''(x) = -\frac{2}{9}x^{-5/3} = -\dfrac{2}{9x^{5/3}}$, so $f''(0)$ is undefined.

108. True. The given conditions imply that $f''(0) < 0$ and the Second Derivative Test gives the desired conclusion.

109. $f(x) = ax^2 + bx + c$, so $f'(x) = 2ax + b$ and $f''(x) = 2a$. Thus, $f''(x) > 0$ if $a > 0$, and the parabola opens upward. If $a < 0$, then $f''(x) < 0$ and the parabola opens downward.

110. **a.** $f'(x) = 3x^2$, $g'(x) = 4x^3$, and $h'(x) = -4x^3$. Setting $f'(x) = 0$, $g'(x) = 0$, and $h'(x) = 0$ gives $x = 0$ as a critical number of each function.

b. $f''(x) = 6x$, $g''(x) = 12x^2$, and $h''(x) = -12x^2$, so that $f''(0) = 0$, $g''(0) = 0$, and $h''(0) = 0$. Thus, the Second Derivative Test yields no conclusion in these cases.

c. Because $f'(x) > 0$ for both $x > 0$ and $x < 0$, $f'(x)$ does not change sign as we move across the critical number $x = 0$. Thus, by the First Derivative Test, f has no extremum at 0. Next, $g'(x)$ changes sign from negative to positive as we move across 0, showing that g has a relative minimum there. Finally, we see that $h'(x) > 0$ for $x < 0$ and $h'(x) < 0$ for $x > 0$, so h has a relative maximum at $x = 0$.

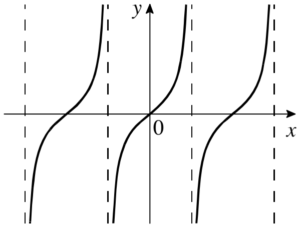
4.3 Curve Sketching

Concept Questions page 298

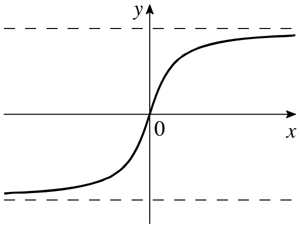
1. **a.** See the definition on page 293 of the text.

   b. See the definition on page 295 of the text.

2. **a.** There is no restriction to the number of vertical asymptotes the graph of a function can have.



   b. The graph of a function can have at most two horizontal asymptotes.



3. See the procedure given on page 292 of the text.

4. See the procedure given on page 295 of the text.

Exercises page 298

1. $y = 0$ is a horizontal asymptote.

2. $y = 0$ is a horizontal asymptote and $x = -1$ is a vertical asymptote.

3. $y = 0$ is a horizontal asymptote and $x = 0$ is a vertical asymptote.

4. $y = 0$ is a horizontal asymptote.

5. $y = 0$ is a horizontal asymptote and $x = -1$ and $x = 1$ are vertical asymptotes.

6. $y = 0$ is a horizontal asymptote.

7. $y = 3$ is a horizontal asymptote and $x = 0$ is a vertical asymptote.

8. $y = 0$ is a horizontal asymptote and $x = -2$ is a vertical asymptote.

9. $y = 1$ and $y = -1$ are horizontal asymptotes.

10. $y = 1$ is a horizontal asymptote and $x = \pm 1$ are vertical asymptotes.

11. $\lim_{x\to\infty} \frac{1}{x} = 0$, and so $y = 0$ is a horizontal asymptote. Next, since the numerator of the rational expression is not equal to zero and the denominator is zero at $x = 0$, we see that $x = 0$ is a vertical asymptote.

12. $\lim_{x\to\infty} \frac{1}{x+2} = 0$, and so $y = 0$ is a horizontal asymptote, Next, observe that the numerator of the rational function is not equal to zero but the denominator is equal to zero at $x = -2$, and so $x = -2$ is a vertical asymptote.

13. $f(x) = -\frac{2}{x^2}$, so $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(-\frac{2}{x^2}\right) = 0$. Thus, $y = 0$ is a horizontal asymptote. Next, the denominator of $f(x)$ is equal to zero at $x = 0$. Because the numerator of $f(x)$ is not equal to zero at $x = 0$, we see that $x = 0$ is a vertical asymptote.

14. $\lim_{x\to\infty} \frac{1}{1+2x^2} = 0$ and so $y = 0$ is a horizontal asymptote, Next, observe that the denominator $1 + 2x^2 \neq 0$, and so there is no vertical asymptote.

15. $\lim_{x\to\infty} \frac{x-2}{x+2} = \lim_{x\to\infty} \frac{1-\frac{2}{x}}{1+\frac{2}{x}} = 1$, and so $y = 1$ is a horizontal asymptote. Next, the denominator is equal to zero at $x = -2$ and the numerator is not equal to zero at this number, so $x = -2$ is a vertical asymptote.

16. $\lim_{t\to\infty} \frac{t+1}{2t-1} = \lim_{t\to\infty} \frac{1+\frac{1}{t}}{2-\frac{1}{t}} = \frac{1}{2}$, and so $y = \frac{1}{2}$ is a horizontal asymptote. Next, observe that the denominator of the rational expression is zero at $t = \frac{1}{2}$, but the numerator is not equal to zero at this number, and so $t = \frac{1}{2}$ is a vertical asymptote.

17. $h(x) = x^3 - 3x^2 + x + 1$. $h(x)$ is a polynomial function, and therefore it does not have any horizontal or vertical asymptotes.

18. The function g is a polynomial, and so the graph of g has no horizontal or vertical asymptotes.

19. $\lim_{t\to\infty} \frac{t^2}{t^2-16} = \lim_{t\to\infty} \frac{1}{1-\frac{16}{t^2}} = 1$, and so $y = 1$ is a horizontal asymptote. Next, observe that the denominator of the rational expression $t^2 - 16 = (t+4)(t-4) = 0$ if $t = -4$ or $t = 4$. But the numerator is not equal to zero at these numbers, so $t = -4$ and $t = 4$ are vertical asymptotes.

20. $\lim_{x\to\infty} \frac{x^3}{x^2-4} = \lim_{x\to\infty} \frac{x}{1-\frac{4}{x^2}} = \infty$, and similarly $\lim_{x\to\infty} \frac{x^3}{x^2-4} = -\infty$. Therefore, there is no horizontal asymptote. Next, note that the denominator of $g(x)$ equals zero at $x = \pm 2$. Because the numerator of $g(x)$ is not equal to zero at $x = \pm 2$, we see that $x = -2$ and $x = 2$ are vertical asymptotes.

21. $\lim_{x\to\infty} \frac{3x}{x^2-x-6} = \lim_{x\to\infty} \frac{\frac{3}{x}}{1-\frac{1}{x}-\frac{6}{x^2}} = 0$ and so $y = 0$ is a horizontal asymptote. Next, observe that the denominator $x^2 - x - 6 = (x-3)(x+2) = 0$ if $x = -2$ or $x = 3$. But the numerator $3x$ is not equal to zero at these numbers, so $x = -2$ and $x = 3$ are vertical asymptotes.

22. $\lim_{x\to\infty} \frac{2x}{x^2+x-2} = \lim_{x\to\infty} \frac{\frac{2}{x}}{1+\frac{1}{x}-\frac{2}{x^2}} = 0$, and so $y = 0$ is a horizontal asymptote. Next, observe that the denominator $x^2 + x - 2 = (x+2)(x-1) = 0$, if $x = -2$ or $x = 1$. The numerator is not equal to zero at these numbers, and so $x = -2$ and $x = 1$ are vertical asymptotes.

23. $\lim_{t\to\infty} \left[2 + \frac{5}{(t-2)^2}\right] = 2$, and so $y = 2$ is a horizontal asymptote. Next observe that $\lim_{t\to 2^+} g(t) = \lim_{t\to 2^-} \left[2 + \frac{5}{(t-2)^2}\right] = \infty$, and so $t = 2$ is a vertical asymptote.

24. $\lim_{x\to\infty}\left(1+\frac{2}{x-3}\right)=1$ and $\lim_{x\to-\infty}\left(1+\frac{2}{x-3}\right)=1$, so $y=1$ is a horizontal asymptote. Next, we write $f(x)=1+\frac{2}{x-3}=\frac{x-3+2}{x-3}=\frac{x-1}{x-3}$, and observe that the denominator of $f(x)$ is equal to zero at $x=3$. However, since the numerator of $f(x)$ is not equal to zero at $x=3$, we see that $x=3$ is a vertical asymptote.

25. $\lim_{x\to\infty}\frac{x^2-2}{x^2-4}=\lim_{x\to\infty}\frac{1-\frac{2}{x^2}}{1-\frac{4}{x^2}}=1$, and so $y=1$ is a horizontal asymptote. Next, observe that the denominator $x^2-4=(x+2)(x-2)=0$ if $x=-2$ or 2. Because the numerator x^2-2 is not equal to zero at these numbers, the lines $x=-2$ and $x=2$ are vertical asymptotes.

26. $\lim_{x\to\infty}\frac{2-x^2}{x^2+x}=\lim_{x\to\infty}\frac{\frac{2}{x^2}-1}{1+\frac{1}{x}}=-1$, and so $y=1$ is a horizontal asymptote. Next, observe that the denominator $x^2+x=x(x+1)=0$ if $x=0$ or $x=-1$. Because the numerator $2-x^2$ is not equal to zero at these values of x, we see that $x=0$ and $x=-1$ are vertical asymptotes.

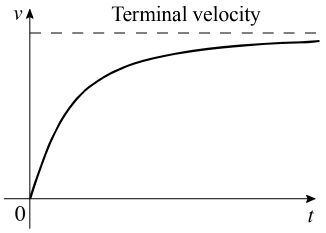
27. $g(x)=\frac{x^3-x}{x(x+1)}$. Rewrite $g(x)$ as $\frac{x^2-1}{x+1}$ for $x\neq 0$, and note that $\lim_{x\to-\infty}g(x)=\lim_{x\to-\infty}\frac{x-\frac{1}{x}}{1+\frac{1}{x}}=-\infty$ and $\lim_{x\to\infty}g(x)=\infty$. Therefore, there is no horizontal asymptote. Next, note that the denominator of $g(x)$ is equal to zero at $x=0$ and $x=-1$. However, since the numerator of $g(x)$ is also equal to zero when $x=0$, we see that $x=0$ is not a vertical asymptote. Also, the numerator of $g(x)$ is equal to zero when $x=-1$, so $x=-1$ is not a vertical asymptote.

28. $\lim_{x\to\infty}\frac{x^4-x^2}{x(x-1)(x+2)}=\lim_{x\to\infty}\left[x\cdot\frac{1-\frac{1}{x^2}}{\left(1-\frac{1}{x}\right)\left(1+\frac{2}{x}\right)}\right]=\infty$, so there is no horizontal asymptote. Next, observe that the denominator is zero at $x=0$, $x=1$, and $x=-2$. Of these values, only $x=-2$ is a vertical asymptote because the numerator is not also equal to zero at this value.

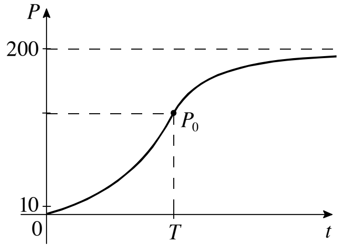
29. f is the derivative function of the function g. Observe that at a relative maximum or minimum of g, $f(x)=0$.

30. f is the derivative function of the function g. Observe that at a relative maximum or minimum of g, $f(x)=0$.

31.



32. a.



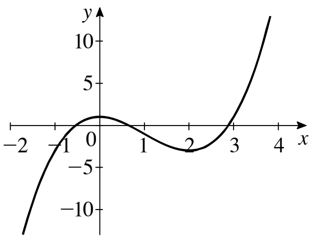
b. f is increasing on $(0,\infty)$.

c. Yes, $P=200$.

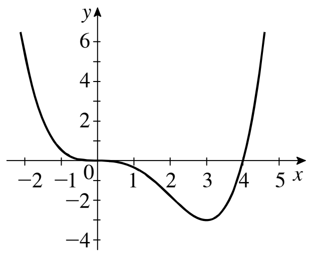
d. Concave upward on $(0,T)$ and concave downward on (T,∞).

e. Yes, at P_0. $P(t)$ is increasing fastest at $t=T$.

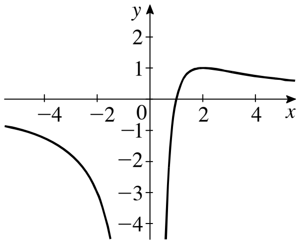
33.



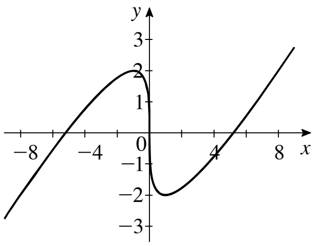
34.



35.

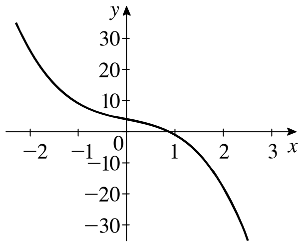


36.



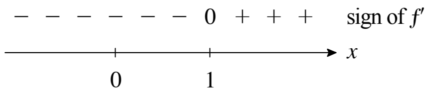
37. $g(x) = 4 - 3x - 2x^3$. We first gather the following information on f.

1. The domain of f is $(-\infty, \infty)$.
2. Setting $x = 0$ gives $y = 4$ as the y-intercept. Setting $y = g(x) = 0$ gives a cubic equation which is not easily solved, and we will not attempt to find the x-intercepts.
3. $\lim\limits_{x \to -\infty} g(x) = \infty$ and $\lim\limits_{x \to \infty} g(x) = -\infty$.
4. The graph of g has no asymptote.
5. $g'(x) = -3 - 6x^2 = -3\left(2x^2 + 1\right) < 0$ for all values of x and so g is decreasing on $(-\infty, \infty)$.
6. The results of step 5 show that g has no critical number and hence no relative extremum.
7. $g''(x) = -12x$. Because $g''(x) > 0$ for $x < 0$ and $g''(x) < 0$ for $x > 0$, we see that g is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$.
8. From the results of step 7, we see that $(0, 4)$ is an inflection point of g.

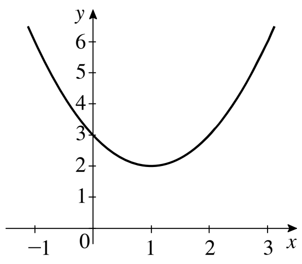


38. $f(x) = x^2 - 2x + 3$. We first gather the following information on f.

1. The domain of f is $(-\infty, \infty)$.
2. Setting $x = 0$ gives the y-intercept as 3. There is no x-intercept because $x^2 - 2x + 3 = 0$ has no real solution.
3. $\lim\limits_{x \to \infty} x^2 - 2x + 3 = \lim\limits_{x \to -\infty} x^2 - 2x + 3 = \infty$.
4. There is no asymptote because $f(x)$ is a polynomial.
5. $f'(x) = 2x - 2 = 2(x - 1) = 0$ if $x = 1$. The sign diagram shows that f is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$.

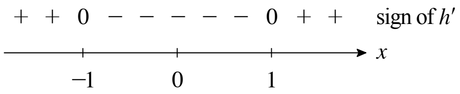


6. The point $(1, 2)$ is a relative minimum.
7. $f''(x) = 2 > 0$ for all x and so the graph of f is concave upward on $(-\infty, \infty)$.
8. Because $f''(x) \neq 0$ for all values of x, there is no inflection point.

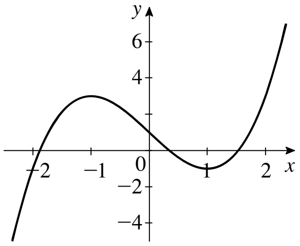


39. $h(x) = x^3 - 3x + 1$. We first gather the following information on h.

1. The domain of h is $(-\infty, \infty)$.
2. Setting $x = 0$ gives 1 as the y-intercept. We will not find the x-intercept.
3. $\lim_{x\to-\infty}(x^3 - 3x + 1) = -\infty$ and $\lim_{x\to\infty}(x^3 - 3x + 1) = \infty$.
4. There is no asymptote because $h(x)$ is a polynomial.
5. $h'(x) = 3x^2 - 3 = 3(x+1)(x-1)$, and we see that $x = -1$ and $x = 1$ are critical numbers. From the sign diagram, we see that h is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$.

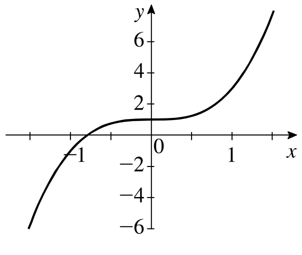


6. The results of step 5 show that $(-1, 3)$ is a relative maximum and $(1, -1)$ is a relative minimum.
7. $h''(x) = 6x$, so $h''(x) < 0$ if $x < 0$ and $h''(x) > 0$ if $x > 0$. Thus, the graph of h is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$.
8. The results of step 7 show that $(0, 1)$ is an inflection point of h.



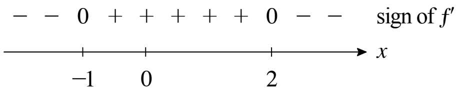
40. $f(x) = 2x^3 + 1$. We first gather the following information on f.

1. The domain of f is $(-\infty, \infty)$.
2. Setting $x = 0$ gives 1 as the y-intercept. Next, observe that $2x^3 = -1$ implies that $x^3 = -\frac{1}{2}$, so $x = -\frac{1}{\sqrt[3]{2}} \approx -0.8$ is the x-intercept.
3. $\lim_{x\to\infty}(2x^3 + 1) = \infty$ and $\lim_{x\to-\infty}(2x^3 + 1) = -\infty$.
4. Because $f(x)$ is a polynomial, there is no asymptote.
5. $f'(x) = 6x^2 = 0$ if $x = 0$, a critical number of f. Because $f'(x) > 0$ for all $x \neq 0$, we see that f is increasing on $(-\infty, \infty)$.
6. Using the results of step 5, we see that f has no relative extremum.
7. $f''(x) = 12x = 0$ if $x = 0$. Because $f''(x) < 0$ if $x < 0$ and $f''(x) > 0$ if $x > 0$, we see that f is concave downward if $x < 0$ and concave upward if $x > 0$.
8. The results of step 7 show that $(0, 1)$ is an inflection point.

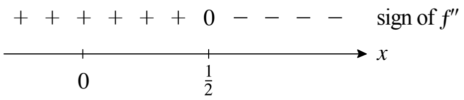


41. $f(x) = -2x^3 + 3x^2 + 12x + 2$. We first gather the following information on f.

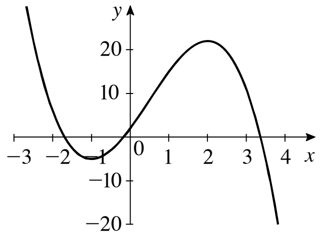
1. The domain of f is $(-\infty, \infty)$.
2. Setting $x = 0$ gives 2 as the y-intercept.
3. $\lim_{x \to -\infty} \left(-2x^3 + 3x^2 + 12x + 2\right) = \infty$ and $\lim_{x \to \infty} \left(-2x^3 + 3x^2 + 12x + 2\right) = -\infty$
4. There is no asymptote because $f(x)$ is a polynomial function.
5. $f'(x) = -6x^2 + 6x + 12 = -6\left(x^2 - x - 2\right) = -6(x-2)(x+1) = 0$ if $x = -1$ or $x = 2$, the critical numbers of f. From the sign diagram, we see that f is decreasing on $(-\infty, -1)$ and $(2, \infty)$ and increasing on $(-1, 2)$.



6. The results of step 5 show that $(-1, -5)$ is a relative minimum and $(2, 22)$ is a relative maximum.
7. $f''(x) = -12x + 6 = 0$ if $x = \frac{1}{2}$. The sign diagram of f''shows that the graph of f is concave upward on $\left(-\infty, \frac{1}{2}\right)$ and concave downward on $\left(\frac{1}{2}, \infty\right)$.

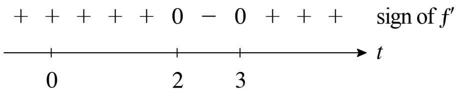


8. The results of step 7 show that $\left(\frac{1}{2}, \frac{17}{2}\right)$ is an inflection point.

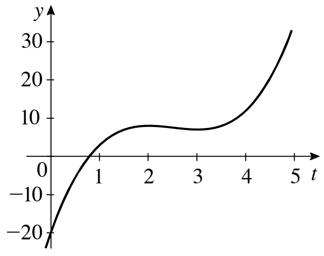


42. $f(t) = 2t^3 - 15t^2 + 36t - 20$. We first gather the following information on f.

1. The domain of f is $(-\infty, \infty)$.
2. Setting $t = 0$ gives -20 as the y-intercept. Setting $y = f(t) = 0$ leads to a cubic equation which is not easily solved and we will not attempt to find the t-intercepts.
3. $\lim_{t \to -\infty} f(t) = -\infty$ and $\lim_{t \to \infty} f(t) = \infty$.
4. f has no asymptote.
5. $f'(t) = 6t^2 - 30t + 36 = 6\left(t^2 - 5t + 6\right) = 6(t-3)(t-2)$. The sign diagram for f' shows that f is increasing on $(-\infty, 2)$ and $(3, \infty)$ and decreasing on $(2, 3)$.

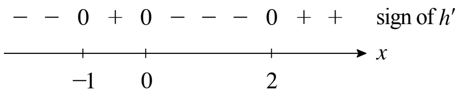


6. The results of step 5 show that $(2, 8)$ is a relative maximum and $(3, 7)$ is a relative minimum.
7. $f''(t) = 12t - 30 = 6(2t - 5)$. Setting $f''(t) = 0$ gives $t = \frac{5}{2}$ as a candidate for an inflection point of f. Because $f''(t) < 0$ for $t < \frac{5}{2}$ and $f''(t) > 0$ for $t > \frac{5}{2}$, we see that f is concave downward on $\left(-\infty, \frac{5}{2}\right)$ and concave upward on $\left(\frac{5}{2}, \infty\right)$.
8. From the results of step 7, we see that $\left(\frac{5}{2}, \frac{15}{2}\right)$ is an inflection point of f.

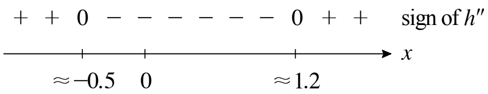


43. $h(x) = \frac{3}{2}x^4 - 2x^3 - 6x^2 + 8$. We first gather the following information on h.

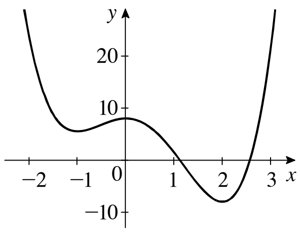
1. The domain of h is $(-\infty, \infty)$.
2. Setting $x = 0$ gives 8 as the y-intercept.
3. $\lim_{x\to-\infty} h(x) = \lim_{x\to\infty} h(x) = \infty$
4. There is no asymptote.
5. $h'(x) = 6x^3 - 6x^2 - 12x = 6x\left(x^2 - x - 2\right) = 6x(x-2)(x+1) = 0$ if $x = -1, 0$, or 2, and these are the critical numbers of h. The sign diagram of h' shows that h is increasing on $(-1, 0)$ and $(2, \infty)$ and decreasing on $(-\infty, -1)$ and $(0, 2)$.



6. The results of step 5 show that $\left(-1, \frac{11}{2}\right)$ and $(2, -8)$ are relative minima of h and $(0, 8)$ is a relative maximum of h.
7. $h''(x) = 18x^2 - 12x - 12 = 6\left(3x^2 - 2x - 2\right)$. The zeros of h'' are $x = \frac{2\pm\sqrt{4+24}}{6} \approx -0.5$ or 1.2. The sign diagram of h'' shows that the graph of h is concave upward on $(-\infty, -0.5)$ and $(1.2, \infty)$ and concave downward on $(-0.5, 1.2)$.

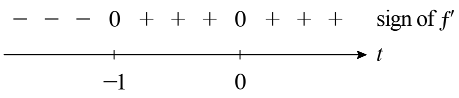


8. The results of step 7 also show that $(-0.5, 6.8)$ and $(1.2, -1)$ are inflection points.

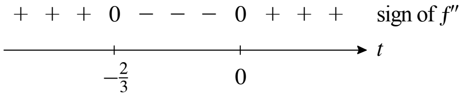


44. $f(t) = 3t^4 + 4t^3 = t^3(3t+4)$. We first gather the following information on f.

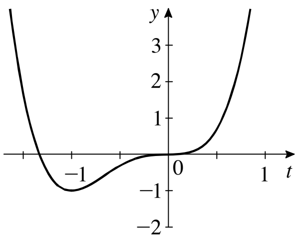
1. The domain of f is $(-\infty, \infty)$.
2. Setting $t = 0$ gives 0 as the y-intercept. Next, setting $y = f(t) = 0$ gives $3t^4 + 4t^3 = t^3(3t+4) = 0$ and $t = -\frac{4}{3}$ and $t = 0$ as the t-intercepts.
3. $\lim_{t\to\infty} f(t) = \infty$ and $\lim_{t\to-\infty} f(t) = \infty$.
4. There is no asymptote.
5. $f'(t) = 12t^3 + 12t^2 = 12t^2(t+1)$. From the sign diagram for f', we see that f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.



6. From the results of step 5, we see that has a relative minimum at $(-1, -1)$.
7. $f''(t) = 36t^2 + 24t = 12t(3t+2)$. Setting $f''(t) = 0$ gives $t = -\frac{2}{3}$ and $t = 0$ as candidates for inflection points of f. The sign diagram for f'' shows that f is concave upward on $\left(-\infty, -\frac{2}{3}\right)$ and $(0, \infty)$ and concave downward on $\left(-\frac{2}{3}, 0\right)$.

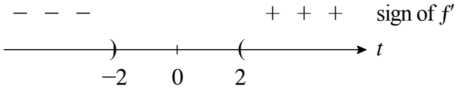


8. The results of step 7 imply that $\left(-\frac{2}{3}, -\frac{16}{27}\right)$ and $(0, 0)$ are inflection points of f.



45. $f(t) = \sqrt{t^2 - 4}$. We first gather the following information on f.

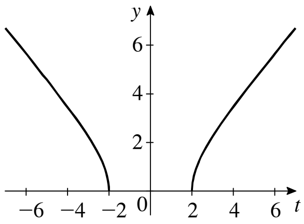
1. The domain of f is found by solving $t^2 - 4 \geq 0$ to obtain $(-\infty, -2] \cup [2, \infty)$.
2. Because $t \neq 0$, there is no y-intercept. Next, setting $y = f(t) = 0$ gives the t-intercepts as -2 and 2.
3. $\lim_{t \to -\infty} f(t) = \lim_{t \to \infty} f(t) = \infty$.
4. There is no asymptote.
5. $f'(t) = \frac{1}{2}\left(t^2 - 4\right)^{-1/2}(2t) = t\left(t^2 - 4\right)^{-1/2} = \dfrac{t}{\sqrt{t^2 - 4}}$. Setting $f'(t) = 0$ gives $t = 0$. But $t = 0$ is not in the domain of f and so there is no critical number. From the sign diagram for f', we see that f is increasing on $(2, \infty)$ and decreasing on $(-\infty, -2)$.



6. From the results of step 5 we see that there is no relative extremum.
7. $f''(t) = \left(t^2 - 4\right)^{-1/2} + t\left(-\frac{1}{2}\right)\left(t^2 - 4\right)^{-3/2}(2t)$

$$= \left(t^2 - 4\right)^{-3/2}\left(t^2 - 4 - t^2\right) = -\frac{4}{\left(t^2 - 4\right)^{3/2}}.$$

8. Because $f''(t) < 0$ for all t in the domain of f, we see that f is concave downward everywhere. From the results of step 7, we see that there is no inflection point.



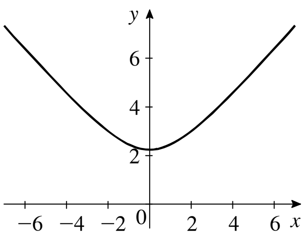
46. $f(x) = \sqrt{x^2 + 5}$. We first gather the following information on f.

1. The domain of f is $(-\infty, \infty)$.
2. Setting $x = 0$ gives $\sqrt{5} \approx 2.2$ as the y-intercept.
3. $\lim_{x \to -\infty} \sqrt{x^2 + 5} = \lim_{x \to \infty} \sqrt{x^2 + 5} = \infty$.
4. The results of step 3 show that there is no horizontal asymptote. There is also no vertical asymptote.
5. $f'(x) = \frac{1}{2}\left(x^2 + 5\right)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2 + 5}}$ and this shows that $x = 0$ is a critical number of f. Because $f'(x) < 0$ if $x < 0$ and $f'(x) > 0$ is $x > 0$, we see that f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.
6. The results of step 5 show that $\left(0, \sqrt{5}\right)$ is a relative minimum of f.
7. $f''(x) = \dfrac{d}{dx}x\left(x^2 + 5\right)^{-1/2} = \left(x^2 + 5\right)^{-1/2} + x\left(-\frac{1}{2}\right)\left(x^2 + 5\right)^{-3/2}(2x) = \left(x^2 + 5\right)^{-3/2}\left[\left(x^2 + 5\right) - x^2\right]$

$$= \frac{5}{\left(x^2 + 5\right)^{3/2}}.$$

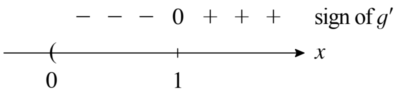
This expression is positive for all values of x, so the graph of f is concave upward on $(-\infty, \infty)$.

8. The results of step 7 also show that there is no inflection point.

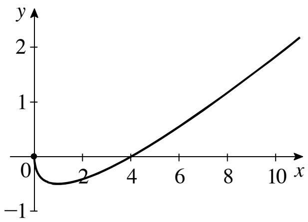


47. $g(x) = \frac{1}{2}x - \sqrt{x}$. We first gather the following information on g.

1. The domain of g is $[0, \infty)$.
2. The y-intercept is 0. To find the x-intercept(s), set $y = 0$, giving $\frac{1}{2}x - \sqrt{x} = 0$, $x = 2\sqrt{x}$, $x^2 = 4x$, $x(x-4) = 0$, and so $x = 0$ or $x = 4$.
3. $\lim_{x\to\infty} \left(\frac{1}{2}x - \sqrt{x}\right) = \lim_{x\to\infty} \frac{1}{2}x\left(1 - \frac{2}{\sqrt{x}}\right) = \infty$.
4. There is no asymptote.
5. $g'(x) = \frac{1}{2} - \frac{1}{2}x^{-1/2} = \frac{1}{2}x^{-1/2}\left(x^{1/2} - 1\right) = \frac{\sqrt{x} - 1}{2\sqrt{x}}$, which is zero when $x = 1$. From the sign diagram for g', we see that g is decreasing on $(0, 1)$ and increasing on $(1, \infty)$.

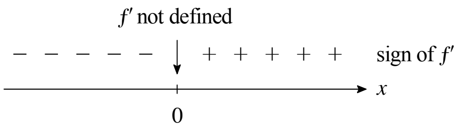


6. From the results of part 5, we see that $g(1) = -\frac{1}{2}$ is a relative minimum.
7. $g''(x) = \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^{-3/2} = \frac{1}{4x^{3/2}} > 0$ for $x > 0$, and so g is concave upward on $(0, \infty)$.
8. There is no inflection point.

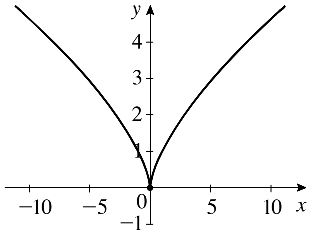


48. $f(x) = \sqrt[3]{x^2}$. We first gather the following information on f.

1. The domain of f is $(-\infty, \infty)$ because $x^2 \geq 0$ for all x.
2. Setting $x = 0$ gives the y-intercept as 0. Similarly, setting $y = 0$ gives 0 as the x-intercept.
3. $\lim_{x\to-\infty} \sqrt[3]{x^2} = \lim_{x\to\infty} \sqrt[3]{x^2} = \infty$.
4. There is no asymptote.
5. $f'(x) = \frac{d}{dx}x^{2/3} = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$. The sign diagram of f' shows that f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.



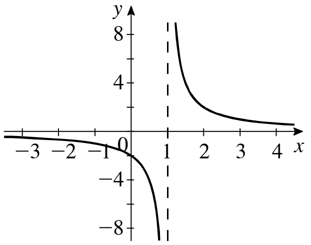
6. Because f has no critical number, f has no relative extremum.
7. $f''(x) = \frac{d}{dx}\left(\frac{2}{3}x^{-1/3}\right) = -\frac{2}{9}x^{-4/3} = -\frac{2}{9x^{4/3}} > 0$ for all $x \neq 0$, and so f is concave downward on $(-\infty, 0)$ and $(0, \infty)$.
8. Because $f''(x) \neq 0$, there is no inflection point.



49. $g(x) = \dfrac{2}{x-1}$. We first gather the following information on g.

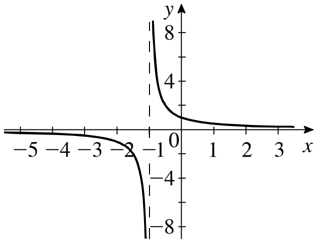
1. The domain of g is $(-\infty, 1) \cup (1, \infty)$.
2. Setting $x = 0$ gives -2 as the y-intercept. There is no x-intercept because $\dfrac{2}{x-1} \neq 0$ for all x.
3. $\lim\limits_{x \to -\infty} \dfrac{2}{x-1} = 0$ and $\lim\limits_{x \to \infty} \dfrac{2}{x-1} = 0$.
4. The results of step 3 show that $y = 0$ is a horizontal asymptote. Furthermore, the denominator of $g(x)$ is equal to zero at $x = 1$ but the numerator is not equal to zero there. Therefore, $x = 1$ is a vertical asymptote.
5. $g'(x) = -2(x-1)^{-2} = -\dfrac{2}{(x-1)^2} < 0$ for all $x \neq 1$ and so g is decreasing on $(-\infty, 1)$ and $(1, \infty)$.
6. Because g has no critical number, there is no relative extremum.
7. $g''(x) = \dfrac{4}{(x-1)^3}$ and so $g''(x) < 0$ if $x < 1$ and $g''(x) > 0$ if $x > 1$.

   Therefore, the graph of g is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.
8. Because $g''(x) \neq 0$, there is no inflection point.



50. $f(x) = \dfrac{1}{x+1}$. We first gather the following information on f.

1. Because the denominator is zero when $x = -1$, we see that the domain of f is $(-\infty, -1) \cup (-1, \infty)$.
2. Setting $x = 0$ gives the y-intercept as 1. Because $y \neq 0$, there is no x-intercept.
3. $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to \infty} f(x) = 0$.
4. From the results of step 3, we see that $y = 0$ is a horizontal asymptote of f. Next, setting the denominator of f equal to zero gives $x = -1$. Furthermore, $\lim\limits_{x \to -1^-} f(x) = -\infty$ and $\lim\limits_{x \to -1^+} f(x) = \infty$, and so $x = -1$ is a vertical asymptote of f.
5. $f'(x) = -\dfrac{1}{(x+1)^2}$. Note that $f'(x)$ is not defined at $x = -1$. Because $f'(x) < 0$ whenever x is defined, we see that f is decreasing everywhere.
6. The results of step 5 show that there is no critical numbers ($x = -1$ is not in the domain of f.) Thus, there is no relative extremum.
7. $f''(x) = \dfrac{2}{(x+1)^3}$. We see that $f''(x) < 0$ for $x < -1$ and $f''(x) > 0$ for $x > -1$. Therefore, f is concave downward on $(-\infty, -1)$ and concave upward on $(-1, \infty)$.
8. Because f' has no critical number, f has no inflection point.

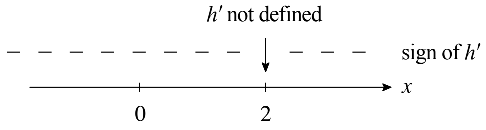


51. $h(x) = \dfrac{x+2}{x-2}$. We first gather the following information on h.

1. The domain of h is $(-\infty, 2) \cup (2, \infty)$.
2. Setting $x = 0$ gives $y = -1$ as the y-intercept. Next, setting $y = 0$ gives $x = -2$ as the x-intercept.
3. $\lim\limits_{x \to \infty} h(x) = \lim\limits_{x \to -\infty} \dfrac{1 + \frac{2}{x}}{1 - \frac{2}{x}} = \lim\limits_{x \to -\infty} h(x) = 1$.

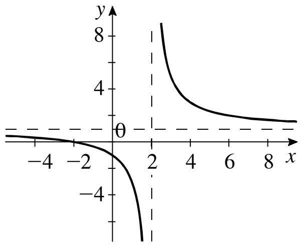
4. Setting $x - 2 = 0$ gives $x = 2$. Furthermore, $\lim_{x \to 2^+} \frac{x+2}{x-2} = \infty$, so $x = 2$ is a vertical asymptote of h. Also, from the results of step 3, we see that $y = 1$ is a horizontal asymptote of h.

5. $h'(x) = \frac{(x-2)(1) - (x+2)(1)}{(x-2)^2} = -\frac{4}{(x-2)^2}$. We see that h has no critical number. (Note that $x = 2$ is not in the domain of h.) The sign diagram of h' shows that h is decreasing on $(-\infty, 2)$ and $(2, \infty)$.



6. From the results of step 5, we see that there is no relative extremum.

7. $h''(x) = \frac{8}{(x-2)^3}$. Note that $x = 2$ is not a candidate for an inflection point because $h(2)$ is not defined. Because $h''(x) < 0$ for $x < 2$ and $h''(x) > 0$ for $x > 2$, we see that h is concave downward on $(-\infty, 2)$ and concave upward on $(2, \infty)$.



8. From the results of step 7, we see that there is no inflection point.

52. $g(x) = \frac{x}{x-1}$. We first gather the following information on g.

1. The domain of g is $(-\infty, 1) \cup (1, \infty)$.

2. Setting $x = 0$ gives 0 as the y-intercept. Similarly, setting $y = 0$ gives 0 as the x-intercept.

3. $\lim_{x \to -\infty} \frac{x}{x-1} = 1$ and $\lim_{x \to \infty} \frac{x}{x-1} = 1$.

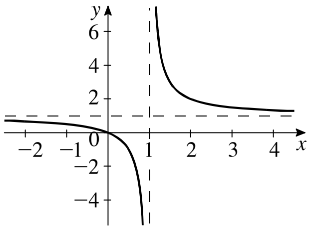
4. The results of step 3 show that $y = 1$ is a horizontal asymptote. Next, because the denominator is zero at $x = 1$ but the numerator is not equal to zero at this value of x, we see that $x = 1$ is a vertical asymptote of g.

5. $g'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = -\frac{1}{(x-1)^2} < 0$ if $x \neq 1$ and so f is decreasing on $(-\infty, 1)$ and $(1, \infty)$.

6. Because $g'(x) \neq 0$ for all x, there is no critical number and so g has no relative extremum.

7. $g''(x) = \frac{2}{(x-1)^3}$ and so $g''(x) < 0$ if $x < 1$ and $g''(x) > 0$ if $x > 1$.

   Therefore, the graph of g is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.



8. Because $g''(x) \neq 0$ for all x, we see that there is no inflection point.

53. $f(t) = \frac{t^2}{1+t^2}$. We first gather the following information on f.

1. The domain of f is $(-\infty, \infty)$.

2. Setting $t = 0$ gives the y-intercept as 0. Similarly, setting $y = 0$ gives the t-intercept as 0.

3. $\lim_{t \to -\infty} \frac{t^2}{1+t^2} = \lim_{t \to \infty} \frac{t^2}{1+t^2} = 1$.

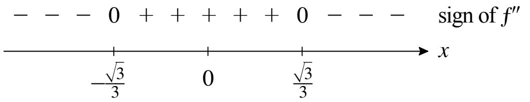
4. The results of step 3 show that $y = 1$ is a horizontal asymptote. There is no vertical asymptote since the denominator is never zero.

5. $f'(t) = \dfrac{(1+t^2)(2t) - t^2(2t)}{(1+t^2)^2} = \dfrac{2t}{(1+t^2)^2} = 0$, if $t = 0$, the only critical number of f. Because $f'(t) < 0$ if $t < 0$ and $f'(t) > 0$ if $t > 0$, we see that f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

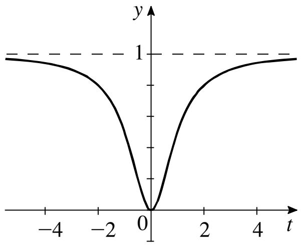
6. The results of step 5 show that $(0, 0)$ is a relative minimum.

7. $f''(t) = \dfrac{(1+t^2)^2(2) - 2t(2)(1+t^2)(2t)}{(1+t^2)^4} = \dfrac{2(1+t^2)\left[(1+t^2) - 4t^2\right]}{(1+t^2)^4} = \dfrac{2(1-3t^2)}{(1+t^2)^3} = 0$ if $t = \pm\frac{\sqrt{3}}{3}$.

   The sign diagram of f'' shows that f is concave downward on $\left(-\infty, -\frac{\sqrt{3}}{3}\right)$ and $\left(\frac{\sqrt{3}}{3}, \infty\right)$ and concave upward on $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.



8. The results of step 7 show that $\left(-\frac{\sqrt{3}}{3}, \frac{1}{4}\right)$ and $\left(\frac{\sqrt{3}}{3}, \frac{1}{4}\right)$ are inflection points.



54. $g(x) = \dfrac{x}{x^2 - 4} = \dfrac{x}{(x+2)(x-2)}$. We first gather the following information on g.

1. The denominator of $g(x)$ is zero when $x = \pm 2$, and so the domain of g is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

2. Setting $x = 0$ gives 0 as the y-intercept and setting $y = 0$ gives 0 as the x-intercept.

3. $\displaystyle\lim_{x\to-\infty} \frac{x}{x^2-4} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1 - \frac{4}{x^2}} = 0$. Similarly, $\displaystyle\lim_{x\to\infty} g(x) = 0$.

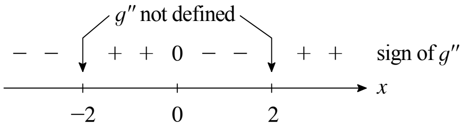
4. From the results of step 3, we see that $y = 0$ is a horizontal asymptote of g. Next, observe that the denominator of $g(x)$ is zero when $x = \pm 2$. Now $\displaystyle\lim_{x\to-2^-} \frac{x}{(x+2)(x-2)} = -\infty$, $\displaystyle\lim_{x\to-2^+} \frac{x}{(x+2)(x-2)} = \infty$, $\displaystyle\lim_{x\to 2^-} \frac{x}{(x+2)(x-2)} = -\infty$, and $\displaystyle\lim_{x\to 2^+} \frac{x}{(x+2)(x-2)} = \infty$. Therefore, $x = -2$ and $x = 2$ are vertical asymptotes.

5. $g'(x) = \dfrac{(x^2-4)(1) - x(2x)}{(x^2-4)^2} = -\dfrac{x^2+4}{(x^2-4)^2}$. Because $g'(x) < 0$ whenever it is defined, we see that g is decreasing everywhere.

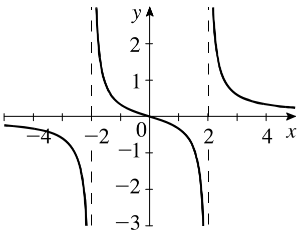
6. From the results of step 5, we see that there is no relative extremum.

7. $g''(x) = \dfrac{(x^2-4)^2(-2x) + (x^2+4)2(x^2-4)(2x)}{(x^2-4)^4} = \dfrac{2x(x^2-4)(-x^2+4+2x^2+8)}{(x^2-4)^4} = \dfrac{2x(x^2+12)}{(x^2-4)^3}$.

   Setting $g''(x) = 0$ gives $x = 0$ as the only candidate for a point of inflection. The sign diagram for g'' shows that g is concave upward on $(-2, 0)$ and $(2, \infty)$ and concave downward on $(-\infty, -2)$ and $(0, 2)$.

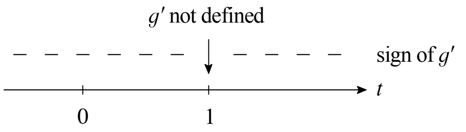


8. From the results of step 7, we see that (0, 0) is an inflection point of g.



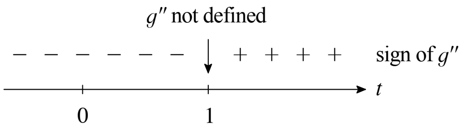
55. $g(t) = -\frac{t^2 - 2}{t - 1}$. We first gather the following information on g.

1. The domain of g is $(-\infty, 1) \cup (1, \infty)$
2. Setting $t = 0$ gives -2 as the y-intercept.
3. $\lim_{t \to -\infty}\left(-\frac{t^2 - 2}{t - 1}\right) = \infty$ and $\lim_{t \to \infty}\left(-\frac{t^2 - 2}{t - 1}\right) = -\infty$.
4. There is no horizontal asymptotes. The denominator is equal to zero at $t = 1$ at which number the numerator is not equal to zero. Therefore $t = 1$ is a vertical asymptote.
5. $$g'(t) = -\frac{(t-1)(2t) - (t^2 - 2)(1)}{(t-1)^2}$$ $$= -\frac{t^2 - 2t + 2}{(t-1)^2} \neq 0 \text{ for all values of } t.$$



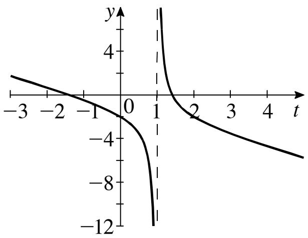
The sign diagram of g' shows that g is decreasing on $(-\infty, 1)$ and $(1, \infty)$.

6. Because there is no critical number, g has no relative extremum.
7. $$g''(t) = -\frac{(t-1)^2(2t-2) - (t^2 - 2t + 2)(2)(t-1)}{(t-1)^4}$$ $$= \frac{-2(t-1)(t^2 - 2t + 1 - t^2 + 2t - 2)}{(t-1)^4} = \frac{2}{(t-1)^3}.$$



The sign diagram of g''shows that the graph of g is concave upward on $(1, \infty)$ and concave downward on $(-\infty, 1)$.

8. There is no inflection point because $g''(x) \neq 0$ for all x.

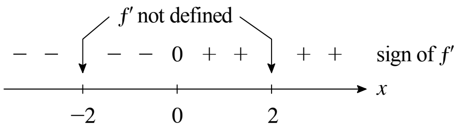


56. $f(x) = \frac{x^2 - 9}{x^2 - 4}$. We first gather the following information on f.

1. The domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.
2. The y-intercept is $\frac{9}{4}$ and the x-intercepts are -3 and 3.
3. $\lim_{x \to \infty} \frac{x^2 - 9}{x^2 - 4} = \lim_{x \to \infty} \frac{1 - \frac{9}{x^2}}{1 - \frac{4}{x^2}} = 1$, and similarly $\lim_{x \to -\infty} \frac{x^2 - 9}{x^2 - 4} = 1$.

4. From the results of step 3, we see that $y = 1$ is a horizontal asymptote. Next, $x^2 - 4 = 0$ implies $x = \pm 2$. Because the numerator $x^2 - 9$ is not zero at $x = \pm 2$, we see that $x = -2$ and $x = 2$ are vertical asymptotes.

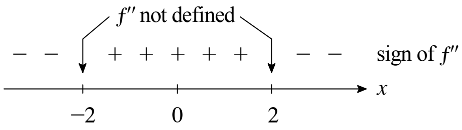
5. $f'(x) = \dfrac{\left(x^2-4\right)(2x) - \left(x^2-9\right)(2x)}{\left(x^2-4\right)^2} = \dfrac{10x}{\left(x^2-4\right)^2}$ is equal to 0 at $x = 0$ and is discontinuous at $x = \pm 2$. From the sign diagram for f', we see that f is increasing on $(0, 2)$ and $(2, \infty)$ and decreasing on $(-\infty, -2)$ and $(-2, 0)$.



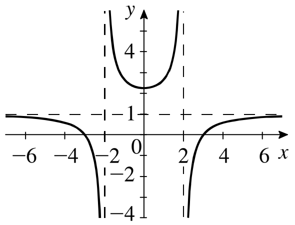
6. The point $\left(0, \frac{9}{4}\right)$ is a relative minimum.

7. $f''(x) = \dfrac{\left(x^2-4\right)^2(10) - (10x)(2)\left(x^2-4\right)(2x)}{\left(x^2-4\right)^4} = \dfrac{10\left(x^2-4\right)\left(x^2-4-4x^2\right)}{\left(x^2-4\right)^4} = \dfrac{-10\left(3x^2+4\right)}{\left(x^2-4\right)^3}$,

which is not defined at $x = \pm 2$. From the sign diagram for f'', we see that f is concave upward on $(-2, 2)$ and concave downward on $(-\infty, -2)$ and $(2, \infty)$.



8. There is no inflection point. Note that both $x = -2$ and $x = 2$ lie outside the domain of f.



57. $g(t) = \dfrac{t^2}{t^2-1}$. We first gather the following information on g.

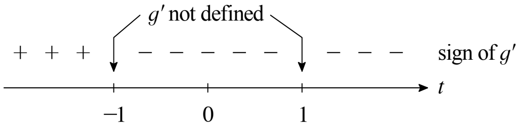
1. Because $t^2 - 1 = 0$ if $t = \pm 1$, we see that the domain of g is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

2. Setting $t = 0$ gives 0 as the y-intercept. Setting $y = 0$ gives 0 as the t-intercept.

3. $\lim\limits_{t\to-\infty} g(t) = \lim\limits_{t\to\infty} g(t) = 1$.

4. The results of step 3 show that $y = 1$ is a horizontal asymptote. Because the denominator (but not the numerator) is zero at $t = \pm 1$, we see that $t = \pm 1$ are vertical asymptotes.

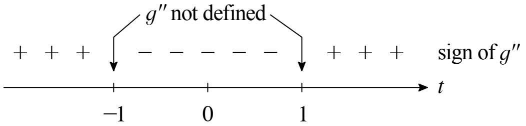
5. $g'(t) = \dfrac{\left(t^2-1\right)(2t) - \left(t^2\right)(2t)}{\left(t^2-1\right)^2} = -\dfrac{2t}{\left(t^2-1\right)^2} = 0$ if $t = 0$. From the sign diagram of g', we see that g is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.



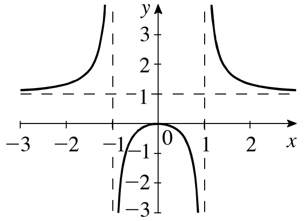
6. From the results of step 5, we see that g has a relative maximum at $t = 0$.

7. $$g''(t) = \frac{\left(t^2-1\right)^2(-2)-(-2t)(2)\left(t^2-1\right)(2t)}{\left(t^2-1\right)^4} = \frac{2\left(t^2-1\right)\left[-\left(t^2-1\right)+4t^2\right]}{\left(t^2-1\right)^4} = \frac{2\left(-t^2+1+4t^2\right)}{\left(t^2-1\right)^3}$$
$$= \frac{2\left(3t^2+1\right)}{\left(t^2-1\right)^3}.$$

From the sign diagram, we see that the graph of g is concave upward on $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$.



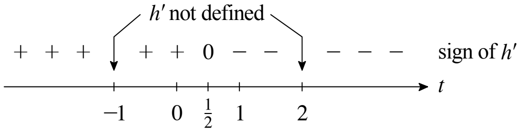
8. Because g is undefined at ±1, the graph of g has no inflection point.



58. $h(x) = \dfrac{1}{x^2-x-2}$. We first gather the following information on h.

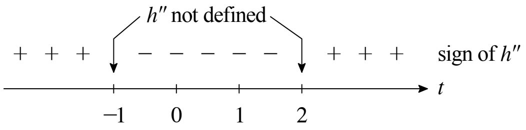
1. Because $x^2-x-2 = (x-2)(x+1) = 0$ if $x=-1$ or $x=2$, we see that the domain of h is $(-\infty,-1)\cup(-1,2)\cup(2,\infty)$.
2. Setting $x=0$ gives $-\frac{1}{2}$ as the y-intercept.
3. $\lim\limits_{x\to-\infty} h(x) = \lim\limits_{x\to\infty} h(x) = 0$.
4. The results of step 3 show that $y=0$ is a horizontal asymptote. Furthermore, the denominator is equal to zero at $x=-1$ and $x=2$, where the numerator is not equal to zero. Therefore, $x=-1$ and $x=2$ are vertical asymptotes.
5. $$h'(x) = \frac{d}{dx}\left(x^2-x-2\right)^{-1} = -\left(x^2-x-2\right)^{-2}(2x-1) = \frac{1-2x}{\left(x^2-x-2\right)^2}.$$

   Setting $h'(x)=0$ gives $x=\frac{1}{2}$ as a critical number. The sign diagram of h' shows us that h is increasing on $(-\infty,-1)$ and $\left(-1,\frac{1}{2}\right)$ and decreasing on $\left(\frac{1}{2},2\right)$ and $(2,\infty)$.

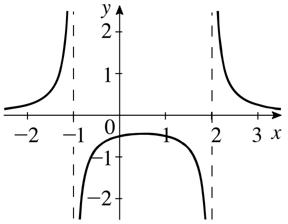


6. The results of step 5 show that $\left(\frac{1}{2},-\frac{4}{9}\right)$ is a relative maximum.
7. $$h''(x) = \frac{\left(x^2-x-2\right)^2(-2)-(1-2x)2\left(x^2-x-2\right)(2x-1)}{\left(x^2-x-2\right)^4}$$
$$= \frac{2\left(x^2-x-2\right)\left[-\left(x^2-x-2\right)+(2x-1)^2\right]}{\left(x^2-x-2\right)^4} = \frac{6\left(x^2-x+1\right)}{\left(x^2-x-2\right)^3}.$$

   $h''(x)$ has no zero and is discontinuous at $x=-1$ and $x=2$. The sign diagram of h'' shows that the graph of h is concave upward on $(-\infty,-1)$ and $(2,\infty)$ and concave downward on $(-1,2)$.



8. Because $h''(x) \neq 0$, there is no inflection point.



59. $h(x) = (x-1)^{2/3} + 1$. We begin by obtaining the following information on h.

1. The domain of h is $(-\infty, \infty)$.

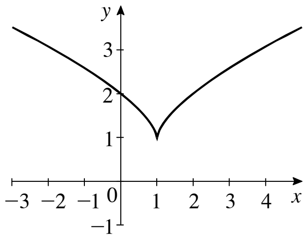
2. Setting $x = 0$ gives 2 as the y-intercept; since $h(x) \neq 0$ there is no x-intercept.

3. $\lim_{x\to\infty} \left[(x-1)^{2/3} + 1\right] = \infty$ and $\lim_{x\to-\infty} \left[(x-1)^{2/3} + 1\right] = \infty$.

4. There is no asymptote.

5. $h'(x) = \frac{2}{3}(x-1)^{-1/3}$ and is positive if $x > 1$ and negative if $x < 1$. Thus, h is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$.

6. From step 5, we see that h has a relative minimum at $(1, 1)$.



7. $h''(x) = \frac{2}{3}\left(-\frac{1}{3}\right)(x-1)^{-4/3} = -\frac{2}{9}(x-1)^{-4/3} = -\dfrac{2}{(x-1)^{4/3}}$.

Because $h''(x) < 0$ on $(-\infty, 1)$ and $(1, \infty)$, we see that h is concave downward on $(-\infty, 1)$ and $(1, \infty)$. Note that $h''(x)$ is not defined at $x = 1$.

8. From the results of step 7, we see h has no inflection point.

60. $g(x) = (x+2)^{3/2} + 1$. We first gather the following information on g.

1. The domain of g is $[-2, \infty)$.

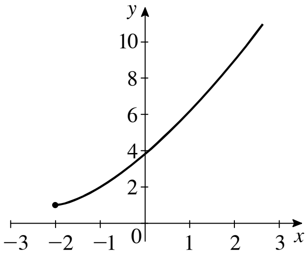
2. Setting $x = 0$ gives $2^{3/2} + 1 \approx 3.8$ as the y-intercept.

3. $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} (x+2)^{3/2} + 1 = \infty$.

4. There is no asymptote.

5. $g'(x) = \frac{3}{2}(x+2)^{1/2} \geq 0$ if $x \geq -2$, and so g is increasing on $(-2, \infty)$.

6. There is no relative extremum because $g'(x) \neq 0$ on $(-2, \infty)$.



7. $g''(x) = \frac{3}{4}(x+2)^{-1/2} = \dfrac{3}{4\sqrt{x+2}} > 0$ if $x > 0$, and so the graph of g is concave upward on $(2, \infty)$.

8. There is no inflection point since $g''(x) \neq 0$.

61. a. The denominator of $C(x)$ is equal to zero if $x = 100$. Also, $\lim_{x\to 100^-} \dfrac{0.5x}{100-x} = \infty$ and $\lim_{x\to 100^+} \dfrac{0.5x}{100-x} = -\infty$. Therefore, $x = 100$ is a vertical asymptote of C.

b. No, because the denominator is equal to zero in that case.

62. a. $\lim_{x\to\infty} \overline{C}(x) = \lim_{x\to\infty} \left(2.2 + \dfrac{2500}{x}\right) = 2.2$, and so $y = 2.2$ is a horizontal asymptote.

b. The limiting value is 2.2, or \$2.20 per disc.

63. **a.** Because $\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \frac{0.2t}{t^2+1} = \lim_{t\to\infty} \left(\frac{0.2}{t+\frac{1}{t^2}}\right) = 0$, $y = 0$ is a horizontal asymptote.

b. Our results reveal that as time passes, the concentration of the drug decreases and approaches zero.

64. **a.** $\lim_{x\to\infty} \frac{ax}{x+b} = \lim_{x\to\infty} \frac{a}{1+\frac{b}{x}} = a$, so the horizontal asymptote is $V = a$.

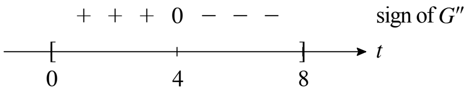
b. The initial speed of the reaction approaches a moles/liter/sec as the amount of substrate becomes arbitrarily large.

65. $G(t) = -0.2t^3 + 2.4t^2 + 60$. We first gather the following information on G.

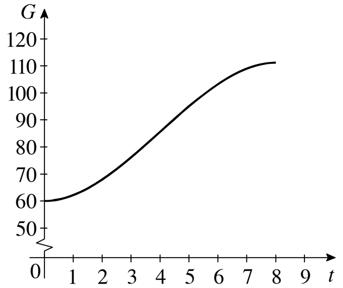
1. The domain of G is restricted to [0, 8].
2. Setting $t = 0$ gives 60 as the y-intercept.

Step 3 is unnecessary in this case because of the restricted domain.

4. There is no asymptote because G is a polynomial function.
5. $G'(t) = -0.6t^2 + 4.8t = -0.6t(t-8) = 0$ if $t = 0$ or $t = 8$, critical numbers of G. But $G'(t) > 0$ on (0, 8), so G is increasing on its domain.
6. The results of step 5 tell us that there is no relative extremum.
7. $G''(t) = -1.2t + 4.8 = -1.2(t-4)$. The sign diagram of G'' shows that G is concave upward on (0, 4) and concave downward on (4, 8).



8. The results of step 7 show that (4, 85.6) is an inflection point.

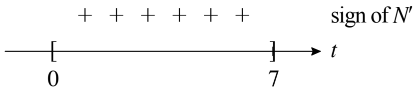


66. $N(t) = -0.1t^3 + 1.5t^2 + 80$, $0 \le t \le 7$. We first gather the following information on N.

1. The domain of N is [0, 7].
2. Setting $t = 0$ gives 80 as the y-intercept.

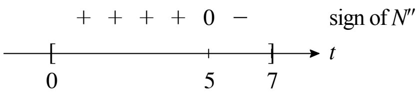
Step 3 is unnecessary in this case because of the restricted domain.

4. There is no asymptote because N is a polynomial function.
5. $N'(t) = -0.3t^2 + 3t = -0.3t(t-10) = 0$ if $t = 0$ or $t = 10$, but $t = 10$ lies outside the domain of N. The sign diagram of N' shows that N is increasing on (0, 7).

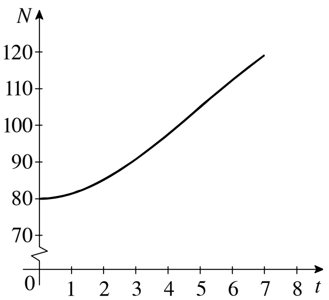


6. The results of step 5 tell us that $f(0) = 80$ is a relative minimum.

7. $N''(t) = -0.6t + 3 = -0.6(t-5)$. The sign diagram of N'' shows that N is concave upward on $(0, 5)$ and concave downward on $(5, 7)$.



8. The results of step 7 show that $(5, 105)$ is an inflection point. Because N is concave upward on $(0, 5)$ and concave downward on $(5, 7)$ it follows that N' (the rate of increase of N) was increasing from 0 to 5 and decreasing thereafter. This indicates that the program is working.

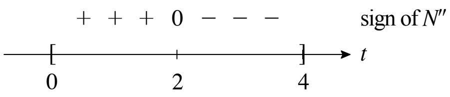


67. $N(t) = -\frac{1}{2}t^3 + 3t^2 + 10t$, $0 \le t \le 4$. We first gather the following information on N.

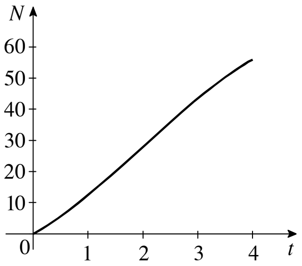
1. The domain of N is restricted to $[0, 4]$.
2. The y-intercept is 0.

Step 3 does not apply because the domain of $N(t)$ is $[0, 4]$.

4. There is no asymptote.
5. $N'(t) = -\frac{3}{2}t^2 + 6t + 10 = -\frac{1}{2}\left(3t^2 - 12t - 20\right)$ is never zero. Therefore, N is increasing on $(0, 4)$.
6. There is no relative extremum in $(0, 4)$.
7. $N''(t) = -3t + 6 = -3(t-2) = 0$ at $t = 2$. From the sign diagram of N'', we see that N is concave upward on $(0, 2)$ and concave downward on $(2, 4)$.

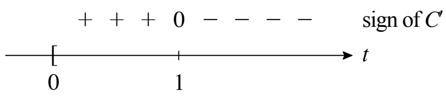


8. The point $(2, 28)$ is an inflection point.



68. $C(t) = \dfrac{0.2t}{t^2+1}$. We first gather the following information on C.

1. The domain of C is $[0, \infty)$.
2. If $t = 0$, then $y = 0$. Also, if $y = 0$, then $t = 0$.
3. $\displaystyle\lim_{t\to\infty} \frac{0.2t}{t^2+1} = 0$.
4. The results of step 3 imply that $y = 0$ is a horizontal asymptote.
5. $C'(t) = \dfrac{\left(t^2+1\right)(0.2) - 0.2t(2t)}{\left(t^2+1\right)^2} = \dfrac{0.2\left(t^2+1-2t^2\right)}{\left(t^2+1\right)^2} = \dfrac{0.2\left(1-t^2\right)}{\left(t^2+1\right)^2} = 0$ at $t = \pm 1$, so $t = 1$ is a critical number of C. The sign diagram of C' shows that C is decreasing on $(1, \infty)$ and increasing on $(0, 1)$.

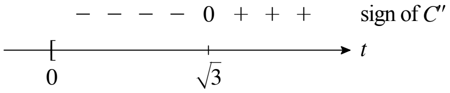


6. The results of step 5 tell us that $(1, 0.1)$ is a relative maximum.

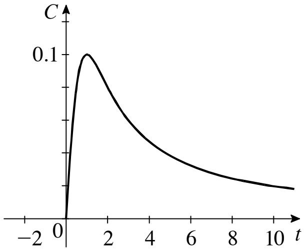
7. $C''(t) = 0.2\left[\dfrac{(t^2+1)^2(-2t) - (1-t^2)\,2\,(t^2+1)(2t)}{(t^2+1)^4}\right] = \dfrac{0.2\,(t^2+1)(2t)(-t^2-1-2+2t^2)}{(t^2+1)^4}$

$= \dfrac{0.4t\,(t^2-3)}{(t^2+1)^3}$.

The sign diagram of C'' shows that the graph of C is concave downward on $(0, \sqrt{3})$ and concave upward on $(\sqrt{3}, \infty)$.



8. The results of step 7 show that $(\sqrt{3}, 0.05\sqrt{3})$ is an inflection point.



69. $T(x) = \dfrac{120x^2}{x^2+4}$. We first gather the following information on T.

1. The domain of T is $[0, \infty)$.

2. Setting $x = 0$ gives 0 as the y-intercept.

3. $\displaystyle\lim_{x\to\infty} \frac{120x^2}{x^2+4} = 120$.

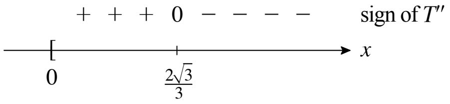
4. The results of step 3 show that $y = 120$ is a horizontal asymptote.

5. $T'(x) = 120\left[\dfrac{(x^2+4)\,2x - x^2(2x)}{(x^2+4)^2}\right] = \dfrac{960x}{(x^2+4)^2}$. Because $T'(x) > 0$ if $x > 0$, we see that T is increasing on $(0, \infty)$.

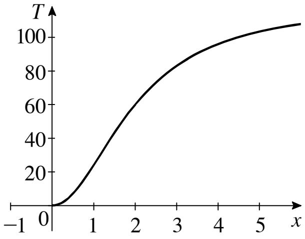
6. There is no relative extremum.

7. $T''(x) = 960\left[\dfrac{(x^2+4)^2 - x\,(2)\,(x^2+4)(2x)}{(x^2+4)^4}\right] = \dfrac{960\,(x^2+4)\left[(x^2+4) - 4x^2\right]}{(x^2+4)^4} = \dfrac{960\,(4-3x^2)}{(x^2+4)^3}$.

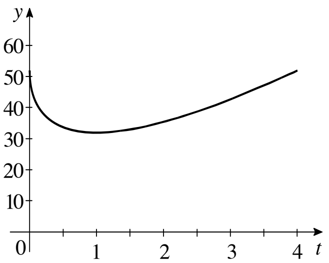
The sign diagram for T'' shows that T is concave downward on $\left(\frac{2\sqrt{3}}{3}, \infty\right)$ and concave upward on $\left(0, \frac{2\sqrt{3}}{3}\right)$.



8. We see from the results of step 7 that $\left(\frac{2\sqrt{3}}{3}, 30\right)$ is an inflection point.



70. Using the curve-sketching guidelines, we obtain the following graph of $f(t) = 20t - 40\sqrt{t} + 52$. The speed of traffic flow decreases until it reaches a minimum of 32 mph at 7 a.m., and then increases again to 52 mph.

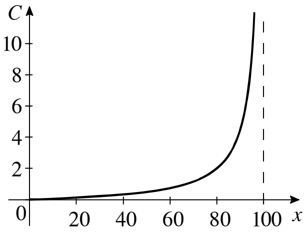


71. $C(x) = \dfrac{0.5x}{100 - x}$. We first gather the following information on C.

1. The domain of C is $[0, 100)$.
2. Setting $x = 0$ gives the y-intercept as 0.

Because of the restricted domain, we omit steps 3 and 4.

5. $C'(x) = 0.5\left[\dfrac{(100 - x)(1) - x(-1)}{(100 - x)^2}\right] = \dfrac{50}{(100 - x)^2} > 0$ for all $x \neq 100$. Therefore C is increasing on $(0, 100)$.
6. There is no relative extremum.
7. $C''(x) = -\dfrac{100}{(100 - x)^3}$, so $C''(x) > 0$ if $x < 100$ and the graph of C is concave upward on $(0, 100)$.
8. There is no inflection point.



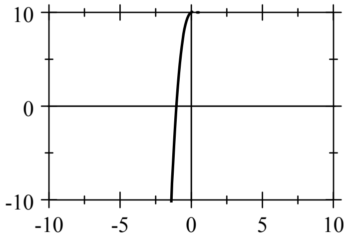
72. False. Consider $f(x) = \begin{cases} 0 & x \leq 0 \\ 1/x & x > 0 \end{cases}$ f has a vertical asymptote at $x = 0$, but $\lim_{x \to 0^-} f(x) = 0$.

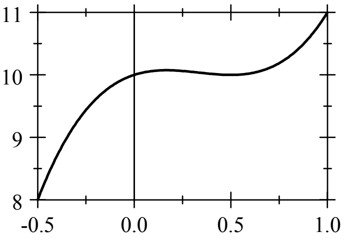
73. False. Consider $f(x) = \begin{cases} 0 & x \leq 0 \\ 1/x & x > 0 \end{cases}$ The graph of f intersects its vertical asymptote at the point $(0, 0)$.

74. False. Consider $f(x) = \begin{cases} \dfrac{2x}{x^2 + 4} & x \leq 2 \\ -\frac{3}{2}x + \frac{7}{2} & x > 2 \end{cases}$ Since $\lim_{x \to \infty} f(x) = 0$, we see that $y = 0$ is a horizontal asymptote of f. But $f(0) = f\left(\frac{7}{3}\right) = 0$, so the graph of f intersects its asymptote at $(0, 0)$ and $\left(\frac{7}{3}, 0\right)$.

Using Technology page 304

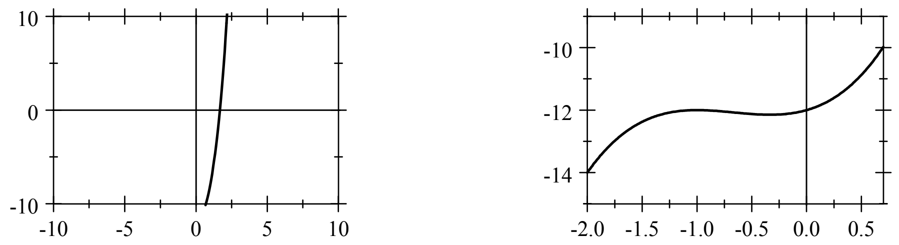
1.





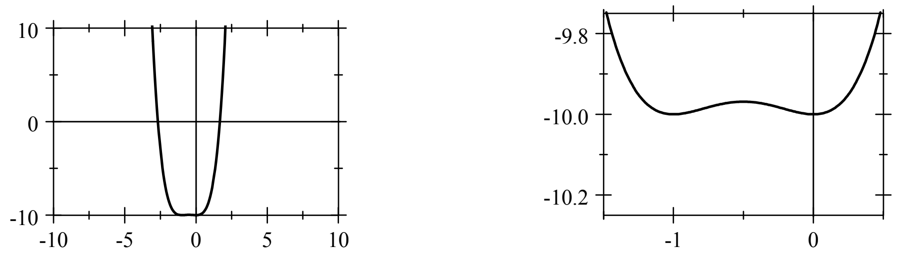
$f(x) = 4x^3 - 4x^2 + x + 10$, so $f'(x) = 12x^2 - 8x + 1 = (6x - 1)(2x - 1) = 0$ if $x = \frac{1}{6}$ or $x = \frac{1}{2}$. The second graph shows that f has a maximum at $x = \frac{1}{6}$ and a minimum at $x = \frac{1}{2}$.

2.



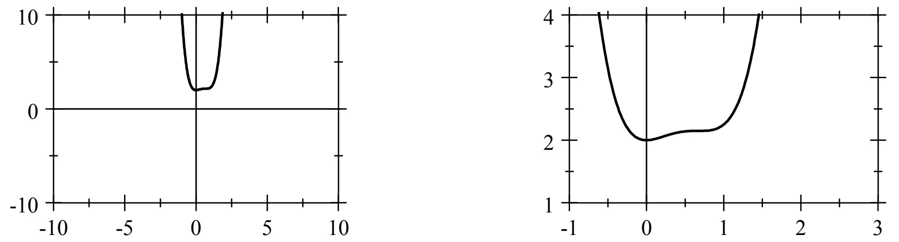
$f(x) = x^3 + 2x^2 + x - 12$, so $f'(x) = 3x^2 + 4x + 1 = (x+1)(3x+1) = 0$ if $x = -1$ or $-\frac{1}{3}$. The second graph shows that f has a maximum at $x = -1$ and a minimum at $x = -\frac{1}{3}$.

3.



$f(x) = \frac{1}{2}x^4 + x^3 + \frac{1}{2}x^2 - 10$, so $f'(x) = 2x^3 + 3x^2 + x = x(x+1)(2x+1) = 0$ if $x = -1, -\frac{1}{2}$, or 0. The second graph shows that x has minima at $x = -1$ and $x = 0$ and a maximum at $x = -\frac{1}{2}$.

4.



$f(x) = 2.25x^4 - 4x^3 + 2x^2 + 2$, so $f'(x) = 9x^3 - 12x^2 + 4x = x(3x-2)^2$ if $x = 0$ or $\frac{2}{3}$. The second graph shows that f has a minimum at $x = 0$ and no extremum at $\frac{2}{3}$.

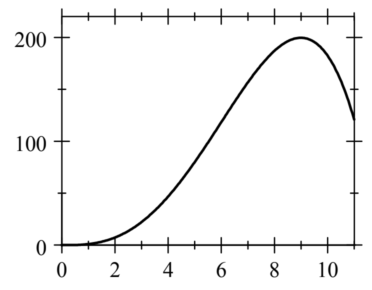
5. -0.9733, 2.3165, and 4.6569.

6. -1.5962, 0.6647, 1.8137, and 3.1178.

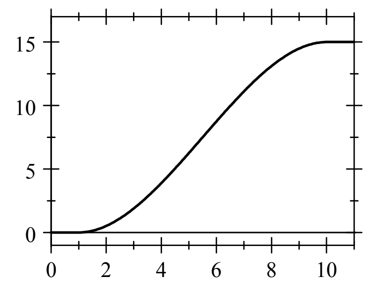
7. 1.5142

8. 0.7071

9.



10.



4.4 Optimization I

Concept Questions page 313

1. **a.** A function f has an absolute maximum at a if $f(x) \le f(a)$ for all x in the domain of f.

   b. A function f has an absolute minimum at a if $f(x) \ge f(a)$ for all x in the domain of f.

2. See the procedure given on page 307 of the text.

Exercises page 313

1. f has no absolute extremum.

2. f has an absolute minimum at $\left(-2, -\frac{1}{2}\right)$ and an absolute maximum at $\left(2, \frac{1}{2}\right)$.

3. f has an absolute minimum at $(0, 0)$.

4. f has an absolute minimum at $(0, 0)$ and no absolute maximum.

5. f has an absolute minimum at $(0, -2)$ and an absolute maximum at $(1, 3)$.

6. f has no absolute extremum.

7. f has an absolute minimum at $\left(\frac{3}{2}, -\frac{27}{16}\right)$ and an absolute maximum at $(-1, 3)$.

8. f has an absolute minimum at $(0, -3)$ and an absolute maximum at $(3, 1)$.

9. The graph of $f(x) = 2x^2 + 3x - 4$ is a parabola that opens upward. Therefore, the vertex of the parabola is the absolute minimum of f. To find the vertex, we solve the equation $f'(x) = 4x + 3 = 0$, finding $x = -\frac{3}{4}$. We conclude that the absolute minimum value is $f\left(-\frac{3}{4}\right) = -\frac{41}{8}$.

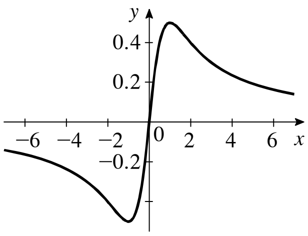
10. The graph of $g(x) = -x^2 + 4x + 3$ is a parabola that opens downward. Therefore, the vertex of the parabola is the absolute maximum of f. To find the vertex, we solve the equation $g'(x) = -2x + 4 = 0$, finding $x = 2$. We conclude that the absolute maximum value is $f(2) = 7$.

11. Because $\lim_{x\to-\infty} x^{1/3} = -\infty$ and $\lim_{x\to\infty} x^{1/3} = \infty$, we see that h is unbounded. Therefore, it has no absolute extremum.

12. From the graph of f (see Figure 15(b) on page 259 of the text), we see that $(0, 0)$ is an absolute minimum of f. There is no absolute maximum because $\lim_{x\to\infty} x^{2/3} = \infty$.

13. $f(x) = \dfrac{1}{1+x^2}$. Using the techniques of graphing, we sketch the graph of f (see Figure 40 on page 278 of the text). The absolute maximum of f is $f(0) = 1$. Alternatively, observe that $1 + x^2 \ge 1$ for all real values of x. Therefore, $f(x) \le 1$ for all x, and we see that the absolute maximum is attained when $x = 0$.

14. $f(x) = \dfrac{x}{1+x^2}$. Because f is defined for all x in $(-\infty, \infty)$, we use the graphical method. Using the techniques of graphing, we sketch the graph of f. From the graph we see that f has an absolute maximum at $\left(1, \frac{1}{2}\right)$ and an absolute minimum at $\left(-1, -\frac{1}{2}\right)$.



15. $f(x) = x^2 - 2x - 3$ and $f'(x) = 2x - 2 = 0$, so $x = 1$ is a critical number. From the table, we conclude that the absolute maximum value is $f(-2) = 5$ and the absolute minimum value is $f(1) = -4$.

x	-2	1	3
$f(x)$	5	-4	0

16. $g(x) = x^2 - 2x - 3$, so $g'(x) = 2x - 2 = 0$ implies that $x = 1$ is a critical number. From the table, we conclude that g has an absolute minimum at $(1, -4)$ and an absolute maximum at $(4, 5)$.

x	0	1	4
$f(x)$	-3	-4	5

17. $f(x) = -x^2 + 4x + 6$; The function f is continuous and defined on the closed interval $[0, 5]$. $f'(x) = -2x + 4$, and so $x = 2$ is a critical number. From the table, we conclude that $f(2) = 10$ is the absolute maximum value and $f(5) = 1$ is the absolute minimum value.

x	0	2	5
$f(x)$	6	10	1

18. $f(x) = -x^2 + 4x + 6$; The function f is continuous and defined on the closed interval $[3, 6]$. $f'(x) = -2x + 4$, so $x = 2$ is a critical number. But this point lies outside the given interval. From the table, we conclude that $f(3) = 9$ is the absolute maximum value and $f(6) = -6$ is the absolute minimum value.

x	3	6
$f(x)$	9	-6

19. The function $f(x) = x^3 + 3x^2 - 1$ is continuous and defined on the closed interval $[-3, 2]$ and differentiable in $(-3, 2)$. The critical numbers of f are found by solving $f'(x) = 3x^2 + 6x = 3x(x+2) = 0$, giving $x = -2$ and $x = 0$. From the table, we see that the absolute maximum value of f is $f(2) = 19$ and the absolute minimum value is $f(-3) = f(0) = -1$.

x	-3	-2	0	2
$f(x)$	-1	3	-1	19

20. The function $g(x) = x^3 + 3x^2 - 1$ is continuous on the closed interval $[-3, 1]$ and differentiable in $(-3, 1)$. The critical numbers of g are found by solving $g'(x) = 3x^2 + 6x = 3x(x+2) = 0$, giving $x = -2$ and $x = 0$. From the table we see that the absolute maximum value of g is $g(1) = g(-2) = 3$ and the absolute minimum value of g is $g(-3) = g(0) = -1$.

x	-3	-2	0	1
$g(x)$	-1	3	-1	3

21. The function $g(x) = 3x^4 + 4x^3$ is continuous on the closed interval $[-2, 1]$ and differentiable in $(-2, 1)$. The critical numbers of g are found by solving $g'(x) = 12x^3 + 12x^2 = 12x^2(x + 1) = 0$, giving $x = 0$ and $x = -1$. From the table, we see that $g(-2) = 16$ is the absolute maximum value of g and $g(-1) = -1$ is the absolute minimum value of g.

x	-2	-1	0	1
$g(x)$	16	-1	0	7

22. $f(x) = \frac{1}{2}x^4 - \frac{2}{3}x^3 - 2x^2 + 3$ is continuous on the closed interval $[-2, 3]$ and differentiable in the open interval $(-2, 3)$. The critical numbers of f are found by solving $f'(x) = 2x^3 - 2x^2 - 4x = 2x\left(x^2 - x - 2\right) = 2x(x - 2)(x + 1) = 0$, giving $x = -1$, 0, and 2 as critical numbers. From the table we see that the absolute maximum value of f is $f(-2) = \frac{25}{3}$ and the absolute minimum value of f is $f(2) = -\frac{7}{3}$.

x	-2	-1	0	2	3
$f(x)$	$\frac{25}{3}$	$\frac{13}{6}$	3	$-\frac{7}{3}$	$\frac{15}{2}$

23. $f(x) = \dfrac{x + 1}{x - 1}$ on [2, 4]. Next, we compute $f'(x) = \dfrac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2} = -\dfrac{2}{(x - 1)^2}$. Because there is no critical number ($x = 1$ is not in the domain of f), we need only test the endpoints. We conclude that $f(4) = \frac{5}{3}$ is the absolute minimum value and $f(2) = 3$ is the absolute maximum value.

24. $g(t) = \dfrac{t}{t - 1}$, so $g'(t) = \dfrac{(t - 1) - t}{(t - 1)^2} = -\dfrac{1}{(t - 1)^2}$. Because there is no critical number ($t = 1$ is not in the domain of g), we need only test the endpoints. We conclude that $g(2) = 2$ is the absolute maximum value and $g(4) = \frac{4}{3}$ is the absolute minimum value.

25. $f(x) = 4x + \dfrac{1}{x}$ is continuous on $[1, 4]$ and differentiable in $(1, 4)$. To find the critical numbers of f, we solve $f'(x) = 4 - \dfrac{1}{x^2} = 0$, obtaining $x = \pm\frac{1}{2}$. Because these critical numbers lie outside the interval $[1, 4]$, they are not candidates for the absolute extrema of f. Evaluating f at the endpoints of the interval $[1, 4]$, we find that the absolute maximum value of f is $f(4) = \frac{65}{4}$, and the absolute minimum value of f is $f(1) = 5$.

26. $f(x) = 9x - \dfrac{1}{x}$ is continuous on $[1, 3]$ and differentiable in $(1, 3)$. To find the critical numbers of f, we solve $f'(x) = 9 + \frac{1}{x^2} = 0$, obtaining $x^2 = -\frac{1}{9}$ which has no solution. Evaluating f at the endpoints of the interval $[1, 3]$, we find that the absolute minimum value is $f(1) = 8$ and the absolute maximum value is $f(3) = \frac{80}{3}$.

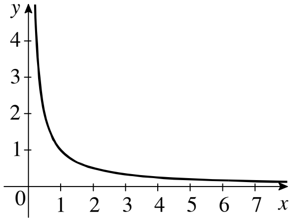
27. $f(x) = \frac{1}{2}x^2 - 2\sqrt{x} = \frac{1}{2}x^2 - 2x^{1/2}$. To find the critical numbers of f, we solve $f'(x) = x - x^{-1/2} = 0$, or $x^{3/2} - 1 = 0$, obtaining $x = 1$. From the table, we conclude that $f(3) \approx 1.04$ is the absolute maximum value and $f(1) = -\frac{3}{2}$ is the absolute minimum value.

x	0	1	3
$f(x)$	0	$-\frac{3}{2}$	$\frac{9}{2} - 2\sqrt{3} \approx 1.04$

28. The function $g(x) = \frac{1}{8}x^2 - 4\sqrt{x} = \frac{1}{8}x^2 - 4x^{1/2}$ is continuous on the closed interval $[0, 9]$ and differentiable in $(0, 9)$. To find the critical numbers of g, we first compute $g'(x) = \frac{1}{4}x - 2x^{-1/2} = \frac{1}{4}x^{-1/2}\left(x^{3/2} - 8\right)$. Setting $g'(x) = 0$, we have $x^{3/2} = 8$, or $x = 4$. From the table, we conclude that $g(4) = -6$ is the absolute minimum value and $g(0) = 0$ is the absolute maximum value of g.

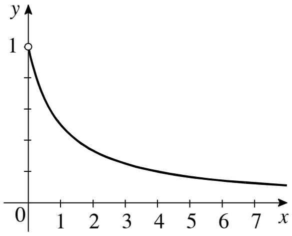
x	0	4	9
$f(x)$	0	-6	$-\frac{15}{8}$

29.



From the graph of $f(x) = \dfrac{1}{x}$ for $x > 0$, we conclude that f has no absolute extremum.

30.



From the graph of $g(x) = \dfrac{1}{x+1}$ for $x > 0$, we conclude that g has no absolute extremum.

31. $f(x) = 3x^{2/3} - 2x$. The function f is continuous on $[0, 3]$ and differentiable on $(0, 3)$. To find the critical numbers of f, we solve $f'(x) = 2x^{-1/3} - 2 = 0$, obtaining $x = 1$ as the critical number. From the table, we conclude that the absolute maximum value is $f(1) = 1$ and the absolute minimum value is $f(0) = 0$.

x	0	1	3
$f(x)$	0	1	$3^{5/3} - 6 \approx 0.24$

32. $g(x) = x^2 + 2x^{2/3}$, so $g'(x) = 2x + \frac{4}{3}x^{-1/3} = \frac{2}{3}x^{-1/3}\left(3x^{4/3} + 2\right)$ is never zero, but $g'(x)$ is not defined at $x = 0$, which is a critical number of g. From the table, we conclude that $g(-2) = g(2) = 4 + 2^{5/3}$ give the absolute maximum value and $g(0) = 0$ gives the absolute minimum value.

x	-2	0	2
$g(x)$	$4 + 2^{5/3}$	0	$4 + 2^{5/3}$

33. $f(x) = x^{2/3}\left(x^2 - 4\right)$, so $f'(x) = x^{2/3}(2x) + \frac{2}{3}x^{-1/3}\left(x^2 - 4\right) = \frac{2}{3}x^{-1/3}\left[3x^2 + \left(x^2 - 4\right)\right] = \dfrac{8\left(x^2 - 1\right)}{3x^{1/3}} = 0$. Observe that f' is not defined at $x = 0$. Furthermore, $f'(x) = 0$ at $x \pm 1$. So the critical numbers of f are -1 and 0, and 1. From the table, we see that f has absolute minima at $(-1, -3)$ and $(1, -3)$ and absolute maxima at $(0, 0)$ and $(2, 0)$.

x	-1	0	1	2
$f(x)$	-3	0	-3	0

34. The function is the same as that of Exercise 33. From the table, we see that f has a absolute minima at $(-1, -3)$ and $(1, -3)$ and an absolute maximum at $\left(3, 5 \cdot 3^{2/3}\right)$.

x	-1	0	1	3
$f(x)$	-3	0	-3	$5 \cdot 3^{2/3}$

35. $f(x) = \dfrac{x}{x^2+2}$. To find the critical numbers of f, we solve $f'(x) = \dfrac{(x^2+2) - x(2x)}{(x^2+2)^2} = \dfrac{2-x^2}{(x^2+2)^2} = 0$, obtaining $x = \pm\sqrt{2}$. Because $x = -\sqrt{2}$ lies outside $[-1, 2]$, $x = \sqrt{2}$ is the only critical number in the given interval.

From the table, we conclude that $f\left(\sqrt{2}\right) = \frac{\sqrt{2}}{4} \approx 0.35$ is the absolute maximum value and $f(-1) = -\frac{1}{3}$ is the absolute minimum value.

x	-1	$\sqrt{2}$	2
$f(x)$	$-\frac{1}{3}$	$\frac{\sqrt{2}}{4} \approx 0.35$	$\frac{1}{3}$

36. $f'(x) = \dfrac{d}{dx}\left(x^2+2x+5\right)^{-1} = -\left(x^2+2x+5\right)^{-2}(2x+2) = \dfrac{-2(x+1)}{\left(x^2+2x+5\right)^2}$. Setting $f'(x) = 0$ gives $x = -1$ as a critical number. From the table, we see that f has an absolute minimum at $\left(1, \frac{1}{8}\right)$ and an absolute maximum at $\left(-1, \frac{1}{4}\right)$.

x	-2	-1	1
$f(x)$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{8}$

37. The function $f(x) = \dfrac{x}{\sqrt{x^2+1}} = \dfrac{x}{\left(x^2+1\right)^{1/2}}$ is continuous on the closed interval $[-1, 1]$ and differentiable on $(-1, 1)$. To find the critical numbers of f, we first compute $f'(x) = \dfrac{\left(x^2+1\right)^{1/2}(1) - x\left(\frac{1}{2}\right)\left(x^2+1\right)^{-1/2}(2x)}{\left[\left(x^2+1\right)^{1/2}\right]^2} = \dfrac{\left(x^2+1\right)^{-1/2}\left[x^2+1-x^2\right]}{x^2+1} = \dfrac{1}{\left(x^2+1\right)^{3/2}}$, which is never equal to zero. We compute $f(x)$ at the endpoints, and conclude that $f(-1) = -\frac{\sqrt{2}}{2}$ is the absolute minimum value and $f(1) = \frac{\sqrt{2}}{2}$ is the absolute maximum value.

38. $g(x) = x\left(4-x^2\right)^{1/2}$ on $[0, 2]$, so

$$g'(x) = \left(4-x^2\right)^{1/2} + x\left(\tfrac{1}{2}\right)\left(4-x^2\right)^{-1/2}(-2x) = \left(4-x^2\right)^{-1/2}\left(4-x^2-x^2\right) = -\frac{2\left(x^2-2\right)}{\sqrt{4-x^2}}.$$

The critical number of g in $(0, 2)$ is $\sqrt{2}$. From the table, we conclude that $g\left(\sqrt{2}\right) = 2$ is the absolute maximum value and $g(0) = g(2) = 0$ is the absolute minimum value.

x	0	$\sqrt{2}$	2
$g(x)$	0	2	0

39. $h(t) = -16t^2 + 64t + 80$. To find the maximum value of h, we solve $h'(t) = -32t + 64 = -32(t-2) = 0$, giving $t = 2$ as the critical number of h. Furthermore, this value of t gives rise to the absolute maximum value of h since the graph of h is a parabola that opens downward. The maximum height is given by $h(2) = -16(4) + 64(2) + 80 = 144$, or 144 feet.

40. $P(x) = -10x^2 + 1760x - 50{,}000$, so $P'(x) = -20x + 1760 = 0$ if $x = 88$, and this is a critical number of P. Now $P(88) = -10(88)^2 + 1760(88) - 50{,}000 = 27{,}440$. The graph of P is a parabola that opens downward, so the point $(88, 27440)$ is an absolute maximum of P. The maximum monthly profit is \$27,440, when 88 units are rented out.

41. $f(t) = 0.136t^2 + 0.127t + 18.1$, so $f'(t) = 0.272t + 0.127$. Setting $f'(t) = 0$ gives $0.272t = -0.127$, so $t \approx -0.467$. This value of t lies outside the interval $[0, 4]$, so f has no critical number on that interval. From the table, we conclude that the lowest and highest strikeout rates are 18.1% and 20.8%, occurring in 2009 and 2013 respectively.

t	0	4
$f(t)$	18.1	20.784

42. $N(t) = -2.65t^2 + 13.13t + 39.9$, so $N'(t) = -5.3t + 13.13$. Setting $f(t) = 0$ gives $5.3t = 13.13$, so $t \approx 2.48$ is a critical number of N. From the table, we see that global iPod sales peaked at this value of t, in mid-2009, at a sales level of approximately 56.2 million units.

t	0	2.5	4
$N(t)$	39.9	56.2	50.02

43. Observe that f is continuous on $[0, 4]$. Next, we compute $f'(t) = \frac{d}{dt}\left(20t - 40t^{1/2} + 50\right) = 20 - 20t^{-1/2} = 20t^{-1/2}\left(t^{1/2} - 1\right) = 20\frac{\sqrt{t} - 1}{\sqrt{t}}$. Observe that $t = 1$ is the only critical number of f in $(0, 4)$. Because $f(0) = 50$, $f(1) = 30$, and $f(4) = 50$, we conclude that f attains its minimum value of 30 at $t = 1$. This tells us that the traffic is moving at the slowest rate at 7 a.m. and the average speed of a vehicle at that time is 30 mph.

44. The revenue is $R(x) = px = \left(\frac{100{,}000}{250 + x} - 100\right)x = \frac{100{,}000x}{250 + x} - 100x$, so $R'(x) = 100{,}000 \cdot \frac{(250 + x)(1) - x(1)}{(250 + x)^2} - 100 = \frac{25{,}000{,}000}{(250 + x)^2} - 100$. Setting $R'(x) = 0$ gives $25{,}000{,}000 = 100(250 + x)^2$, so $250 + x = \pm\sqrt{250{,}000} = \pm 500$; that is, $x = -750$ or 250.

Thus, R has the critical number 250 on the interval $[0, 750]$. From the table, we see that selling 250 handbags per day gives the maximum daily profit of \$25,000.

x	0	250	750
$R(x)$	0	25,000	0

45. $h(t) = -\frac{1}{3}t^3 + 4t^2 + 20t + 2$, so $h'(t) = -t^2 + 8t + 20 = -\left(t^2 - 8t - 20\right) = -(t - 10)(t + 2) = 0$ if $t = -2$ or $t = 10$. Rejecting the negative root, we take $t = 10$. Next, we compute $h''(t) = -2t + 8$. Because $h''(10) = -20 + 8 = -12 < 0$, the Second Derivative Test indicates that the point $t = 10$ gives a relative maximum. From physical considerations, or from a sketch of the graph of h, we conclude that the rocket attains its maximum altitude at $t = 10$ with a maximum height of $h(10) = -\frac{1}{3}(10)^3 + 4(10)^2 + 20(10) + 2$, or approximately 268.7 ft.

46. $P(x) = -0.000002x^3 + 6x - 400$, so $P'(x) = -0.000006x^2 + 6 = 0$ if $x = \pm 1000$. We reject the negative root. Next, we compute $P''(x) = -0.000012x$. Because $P''(1000) = -0.012 < 0$, the Second Derivative Test shows that $x = 1000$ gives a relative maximum of f. From physical considerations, or from a sketch of the graph of f, we see that the maximum profit is realized if 1000 cases are produced per day. That profit is $P(1000) = -0.000002(1000)^3 + 6(1000) - 400$, or \$3600/day.

47. The revenue is $R(x) = px = -0.00042x^2 + 6x$. Therefore, the profit is $P(x) = R(x) - C(x) = -0.00042x^2 + 6x - \left(600 + 2x - 0.00002x^2\right) = -0.0004x^2 + 4x - 600$.

$P'(x) = -0.0008x + 4 = 0$ if $x = 5000$, a critical number of P. From the table, we see that Phonola should produce 5000 discs/month.

x	0	5000	12,000
$P(x)$	-600	9400	$-10{,}200$

48. The revenue is $R(x) = px = -0.0004x^2 + 10x$ and the profit is $P(x) = R(x) - C(x) = -0.0004x^2 + 10x - \left(400 + 4x + 0.0001x^2\right) = -0.0005x^2 + 6x - 400$. $P'(x) = -0.001x + 6 = 0$ if $x = 6000$, a critical number. Because $P''(x) = -0.001 < 0$ for all x, we see that the graph of P is a parabola that opens downward. Therefore, a level of production of 6000 rackets/day will yield a maximum profit.

49. The cost function is $C(x) = V(x) + 20{,}000 = 0.000001x^3 - 0.01x^2 + 50x + 20{,}000$, so the profit function is

$$\begin{aligned} P(x) &= R(x) - C(x) = -0.02x^2 + 150x - 0.000001x^3 + 0.01x^2 - 50x + 20{,}000 \\ &= -0.000001x^3 - 0.01x^2 + 100x - 20{,}000. \end{aligned}$$

We want to maximize P on $[0, 7000]$. $P'(x) = -0.000003x^2 - 0.02x + 100$. Setting $P'(x) = 0$ gives $3x^2 + 20{,}000x - 100{,}000{,}000 = 0$, so or $x = \dfrac{-20{,}000 \pm \sqrt{20{,}000^2 + 1{,}200{,}000{,}000}}{6} = -10{,}000$ or $3{,}333.33$.

Thus, $x = 3333.33$ is a critical number in the interval $[0, 7500]$. From the table, we see that a level of production of 3,333 pagers per week will yield a maximum profit of \$165,185.20 per week.

x	0	3333.33	7500
$P(x)$	$-20{,}000$	165,185.2	$-254{,}375$

50. $R(x) = px = -0.05x^2 + 600x$, so

$$\begin{aligned} P(x) &= R(x) - C(x) = -0.05x^2 + 600x - \left(0.000002x^3 - 0.03x^2 + 400x + 80{,}000\right) \\ &= -0.000002x^3 - 0.02x^2 + 200x - 80{,}000. \end{aligned}$$

We want to maximize P on $[0, 12000]$. $P'(x) = -0.000006x^2 - 0.04x + 200$, so setting $P'(x) = 0$ gives $3x^2 + 20{,}000x - 100{,}000{,}000 = 0$ or $x = \dfrac{-20{,}000 \pm \sqrt{20{,}000^2 + 1{,}200{,}000{,}000}}{6} = -10{,}000$ or 3333.3. Thus, $x = 3333.3$ is a critical number in the interval $[0, 12000]$. From the table, we see that a level of production of 3333 units will yield a maximum profit.

x	0	3333	12,000
$P(x)$	$-80{,}000$	290,370	$-4{,}016{,}000$

51. The cost function is $C(x) = 0.2\left(0.01x^2 + 120\right)$ and the average cost function is $\overline{C}(x) = \dfrac{C(x)}{x} = 0.2\left(0.01x + \dfrac{120}{x}\right) = 0.002x + \frac{24}{x}$. To find the minimum average cost, we first compute $\overline{C}'(x) = 0.002 - \dfrac{24}{x^2}$. Setting $\overline{C}'(x) = 0$ gives $0.002 - \dfrac{24}{x^2} = 0$, so $x^2 = \dfrac{24}{0.002} = 12{,}000$, and thus $x \approx \pm 110$. We reject the negative root, leaving $x = 110$ as the only critical number of $\overline{C}(x)$. Because $\overline{C}''(x) = 48x^{-3} > 0$ for all $x > 0$, we see that $\overline{C}(x)$ is concave upward on $(0, \infty)$. We conclude that $\overline{C}(110) \approx 0.44$ is the absolute minimum value of $\overline{C}(x)$ and that the average cost is minimized when $x = 110$ units.

52. **a.** $\overline{C}(x) = \dfrac{C(x)}{x} = 0.0025x + 80 + \dfrac{10{,}000}{x}$.

b. $\overline{C}'(x) = 0.0025 - \dfrac{10{,}000}{x^2} = 0$ if $0.0025x^2 = 10{,}000$, or $x = 2000$. Because $\overline{C}''(x) = \dfrac{20{,}000}{x^3}$, we see that $\overline{C}''(x) > 0$ for $x > 0$ and so $\overline{C}$ is concave upward on $(0, \infty)$. Therefore, $x = 2000$ yields a minimum.

c. We solve $\overline{C}(x) = C'(x)$: $0.0025x + 80 + \dfrac{10{,}000}{x} = 0.005x + 80$, so $0.0025x^2 = 10{,}000$ and $x = 2000$.

d. It appears that we can solve the problem in two ways.

53. **a.** $C(x) = 0.000002x^3 + 5x + 400$, so $\overline{C}(x) = \dfrac{C(x)}{x} = 0.000002x^2 + 5 + \dfrac{400}{x}$.

b. $\overline{C}'(x) = 0.000004x - \dfrac{400}{x^2} = \dfrac{0.000004x^3 - 400}{x^2} = \dfrac{0.000004\left(x^3 - 100{,}000{,}000\right)}{x^2}$. Setting $\overline{C}'(x) = 0$ gives $x = 464$, the only critical number of $\overline{C}$. Next, $\overline{C}''(x) = 0.000004 + \dfrac{800}{x^3}$, so $\overline{C}''(464) > 0$ and by the Second Derivative Test, the point $x = 464$ gives rise to a relative minimum. Because $\overline{C}''(x) > 0$ for all $x > 0$, $\overline{C}$ is concave upward on $(0, \infty)$ and $x = 464$ gives rise to an absolute minimum of $\overline{C}$. Thus, the smallest average product cost occurs when the level of production is 464 cases per day.

c. We want to solve the equation $\overline{C}(x) = C'(x)$, that is, $0.000002x^2 + 5 + \dfrac{400}{x} = 0.000006x^2 + 5$, so $0.000004x^3 = 400$, $x^3 = 100{,}000{,}000$, and $x = 464$.

d. The results are as expected.

54. $\overline{C}(x) = \dfrac{C(x)}{x}$, so $\overline{C}'(x) = \dfrac{xC'(x) - C(x)}{x^2} = 0$. This implies that $xC'(x) - C(x) = x^2$, so $\dfrac{C(x)}{x} = C'(x)$. This shows that at a level of production where the average cost is minimized, the average cost $\dfrac{C(x)}{x}$ is equal to the marginal cost $C'(x)$.

55. **a.** $\overline{C}(x) = \dfrac{C(x)}{x} = 0.0025x + 80 + \dfrac{10{,}000}{x}$.

b. Using the result of Exercise 54, we set $\dfrac{C(x)}{x} = \overline{C}(x) = C'(x)$, obtaining $0.0025x + 80 + \dfrac{10{,}000}{x} = 0.005x + 80$. This is the same equation obtained in Exercise 52(b). The lowest average production cost occurs when the production level is 2000 cases per day.

c. The average cost is equal to the marginal cost when the production level is 2000 cases per day.

d. They are the same, as expected.

56. The demand equation is $p = \sqrt{800 - x} = (800 - x)^{1/2}$, so the revenue function is $R(x) = xp = x(800 - x)^{1/2}$. To find the maximum of R, we compute

$$R'(x) = \tfrac{1}{2}(800 - x)^{-1/2}(-1)(x) + (800 - x)^{1/2} = \tfrac{1}{2}(800 - x)^{-1/2}\left[-x + 2(800 - x)\right]$$
$$= \tfrac{1}{2}(800 - x)^{-1/2}(1600 - 3x).$$

Next, $R'(x) = 0$ implies $x = 800$ or $x = \frac{1600}{3}$, the critical numbers of R.

From the table, we conclude that $R\left(\frac{1600}{3}\right) = 8709$ is the absolute maximum value. Therefore, the revenue is maximized by producing $\frac{1600}{3} \approx 533$ dresses.

x	0	800	$\frac{1600}{3}$
$R(x)$	0	0	8709

57. The revenue function is $R(x) = xp = \dfrac{50x}{0.01x^2+1}$. To find the maximum value of R, we compute $R'(x) = \dfrac{\left(0.01x^2+1\right)50 - 50x\,(0.02x)}{\left(0.01x^2+1\right)^2} = -\dfrac{0.5\left(x^2-100\right)}{\left(0.01x^2+1\right)^2}$. Now $R'(x) = 0$ implies $x = -10$, or $x = 10$. The first root is rejected since x must be greater than or equal to zero. Thus, $x = 10$ is the only critical number.

From the table, we conclude that $R(10) = 250$ is the absolute maximum value of R. Thus, the revenue is maximized by selling 10,000 watches.

x	0	10	20
$R(x)$	0	250	200

58. $A(t) = 136\left[1+0.25(t-4.5)^2\right]^{-1} + 28$, so

$$A'(t) = 136(-1)\left[1+0.25(t-4.5)^2\right]^{-2}(0.25)\,2(t-4.5) = -\frac{68(t-4.5)}{\left[1+0.25(t-4.5)^2\right]^2}.$$

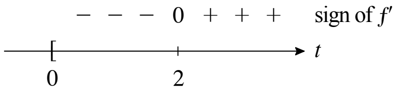
Setting $A'(t) = 0$ gives $t = 4.5$ as a critical number of A. We see that the maximum of A occurs when $t = 4.5$, that is, at 11:30 a.m.

t	0	4.5	11
$A(t)$	50.4	164	39.8

59. $f(t) = 100\left(\dfrac{t^2-4t+4}{t^2+4}\right)$.

a. $f'(t) = 100\left[\dfrac{\left(t^2+4\right)(2t-4) - \left(t^2-4t+4\right)(2t)}{\left(t^2+4\right)^2}\right] = \dfrac{400\left(t^2-4\right)}{\left(t^2+4\right)^2} = \dfrac{400(t-2)(t+2)}{\left(t^2+4\right)^2}$.

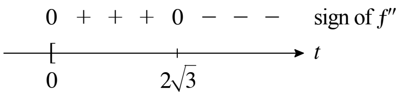
From the sign diagram for f', we see that $t = 2$ gives a relative minimum, and we conclude that the oxygen content is the lowest 2 days after the organic waste has been dumped into the pond.



b. $f''(t) = 400\left[\dfrac{\left(t^2+4\right)^2(2t) - \left(t^2-4\right)2\left(t^2+4\right)(2t)}{(t+4)^4}\right] = 400\left[\dfrac{(2t)\left(t^2+4\right)\left(t^2+4-2t^2+8\right)}{\left(t^2+4\right)^4}\right]$

$= -\dfrac{800t\left(t^2-12\right)}{\left(t^2+4\right)^3}$.

$f''(t) = 0$ when $t = 0$ and $t = \pm 2\sqrt{3}$. We reject $t = 0$ and $t = -2\sqrt{3}$. From the sign diagram for f'', we see that $t = 2\sqrt{3}$ gives an inflection point of f and we conclude that this is an absolute maximum. Therefore, the rate of oxygen regeneration is greatest 3.5 days after the organic waste has been dumped into the pond.



60. $v'(r) = -2kr = 0$ if $r = 0$, so there is no critical number in $(0, R)$. Calculating $v(r)$ at the endpoints, we see that v has an absolute maximum of $v(0) = kR^2$, so the velocity is greatest along the central axis.

61. We compute $\overline{R}'(x) = \dfrac{xR'(x) - R(x)}{x^2}$. Setting $\overline{R}'(x) = 0$ gives $xR'(x) - R(x) = 0$, or $R'(x) = \dfrac{R(x)}{x} = \overline{R}(x)$, so a critical number of $\overline{R}$ occurs when $\overline{R}(x) = R'(x)$. Next, we compute $\overline{R}''(x) = \dfrac{x^2\left[R'(x) + xR''(x) - R'(x)\right] - \left[xR'(x) - R(x)\right](2x)}{x^4} = \dfrac{R''(x)}{x} < 0$. Thus, by the Second Derivative Test, the critical number does give the maximum revenue.

62. $N(t) = -0.1t^3 + 1.5t^2 + 100$ and $N'(t) = -0.3t^2 + 3t$. We want to maximize the function $N'(t)$. Now $N''(t) = -0.6t + 3$, so setting $N''(t) = 0$ gives $t = 5$ as the critical number of N'. $N'''(5) = -0.6 < 0$ and $t = 5$ does give rise to a maximum for $N'(t)$, that is the growth rate was maximal in 2012, as we wished to show.

63. $G(t) = -0.2t^3 + 2.4t^2 + 60$, so the growth rate is $G'(t) = -0.6t^2 + 4.8t$. To find the maximum growth rate, we compute $G''(t) = -1.2t + 4.8$. Setting $G''(t) = 0$ gives $t = 4$ as a critical number. From the table, we see that G is maximal at $t = 4$; that is, the growth rate is greatest in 2010.

t	0	4	8
$G'(t)$	0	9.6	0

64. $D(t) = -0.038898t^3 + 0.30858t^2 - 0.31849t + 0.22$, so $D'(t) = -0.116694t^2 + 0.61716t - 0.31849$. Setting $D'(t) = 0$ gives $t = \dfrac{-0.61716 \pm \sqrt{(0.61716)^2 - 4(-0.116694)(-0.31849)}}{2(-0.116694)} \approx 0.58$ or 4.71.

From the table, we see that the largest federal budget deficit over the period under consideration was approximately \$1.5 trillion in 2010.

t	0	0.58	4.71	6
$D(t)$	0.22	0.13	1.50	1.02

65. $N(t) = -87.244444t^3 - 2482.35t^2 + 46009.26t + 579185$, so $N'(t) = -261.733328t^2 - 4964.7t + 46009.26$. Setting $N'(t) = 0$ and using the quadratic formula gives
$$t = \frac{-(-4964.7) \pm \sqrt{(-4964.7)^2 - 4(-261.733328)(46009.26)}}{2(-261.73332)} \approx -25.8$$
or 6.82, so 6.82 is an approximate critical number of N. From the table, we see that the number of new prison admissions did indeed peak in 2006 ($t = 6$) at approximately 749,833.

t	2	6.82	11
$N(t)$	660,576	749,833	668,800

66. $f(t) = -0.0004401t^3 + 0.007t^2 + 0.112t + 0.28$, so $f'(t) = -0.0013203t^2 + 0.014t + 0.112$. Setting $f'(t) = 0$ and using the quadratic formula gives $t = \dfrac{-0.014 \pm \sqrt{(0.014)^2 - 4(-0.0013203)(0.112)}}{2(-0.0013203)} \approx -5.325$ or 15.93, so the only critical number of f in the relevant interval is approximately 15.93. From the table, we see that the highest rate of death from AIDS worldwide over the period from 1990 through 2011 was approximately 2.06 million per year in 2006.

t	0	15.93	21
$f(t)$	0.28	2.06	1.64

67. $S'(t) = \frac{d}{dt}\left(0.000989t^3 - 0.0486t^2 + 0.7116t + 1.46\right) = 0.002967t^2 - 0.0972t + 0.7116$. Using the quadratic formula to solve the equation $f'(t) = 0$ gives $t = \dfrac{0.0972 \pm \sqrt{(-0.0972)^2 - 4(0.002967)(0.7116)}}{2(0.002967)} \approx 11.0$ or 21.7. From the table, we see that S has an absolute maximum when $t \approx 11$. Thus, children with superior intelligence have a cortex that reaches maximum thickness around 11 years of age.

t	5	11.0	19
$S(t)$	3.9	4.7	4.2

68. $A'(t) = \dfrac{d}{dt}\left(-0.00005t^3 - 0.000826t^2 + 0.0153t + 4.55\right) = -0.00015t^2 - 0.001652t + 0.0153$. Using the quadratic formula to solve $f'(t) = 0$ with $a = -0.00015$, $b = -0.001652$, and $c = 0.0153$, we have

$$t = \frac{-(-0.001652) \pm \sqrt{(-0.001652)^2 - 4(-0.00015)(0.0153)}}{2(-0.00015)} \approx -17.01 \text{ or } 5.997.$$

From the table, we see that A has an absolute maximum when $t \approx 6$, so the cortex of children of average intelligence reaches a maximum thickness around the time the children are 6 years old.

t	5	6	19
$A(t)$	4.5996	4.601	4.200

69. a. $P(t) = 0.00074t^3 - 0.0704t^2 + 0.89t + 6.04$, so $P'(t) = 0.00222t^2 - 0.1408t + 0.89 = 0$ implies $t = \dfrac{0.1408 \pm \sqrt{(0.1408)^2 - 4(0.00222)(0.89)}}{2(0.00222)} \approx 7.12$ or 56.3. The root 56.3 is rejected because it lies outside the interval [0, 10]. $P''(t) = 0.00444t - 0.1408$ and $P''(7.12) = -0.109 < 0$, and so $t \approx 7.12$ gives a relative maximum. This occurs around 2071.

b. The population will peak at $P(7.12) \approx 9.075$ billion.

70. a. On [0, 3], $f(t) = 0.6t^2 + 2.4t + 7.6$, so $f'(t) = 1.2t + 2.4 = 0$ implies $t = -2$ which lies outside the interval [0, 3]. (We evaluate f at each relevant point below.)

On [3, 5], $f(t) = 3t^2 + 18.8t - 63.2$, so $f'(t) = 6t + 18.8 = 0$ implies $t = -3.13$ which lies outside the interval [3, 5].

On [5, 8], $f(t) = -3.3167t^3 + 80.1t^2 - 642.583t + 1730.8025$, so $f'(t) = -9.9501t^2 + 160.2t - 642.583 = 0$ implies $t = \dfrac{-160.2 \pm \sqrt{160.2^2 - 4(-9.9501)(642.583)}}{2(-9.9501)} \approx 7.58$ or 8.52. Only the critical number $t = 7.58$ lies inside the interval [5, 8].

From the table, we see that the investment peaked when $t = 5$, that is, in the year 2000. The amount invested was \$105.8 billion.

t	0	3	5	7.58	8
$f(t)$	7.6	20.2	105.8	17.8	18.4

b. Investment was lowest (at \$7.6 billion) when $t = 0$.

71. We want to minimize the function $E(v) = \dfrac{aLv^3}{v-u}$. Because $v > u$, the function has no points of discontinuity. To find the critical numbers of $E(v)$, we solve the equation $E'(v) = \dfrac{(v-u)\,3aLv^2 - aLv^3}{(v-u)^2} = \dfrac{aLv^2(2v-3u)}{(v-u)^2} = 0$, obtaining $v = \frac{3}{2}u$ or $v = 0$. Now $v \neq 0$ since $u < v$, so $v = \frac{3}{2}u$ is the only critical number of interest. Because $E'(v) < 0$ if $v < \frac{3}{2}u$ and $E'(v) > 0$ if $v > \frac{3}{2}u$, we see that $v = \frac{3}{2}u$ gives a relative minimum. The nature of the problem suggests that $v = \frac{3}{2}u$ gives the absolute minimum of E (we can verify this by sketching the graph of E). Therefore, the fish must swim at $\frac{3}{2}u$ ft/sec in order to minimize the total energy expended.

72. $R = D^2\left(\dfrac{k}{2} - \dfrac{D}{3}\right) = \dfrac{kD^2}{2} - \dfrac{D^3}{3}$, so $\dfrac{dR}{dD} = \dfrac{2kD}{2} - \dfrac{3D^2}{3} = kD - D^2 = D(k-D)$. Setting $\dfrac{dR}{dD} = 0$, we have $D = 0$ or $k = D$. We consider only $k = D$ because $D > 0$. If $k > 0$, $\dfrac{dR}{dD} > 0$ and if $k < 0$, $\dfrac{dR}{dD} < 0$. Therefore $k = D$ gives a relative maximum. The nature of the problem suggests that $k = D$ gives the absolute maximum of R. We can also verify this by graphing R.

73. $\dfrac{dR}{dD} = kD - D^2$ and $\dfrac{d^2R}{dD^2} = k - 2D$. Setting $\dfrac{d^2R}{dD^2} = 0$, we obtain $k = 2D$, or $D = \dfrac{k}{2}$. Because $\dfrac{d^2R}{dD^2} > 0$ for $k < 2D$ and $\dfrac{d^2R}{dD^2} < 0$ for $k > 2D$, we see that $k = 2D$ provides the relative (and absolute) maximum.

74. $R'(x) = \dfrac{d}{dx}\left[kx(Q-x)\right] = k\dfrac{d}{dx}\left(Qx - x^2\right) = k(Q-2x)$ is continuous everywhere and has a zero at $\frac{1}{2}Q$; this is the only critical number of R in $(0, Q)$. $R(0) = 0$, $R\left(\frac{1}{2}Q\right) = \frac{1}{4}kQ^2$, and $R(Q) = 0$, so the absolute maximum value of R is $R\left(\frac{1}{2}Q\right) = \frac{1}{4}kQ^2$, showing that the rate of chemical reaction is greatest when exactly half of the original substrate has been transformed.

75. Setting $P' = 0$ gives $P' = \dfrac{d}{dR}\left[\dfrac{E^2R}{(R+r)^2}\right] = E^2\left[\dfrac{(R+r)^2 - R(2)(R+r)}{(R+r)^4}\right] = \dfrac{E^2(r-R)}{(R+r)^3} = 0$. Therefore, $R = r$ is a critical number of P. Because $P'' = E^2\dfrac{(R+r)^3(-1) - (r-R)(3)(R+r)^2}{(R+r)^6} = \dfrac{2E^2(R-2r)}{(R+r)^4}$ and $P''(r) = \dfrac{-2E^2r}{(2r)^4} = -\dfrac{E^2}{8r^3} < 0$, the Second Derivative Test and physical considerations both imply that $R = r$ gives a relative maximum value of P. The maximum power is $P = \dfrac{E^2r}{(2r)^2} = \dfrac{E^2}{4r}$ watts.

76. Setting $v' = 0$ gives

$$v' = \frac{d}{dL}\left[k\sqrt{\frac{L}{C} + \frac{C}{L}}\right] = k\frac{d}{dL}\left(\frac{L}{C} + \frac{C}{L}\right)^{1/2} = \frac{k}{2}\left(\frac{L}{C} + \frac{C}{L}\right)^{-1/2}\left(\frac{1}{C} - \frac{C}{L^2}\right) = \frac{k\left(L^2 - C^2\right)}{2CL^2\sqrt{\dfrac{L}{C} + \dfrac{C}{L}}} = 0.$$

Therefore, $L = \pm C$. Because v is not defined for $L = -C$, we reject that root. The length of the wave with minimum velocity is C.

77. $\frac{dx}{dt} = \frac{d}{dt}\left[1.5\,(10-t) - 0.0013\,(10-t)^4\right] = -1.5 - 0.0013\,(4)\,(10-t)^3\,(-1) = -1.5 + 0.0052\,(10-t)^3$ is continuous everywhere and has zeros where $0.0052\left(10-t^3\right) = 1.5$; that is, $(10-t)^3 = \frac{1.5}{0.0052}$, or $t = 10 - \sqrt[3]{\frac{1.5}{0.0052}} \approx 3.4$, and so x has the critical number 3.4 in (0, 10). Now $x\,(0) = 2$, $x\,(3.4) = 7.4$, and $x\,(10) = 0$, showing that after 3.4 minutes, the maximum amount of salt (roughly 7.4 lb) is in the tank.

78. False. Let $f\,(x) = \begin{cases} -x & \text{if } -1 \le x < 0 \\ \frac{1}{2} & \text{if } 0 \le x < 1 \end{cases}$ Then f is discontinuous at $x = 0$, but f has an absolute maximum value of 1, attained at $x = -1$.

79. False. Let $f\,(x) = \begin{cases} |x| & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$ on $[-1, 1]$.

80. True. $f''\,(x) < 0$ on (a, b), so the graph of f is concave downward on (a, b). Therefore, the relative maximum value at $x = c$ must be the absolute maximum value.

81. True. The absolute extrema of f must occur for some x in (a, b) at which $f'\,(x) = 0$, or at an endpoint. Since $f'\,(x) \ne 0$ for all x in (a, b), the absolute extrema of f (and in particular its absolute maximum) must occur at $x = a$ or $x = b$, with value $f\,(a)$ or $f\,(b)$.

82. True. Since $f'\,(x) > 0$ for all x in (a, b), we see that the absolute extrema of f must occur at $x = a$ or $x = b$ (see Exercise 81). But f is increasing on (a, b), which implies that the absolute minimum value of f must occur at the left endpoint a.

83. True. This follows from the Second Derivative Test applied to the function $P = R - C$.

84. False. Consider $f\,(x) = 1/x$ on the interval $(0, \infty)$.

85. Because $f\,(x) = c$ for all x, the function f satisfies $f\,(x) \le c$ for all x and so f has absolute maxima at all values of x. Similarly, f has absolute minima at all values of x.

86. Suppose f is a nonconstant polynomial function. Then $f\,(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_n \ne 0$ and $n \ge 1$.

First, let us suppose that $a_n > 0$. There are two cases to consider:

(1) If n is odd, then $\lim_{x\to-\infty} f\,(x) = -\infty$ and $\lim_{x\to\infty} f\,(x) = \infty$, and so f has no absolute extremum.

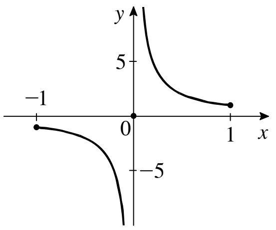
(2) If n is even, then $\lim_{x\to-\infty} f\,(x) = \lim_{x\to\infty} f\,(x) = \infty$, so f cannot have an absolute maximum.

A similar argument is used in the case where $a_n < 0$.

87. **a.** f is not continuous at $x = 0$ because $\lim_{x\to0} f\,(x)$ does not exist.

b. $\lim_{x\to0} f\,(x) = \lim_{x\to0} \frac{1}{x} = -\infty$ and $\lim_{x\to0} f\,(x) = \lim_{x\to0} \frac{1}{x} = \infty$.

c.



88. $f(x)$ can be made as close to -1 as we please by taking x sufficiently close to -1. But the value -1 is never attained, because x must be greater than -1. Similarly, 1 is never attained. Therefore, f has neither an absolute minimum value nor an absolute maximum value.

Using Technology page 320

1. Absolute maximum value 145.9, absolute minimum value -4.3834.

2. Absolute maximum value 26.3997, absolute minimum value -4.4372.

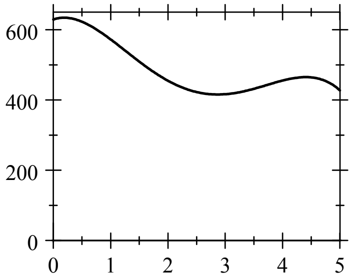
3. Absolute maximum value 16, absolute minimum value -0.1257.

4. Absolute maximum value 11.2016, absolute minimum value 9.

5. Absolute maximum value 2.8889, absolute minimum value 0.

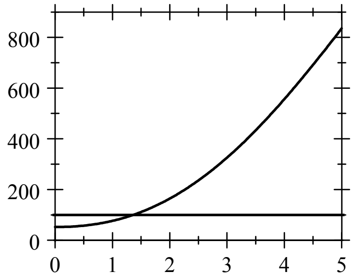
6. No absolute maximum or minimum value.

7. **a.**



b. Using the function for finding the absolute minimum of f on $[0, 5]$, we see that the absolute minimum value of f is approximately 415.56, occurring when $x \approx 2.87$. This proves the assertion.

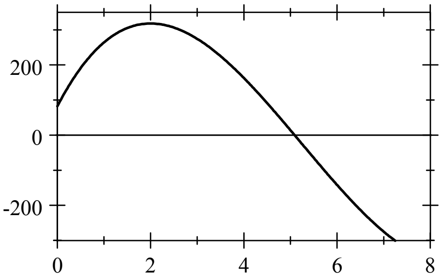
8. **a.** The graphs of $y_1 = g(t)$ and $y_2 = 100$ are shown below.



The graphs intersect at approximately (1.36, 100). This says that the construction loans of peer banks first exceeded the recommended maximum of 100% near the beginning of May 2004.

b. The maximum was approximately $g(5) \approx 836\%$.

9. a. $N(t) = 1.2576t^4 - 26.357t^3 + 127.98t^2 + 82.3t + 43$, so $N'(t) = 5.0304t^3 - 79.071t^2 + 255.96t + 82.3$.

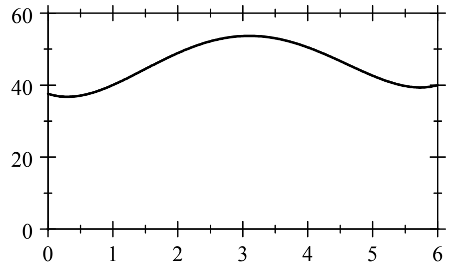


From the graph, we see that $N'(t)$ has a maximum when $t \approx 2$, on February 8.

b. From the graph in part (a), the maximum number of sickouts occurred when $N'(t) = 0$, that is, when $t = 5$. We calculate $N(5) \approx 1145$ canceled flights.

10. a. $A(t) = 0.28636t^4 - 3.4864t^3 + 11.689t^2 - 6.08t + 37.6$.

b.



c. The average account balance was lowest in the second year and highest early in the fourth year.

d. The lowest average balance was approximately \$37,000 (according to the model), and the highest was approximately \$55,500 (according to the figure for the third year).

4.5 Optimization II

Concept Questions page 327

1. We could solve the problem by sketching the graph of f and checking to see if there is an absolute extremum.

2. $S = 2\pi r^2 + 2\pi rh$. From $\pi r^2 h = 54$, we see that $r = \left(\frac{54}{\pi h}\right)^{1/2}$. Therefore,

$$S = 2\pi\left(\frac{54}{\pi h}\right) + 2\pi h\sqrt{\frac{54}{\pi}}\cdot\frac{1}{h^{1/2}} = \frac{108}{h} + 2\sqrt{54\pi}\,h^{1/2}, \text{ so } S' = -\frac{108}{h^2} + 2\sqrt{54\pi}\left(\frac{1}{2}h^{-1/2}\right) = -\frac{108}{h^2} + \frac{\sqrt{54\pi}}{h^{1/2}} = 0$$

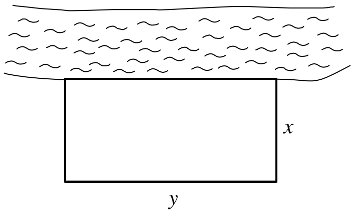
implies $\frac{108}{h^2} = \frac{\sqrt{54\pi}}{h^{1/2}}$, $h^{3/2} = \frac{108}{\sqrt{54\pi}}$. Thus, $h = \left[\frac{108}{(54\pi)^{1/2}}\right]^{2/3} = \frac{108^{2/3}}{(54\pi)^{1/3}} = \left(\frac{108}{54\pi}\right)^{1/3} = \frac{6}{\sqrt[3]{\pi}}$, as obtained in Example 4. Writing S in terms of r seems to be a better choice.

Exercises page 327

1. Let x and y denote the lengths of two adjacent sides of the rectangle. We want to maximize $A = xy$. But the perimeter is $2x + 2y$ and this is equal to 100, so $2x + 2y = 100$, and therefore $y = 50 - x$. Thus, $A = f(x) = x(50 - x) = -x^2 + 50x$, $0 \le x \le 50$. We allow the “degenerate”cases $x = 0$ and $x = 50$. $A' = -2x + 50 = 0$ implies that $x = 25$ is a critical number of f. $A(0) = 0$, $A(25) = 625$, and $A(50) = 0$, so we see that A is maximized for $x = 25$. The required dimensions are 25 ft by 25 ft.

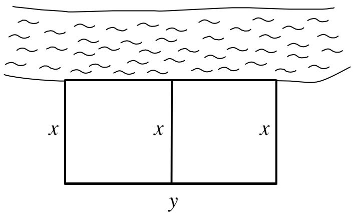
2. Let x and y denote the lengths of two adjacent sides of the rectangle. The quantity to be minimized is the perimeter of the rectangle, $P = 2x + 2y$. But $xy = 144$ and $y = \dfrac{144}{x}$, so $P = f(x) = 2x + \dfrac{288}{x}$ for $x > 0$, and $f'(x) = 2 - \dfrac{288}{x^2} = \dfrac{2x^2 - 288}{x^2}$, which has a zero when $2x^2 - 288 = 0$. The critical number in $(0, \infty)$ is $x = 12$. $f''(x) = \dfrac{576}{x^3} > 0$ on $(0, \infty)$, so f is concave upward. Therefore, $x = 12$ gives the absolute minimum value of f. The corresponding value of y is $y = \dfrac{144}{x} = \dfrac{144}{12} = 12$, so the dimensions are 12 ft by 12 ft.

3. We have $2x + y = 3000$ and we want to maximize the function $A = f(x) = xy = x(3000 - 2x) = 3000x - 2x^2$ on the interval $[0, 1500]$. The critical number of A is obtained by solving $f'(x) = 3000 - 4x = 0$, giving $x = 750$. From the table of values, we conclude that $x = 750$ yields the absolute maximum value of A. Thus, the required dimensions are 750 × 1500 yards. The maximum area is 1,125,000 yd^2.



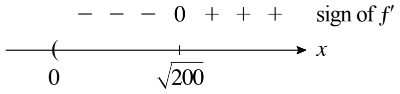
x	0	750	1500
$f(x)$	0	1,125,000	0

4. Let x denote the length of one of the sides. Then $y = 3000 - 3x = 3(1000 - x)$. The area is $A(x) = xy = 3x(1000 - x) = -3x^2 + 3000x$ for $0 \le x \le 1000$. Next, $A'(x) = -6x + 3000 = -6(x - 500)$. Setting $A'(x) = 0$ gives $x = 500$ as the critical number. From the table of values, we see that $f(500) = 750{,}000$ is the absolute maximum value. Next, $y = 3(1000 - 500) = 1500$. Therefore, the required dimensions are 500 × 1500 yd, and the area is 750,000 yd^2.



x	0	500	1000
$A(x)$	0	750,000	0

5. Let x denote the length of the side made of wood and y the length of the side made of steel. The cost of construction is $C = 6(2x) + 3y$, but $xy = 800$, so $y = \dfrac{800}{x}$. Therefore, $C = f(x) = 12x + 3\left(\dfrac{800}{x}\right) = 12x + \dfrac{2400}{x}$. To minimize C, we compute $f'(x) = 12 - \dfrac{2400}{x^2} = \dfrac{12x^2 - 2400}{x^2} = \dfrac{12\left(x^2 - 200\right)}{x^2}$. Setting $f'(x) = 0$ gives $x = \pm\sqrt{200}$ as critical numbers of f. The sign diagram of f' shows that $x = \pm\sqrt{200}$ gives a relative minimum of f. $f''(x) = \dfrac{4800}{x^3} > 0$ if $x > 0$, and so f is concave upward for $x > 0$. Therefore, $x = \sqrt{200} = 10\sqrt{2}$ yields the absolute minimum. Thus, the dimensions of the enclosure should be $10\sqrt{2}$ ft × $40\sqrt{2}$ ft, or 14.1 ft × 56.6 ft.



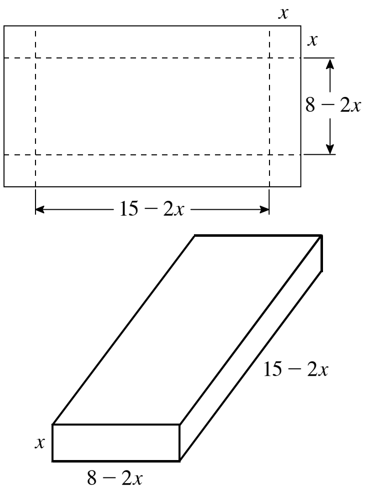
6. The volume of the box is given by

$V = f(x) = (8 - 2x)(15 - 2x)x = 4x^3 - 46x^2 + 120x$.

Because the sides of the box must be nonnegative, we must have $8 - 2x \geq 0$, so $x \leq 4$ and $15 - 2x \geq 0$, so $x \leq \frac{15}{2}$. The problem is equivalent to finding the absolute maximum of f on $[0, 4]$. Now

$$\begin{aligned} f'(x) &= 12x^2 - 92x + 120 \\ &= 4\left(3x^2 - 23x + 30\right) \\ &= 4(3x - 5)(x - 6), \end{aligned}$$

so $f'(x) = 0$ implies $x = \frac{5}{3}$ or $x = 6$. Because $x = 6$ is outside the interval $[0, 4]$, only $x = \frac{5}{3}$ qualifies as a critical number of f. From the table of values, we see that $x = \frac{5}{3}$ gives rise to an absolute maximum of f. Thus, the dimensions which yield the maximum volume are $\frac{14}{3}'' \times \frac{35}{3}'' \times \frac{5}{3}''$. The maximum volume is $\frac{2450}{27}$, or approximately 90.7 cubic inches.

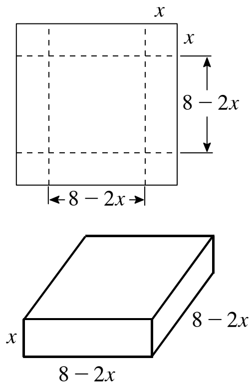


x	0	$\frac{5}{3}$	4
$f(x)$	0	$\frac{2450}{27}$	0

7. Let the dimensions of each square that is cut out be $x'' \times x''$. Then the dimensions of the box are $(8 - 2x)''$ by $(8 - 2x)''$ by x'', and its volume is be $V = f(x) = x(8 - 2x)^2$. We want to maximize f on $[0, 4]$.

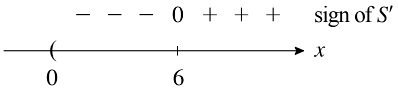
$$\begin{aligned} f'(x) &= (8 - 2x)^2 + x(2)(8 - 2x)(-2) \quad \text{(by the Product Rule)} \\ &= (8 - 2x)[(8 - 2x) - 4x] \\ &= (8 - 2x)(8 - 6x) \\ &= 0 \text{ if } x = 4 \text{ or } \tfrac{4}{3}. \end{aligned}$$

The latter is a critical number in $(0, 4)$. From the table, we see that $x = \frac{4}{3}$ yields an absolute maximum for f, so the dimensions of the box should be $\frac{16}{3}'' \times \frac{16}{3}'' \times \frac{4}{3}''$.



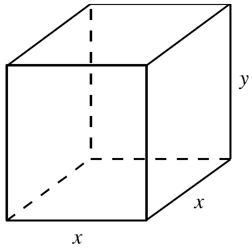
x	0	$\frac{4}{3}$	4
$f(x)$	0	$\frac{1024}{27}$	0

8. Let the dimensions of the box be $x'' \times x'' \times y''$. Because its volume is 108 cubic inches, we have $x^2 y = 108$. We want to minimize $S = x^2 + 4xy$. But $y = 108/x^2$, so we minimize $S = x^2 + 4x\left(\frac{108}{x^2}\right) = x^2 + \frac{432}{x}$ for $x > 0$. Now $S' = 2x - \frac{432}{x^2} = \frac{2\left(x^3 - 216\right)}{x^2}$. Setting $S' = 0$ gives $x = 6$ as a critical number of S. The sign diagram shows that $x = 6$ gives a relative minimum of S. Next, $S'' = 2 + \frac{864}{x^3} > 0$ for $x > 0$, and this says that S is concave upward on $(0, \infty)$. Therefore, $x = 6$ gives an absolute minimum, and so the dimensions of the box should be $6'' \times 6'' \times 3''$.



9. Let x denote the length of a side of the base and y the height of the cup, both measured in inches. Then the cost of constructing the cup is $C = 40x^2 + 15(4xy) = 40x^2 + 60xy$ (cents). The volume of the cup is 36 cubic inches, and so $x^2y = 36$ and $y = \frac{36}{x^2}$. Therefore, $C(x) = 40x^2 + 60x \cdot \frac{36}{x^2} = 40x^2 + \frac{2160}{x}$ and $C'(x) = 80x - \frac{2160}{x^2}$. Setting $C'(x) = 0$ gives $x^3 = 27$, so $x = 3$. Because $C''(3) = \left[80 + \frac{4320}{x^3}\right]_{x=3} > 0$, we see that $x = 3$ gives a relative (and absolute) minimum of C. Also, $y = \frac{36}{3^2} = 4$, so the required dimensions are $3'' \times 3'' \times 4''$.

10. Let x denote the length of the sides of the box and y denote its height. Referring to the figure, we see that the volume of the box is given by $x^2y = 128$. The amount of material used is given by $S = f(x) = 2x^2 + 4xy = 2x^2 + 4x\left(\frac{128}{x^2}\right) = 2x^2 + \frac{512}{x}$ in^2. We want to minimize f subject to the condition that $x > 0$.

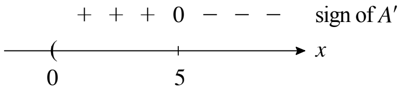


Now $f'(x) = 4x - \frac{512}{x^2} = \frac{4x^3 - 512}{x^2} = \frac{4\left(x^3 - 128\right)}{x^2}$.Setting $f'(x) = 0$ yields $x = 5.04$, a critical number of f. Next, $f''(x) = 4 + \frac{1024}{x^3} > 0$ for all $x > 0$. Thus, the graph of f is concave upward, and so $x = 5.04$ yields an absolute minimum of f. The required dimensions are $5.04'' \times 5.04'' \times 5.04''$.

11. From the given figure, we see that $x^2y = 20$ and $y = 20/x^2$, and so $C = 30x^2 + 10(4xy) + 20x^2 = 50x^2 + 40x\left(\frac{20}{x^2}\right) = 50x^2 + \frac{800}{x}$. To find the critical numbers of C, we solve $C' = 100x - \frac{800}{x^2} = 0$, obtaining $100x^3 = 800$, $x^3 = 8$, and $x = 2$. Next, $C'' = \frac{1600}{x^3} > 0$ for all $x > 0$, so we see that $x = 2$ gives the absolute minimum value of C. Because $y = \frac{20}{4} = 5$, we see that the dimensions are 2 ft × 2 ft × 5 ft.

12. The length plus the girth of the box is $4x + h = 108$ and $h = 108 - 4x$. Then $V = x^2h = x^2(108 - 4x) = 108x^2 - 4x^3$ and $V' = 216x - 12x^2$. We want to maximize V on the interval $[0, 27]$. Setting $V'(x) = 0$ and solving for x, we obtain $x = 18$ and $x = 0$. We calculate $V(0) = 0$, $V(18) = 11{,}664$, and $V(27) = 0$. Thus, the dimensions of the box are $18'' \times 18'' \times 36''$ and its maximum volume is approximately 11,664 in^3.

13. $xy = 50$ and so $y = 50/x$. The printed area is $A = (x-1)(y-2) = (x-1)\left(\frac{50}{x} - 2\right) = (x-1)\left(\frac{50-2x}{x}\right) = -2x + 52 - \frac{50}{x}$, so $A' = -2 + \frac{50}{x^2} = \frac{-2\left(x^2 - 25\right)}{x^2} = 0$ if $x = \pm 5$. From the sign diagram for A', we see that $x = 5$ yields a maximum. Because $A'' = -\frac{100}{x^3} < 0$ for $x > 0$, we see that the graph of A is concave downward on $(0, \infty)$ and so $x = 5$ yields an absolute maximum. The dimensions of the paper should therefore be $5'' \times 10''$.



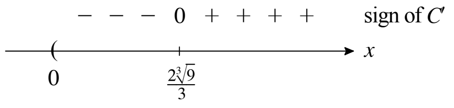
14. We take $2\pi r + \ell = 108$. We want to maximize $V = \pi r^2 \ell = \pi r^2 (-2\pi r + 108) = -2\pi^2 r^3 + 108\pi r^2$ subject to the condition that $0 \le r \le \frac{54}{\pi}$. Now $V'(r) = -6\pi^2 r^2 + 216\pi r = -6\pi r (\pi r - 36) = 0$ implies that $r = 0$ and $r = \frac{36}{\pi}$ are critical numbers of V. From the table, we conclude that the maximum volume occurs when $r = \frac{36}{\pi} \approx 11.5$ inches and $\ell = 108 - 2\pi \left(\frac{36}{\pi}\right) = 36$ inches and the volume of the parcel is $46{,}656/\pi$ in^3.

r	0	$\frac{36}{\pi}$	$\frac{54}{\pi}$
V	0	$\frac{46{,}656}{\pi}$	0

15. Denote the radius and height of the cup (in inches) by r and h respectively. Let k denote the price (in cents per square inch) of the material for the base of the cup. Then the cost of constructing the cup is $C = k\pi r^2 + \frac{3}{8}k(2\pi r h) = k\pi\left(r^2 + \frac{3}{4}rh\right)$. It suffices to minimize $F(r) = \dfrac{C(r)}{k\pi} = r^2 + \dfrac{3rh}{4}$. But $\pi r^2 h = 9\pi$, and so $h = \dfrac{9}{r^2}$. Thus, $F(r) = r^2 + \dfrac{3r}{4}\left(\dfrac{9}{r^2}\right) = r^2 + \dfrac{27}{4r}$. Now $F'(r) = 2r - \dfrac{27}{4r^2} = 0$ gives $8r^3 = 27$, so $r^3 = \frac{27}{8}$ and $r = \frac{3}{2}$. Because $F''(r) = 2 + \dfrac{27}{2r^2}$, we see that $F''\left(\frac{3}{2}\right) > 0$, and so F has a minimum at $r = \frac{3}{2}$. Also, $h = \frac{9}{(3/2)^2} = 4$, and so the required dimensions are a radius of 1.5 inches and a height of 4 inches.

16. Let r and h denote the radius and height of the container. Because its capacity is to be 36 in^3, we have $\pi r^2 h = 36$ or $h = 36/\pi r^2$. We want to minimize $S = 2\pi r^2 + 2\pi r h$ or $S = f(r) = 2\pi r^2 + 2\pi r\left(\dfrac{36}{\pi r^2}\right) = 2\pi r^2 + \dfrac{72}{r}$ over the interval $(0, \infty)$. Now $f'(r) = 4\pi r - \dfrac{72}{\pi r^2} = 0$ gives $4\pi r^3 = 72$, or $r = \left(\frac{18}{\pi}\right)^{1/3}$, as the only critical number of f. Next, observe that $f''(r) = 4\pi + \frac{144}{\pi r} > 0$ for r in $(0, \infty)$. Thus, f is concave upward on $(0, \infty)$ and $r = \left(\frac{18}{\pi}\right)^{1/3}$ gives rise to the absolute minimum of f. We find $h = \dfrac{36}{\pi\left(\frac{18}{\pi}\right)^{2/3}} = \dfrac{2 \cdot 18}{\pi^{1/3} 18^{2/3}} = 2\left(\dfrac{18}{\pi}\right)^{1/3}$ or twice the radius.

17. Let y denote the height and x the width of the cabinet. Then $y = \frac{3}{2}x$. Because the volume is to be 2.4 ft^3, we have $xyd = 2.4$, where d *is* the depth of the cabinet. Thus, $x\left(\frac{3}{2}x\right)d = 2.4$, so $d = \dfrac{2.4(2)}{3x^2} = \dfrac{1.6}{x^2}$. The cost for constructing the cabinet is $C = 40(2xd + 2yd) + 20(2xy) = 80\left[\dfrac{1.6}{x} + \left(\dfrac{3x}{2}\right)\left(\dfrac{1.6}{x^2}\right)\right] + 40x\left(\dfrac{3x}{2}\right) = \dfrac{320}{x} + 60x^2$, so $C'(x) = -\dfrac{320}{x^2} + 120x = \dfrac{120x^3 - 320}{x^2} = 0$ if $x = \sqrt[3]{\frac{8}{3}} = \frac{2}{\sqrt[3]{3}} = \frac{2}{3}\sqrt[3]{9}$. Therefore, $x = \frac{2}{3}\sqrt[3]{9}$ is a critical number of C. The sign diagram shows that $x = \frac{2}{3}\sqrt[3]{9}$ gives a relative minimum. Next, $C''(x) = \dfrac{640}{x^3} + 120 > 0$ for all $x > 0$, telling us that the graph of C is concave upward, so $x = \frac{2}{3}\sqrt[3]{9}$ yields an absolute minimum. The required dimensions are $\frac{2}{3}\sqrt[3]{9}' \times \sqrt[3]{9}' \times \frac{2}{5}\sqrt[3]{9}'$.



18. Because the perimeter of the window is 28 ft, we have $2x + 2y + \pi x = 28$ or $y = \frac{1}{2}(28 - \pi x - 2x)$. We want to maximize $A = 2xy + \frac{1}{2}\pi x^2 = \frac{1}{2}\pi x^2 + x(28 - \pi x - 2x) = \frac{1}{2}\pi x^2 + 28x - \pi x^2 - 2x^2 = 28x - \frac{\pi}{2}x^2 - 2x^2$. Now $A' = 28 - \pi x - 4x = 0$ gives $x = \frac{28}{4+\pi}$ as a critical number of A. Because $A'' = -\pi - 4 < 0$, the point yields a maximum of A. Finally, $y = \dfrac{1}{2}\left(28 - \dfrac{28\pi}{4+\pi} - \dfrac{56}{4+\pi}\right) = \dfrac{1}{2}\left(\dfrac{112 + 28\pi - 28\pi - 56}{4+\pi}\right) = \dfrac{28}{4+\pi}$.

19. Let x denote the number of passengers beyond the 200th. We want to maximize the function $R(x) = (200 + x)(300 - x) = -x^2 + 100x + 60{,}000$. Now $R'(x) = -2x + 100 = 0$ gives $x = 50$, and this is a critical number of R. Because $R''(x) = -2 < 0$, we see that $x = 50$ gives an absolute maximum of R. Therefore, the number of passengers should be 250. The fare will then be \$250/passenger and the revenue will be \$62,500.

20. Let x denote the number of trees beyond 22 per acre. Then the yield is $Y = (36 - 2x)(22 + x) = -2x^2 - 8x + 792$. Next, $Y' = -4x - 8 = 0$ gives $x = -2$ as the critical number of Y. Now $Y'' = -4 < 0$ and so $x = -2$ gives the absolute maximum of Y. So 20 trees/acre should be planted.

21. Let x denote the number of people beyond 20 who sign up for the cruise. Then the revenue is $R(x) = (20 + x)(600 - 4x) = -4x^2 + 520x + 12{,}000$. We want to maximize R on the closed bounded interval $[0, 70]$. $R'(x) = -8x + 520 = 0$ implies $x = 65$, a critical number of R. Evaluating R at this critical number and the endpoints, we see that R is maximized if $x = 65$. Therefore, 85 passengers will result in a maximum revenue of \$28,900. The fare in this case is \$340/passenger.

x	0	65	70
$R(x)$	12,000	28,900	28,800

22. Let x denote the number of bottles beyond 10,000. Then the profit is $P(x) = (10{,}000 + x)(5 - 0.0002x) = -0.0002x^2 + 3x + 50{,}000$. We want to maximize P on $[0, \infty)$. $P'(x) = -0.0004x + 3 = 0$ implies $x = 7500$. Because $P''(x) = -0.0004 < 0$, the graph of P is concave downward, and we see that $x = 7500$ gives the absolute maximum of P. So Phillip should produce 17,500 bottles of wine for a profit of $P(7500) = -0.0002(7500)^2 + 3(7500) + 50{,}000$ or \$61,250.

23. The fuel cost is $x/600$ dollars per mile and the labor cost is $18/x$ dollars per mile. Therefore, the total cost is $C(x) = \dfrac{18}{x} + \dfrac{3x}{600}$. We calculate $C'(x) = -\dfrac{18}{x^2} + \dfrac{3}{600} = 0$, giving $-\dfrac{18}{x^2} = -\dfrac{3}{600}$, $3x^2 = 18(600)$, $x^2 = 3600$, and so $x = 60$. Next, $C''(x) = \dfrac{48}{x^3} > 0$ for all $x > 0$ so C is concave upward. Therefore, $x = 60$ gives the absolute minimum. The most economical speed is 60 mph.

24. Suppose the distance between the two ports is D miles. Then it takes the ship D/v hours to travel from one port to the other. Therefore, the total cost incurred in making the trip is $C = (a + bv^3)\left(\dfrac{1}{v}\right) = \dfrac{a}{v} + bv^2$ dollars. We want to minimize C for $v > 0$. Setting $C' = 0$, we have $C' = -\dfrac{a}{v^2} + 2bv = \dfrac{-a + 2bv^3}{v^2} = 0$, $2bv^3 = a$, and so $v = \left(\dfrac{a}{2b}\right)^{1/3}$. Because $C'' = \dfrac{2a}{v^3} + 2b > 0$ for $v > 0$, the graph of C is concave upward and so the critical number $\sqrt[3]{\dfrac{a}{2b}}$ gives rise to the absolute minimum value of C. So the ship should sail at $\sqrt[3]{\dfrac{a}{2b}}$ mph.

25. We want to maximize $S = kh^2w$. But $h^2 + w^2 = 24^2$, or $h^2 = 576 - w^2$, so $S = f(w) = kw\left(576 - w^2\right) = k\left(576w - w^3\right)$. Now, setting $f'(w) = k\left(576 - 3w^2\right) = 0$ gives $w = \pm\sqrt{192} \approx \pm 13.86$. Only the positive root is a critical number of interest. Next, we find $f''(w) = -6kw$, and in particular, $f''\left(\sqrt{192}\right) = -6\sqrt{192}k < 0$, so that $w \approx 13.86$ gives a relative maximum of f. Because $f''(w) < 0$ for $w > 0$, we see that the graph of f is concave downward on $(0, \infty)$, and so $w = \sqrt{192}$ gives an absolute maximum of f. We find $h^2 = 576 - 192 = 384$ and so $h \approx 19.60$, so the width and height of the log should be approximately 13.86 inches and 19.60 inches, respectively.

26. We want to minimize $S = 3\pi r^2 + 2\pi rh$. But $\pi r^2 h + \frac{2}{3}\pi r^3 = 504\pi$, or $h = \dfrac{1}{r^2}\left(504 - \frac{2}{3}r^3\right)$. Therefore, $S = f(r) = 3\pi r^2 + 2\pi r \cdot \dfrac{1}{r^2}\left(504 - \frac{2}{3}r^3\right) = 3\pi r^2 + \dfrac{1008\pi}{r} - \dfrac{4\pi r^2}{3} = \dfrac{5\pi r^2}{3} + \dfrac{1008\pi}{r}$. Now $f'(r) = \dfrac{10\pi r}{3} - \dfrac{1008\pi}{r^2} = \dfrac{10\pi r^3 - 3024\pi}{3r^2}$, so $f'(r) = 0$ if $r^3 = \frac{3024\pi}{10\pi}$. Thus, $r = \left(\frac{1512}{5}\right)^{1/3} \approx 6.7$ is a critical number of f. Because $f''(r) = \dfrac{10\pi}{3} + \dfrac{2016\pi}{r^3} > 0$ for all r in $(0, \infty)$, we see that $r \approx 6.7$ does yield an absolute minimum of h. Therefore, the radius should be approximately 6.7 ft and the height should be approximately 6.7 ft.

27. We want to minimize $C(x) = 1.50\,(10{,}000 - x) + 2.50\sqrt{3000^2 + x^2}$ subject to $0 \le x \le 10{,}000$. Now

$$C'(x) = -1.50 + 2.5\left(\tfrac{1}{2}\right)\left(9{,}000{,}000 + x^2\right)^{-1/2}(2x) = -1.50 + \frac{2.50x}{\sqrt{9{,}000{,}000 + x^2}} = 0 \text{ if}$$

$2.5x = 1.50\sqrt{9{,}000{,}000 + x^2}$, or $6.25x^2 = 2.25\left(9{,}000{,}000 + x^2\right)$, or $4x^2 = 20{,}250{,}000$, giving $x = 2250$. From the table, we see that $x = 2250$, or 2250 ft, gives the absolute minimum.

x	0	2250	10,000
$C(x)$	22,500	21,000	26,101

28. We need to maximize $\hat{V} = \dfrac{16r^2}{\left(r + \frac{1}{2}\right)^2} - r^2$. Now

$$\hat{V}' = \frac{\left(r + \frac{1}{2}\right)^2(32r) - 16r^2 \cdot 2\left(r + \frac{1}{2}\right)}{\left(r + \frac{1}{2}\right)^4} - 2r = \frac{32r\left(r + \frac{1}{2}\right) - 32r^2\left(r + \frac{1}{2} - r\right)}{\left(r + \frac{1}{2}\right)^3} - 2r = \frac{16r - 2r\left(r + \frac{1}{2}\right)^3}{\left(r + \frac{1}{2}\right)^3}$$

$$= \frac{2r\left[8 - \left(r + \frac{1}{2}\right)^3\right]}{\left(r + \frac{1}{2}\right)^3}.$$

$\hat{V}' = 0$ implies $8 - \left(r + \frac{1}{2}\right)^3 = 0$, $\left(r + \frac{1}{2}\right)^3 = 8$, $r + \frac{1}{2} = 2$, and $r = \frac{3}{2}$. Next,

$$\hat{V}\left(\tfrac{3}{2}\right) = \frac{16\left(\frac{3}{2}\right)^2}{2^2} - \left(\tfrac{3}{2}\right)^2 = \left(\tfrac{3}{2}\right)^2(4 - 1) = 3\left(\tfrac{9}{4}\right) = \tfrac{27}{4}, \text{ so } h = \frac{16}{\left(r + \frac{1}{2}\right)^2} - 1 = \tfrac{16}{4} - 1 = 3.$$

Thus, the dimensions are $r = \frac{3}{2}$ and $h = 3$. From the table, we see that $\hat{V}$ is maximized if $r = \frac{3}{2}$, so the radius is 1.5 ft and the height is 3 ft.

r	0	$\frac{3}{2}$	$\frac{7}{2}$
V	0	$\frac{27}{4}$	0

29. The time of flight is $T = f(x) = \dfrac{12-x}{6} + \dfrac{\sqrt{x^2+9}}{4}$, so $f'(x) = -\dfrac{1}{6} + \dfrac{1}{4}\left(\dfrac{1}{2}\right)\left(x^2+9\right)^{-1/2}(2x) = -\dfrac{1}{6} + \dfrac{x}{4\sqrt{x^2+9}} = \dfrac{3x-2\sqrt{x^2+9}}{12\sqrt{x^2+9}}$. Setting $f'(x) = 0$ gives $3x = 2\sqrt{x^2+9}$, $9x^2 = 4\left(x^2+9\right)$, and $5x^2 = 36$. Therefore, $x = \pm\frac{6}{\sqrt{5}} = \pm\frac{6\sqrt{5}}{5}$. Only the critical number $x = \frac{6\sqrt{5}}{5}$ is of interest. The nature of the problem suggests $x \approx 2.68$ gives an absolute minimum for T.

30. The time taken to get to Q is $T(x) = \dfrac{\sqrt{x^2+1}}{3} + \dfrac{10-x}{4}$ for $0 \le x \le 10$. Next, $T'(x) = \dfrac{1}{3}\dfrac{d}{dx}\left(x^2+1\right)^{1/2} + \dfrac{1}{4}\dfrac{d}{dx}(10-x) = \dfrac{1}{3}\left(\dfrac{1}{2}\right)\left(x^2+1\right)^{-1/2}(2x) - \dfrac{1}{4} = \dfrac{x}{3\sqrt{x^2+1}} - \dfrac{1}{4}$. Setting $T'(x) = 0$ gives $\dfrac{x}{3\sqrt{x^2+1}} = \dfrac{1}{4}$, $4x = 3\sqrt{x^2+1}$, $16x^2 = 9\left(x^2+1\right)$, $7x^2 = 9$, and $x = \frac{3\sqrt{7}}{7}$, since x must be positive. $T(0) = \frac{17}{6} \approx 2.83$, $T\left(\frac{3\sqrt{7}}{7}\right) \approx 2.72$, and $T(10) = \frac{\sqrt{101}}{3} \approx 3.35$. We see that $x = \frac{3\sqrt{7}}{7} \approx 1.134$ yields the absolute minimum value for T, so she should land at the point R located about 1.134 miles from P.

31. The area enclosed by the rectangular region of the racetrack is $A = (\ell)(2r) = 2r\ell$. The length of the racetrack is $2\pi r + 2\ell$, and is equal to 1760. That is, $2(\pi r + \ell) = 1760$, and $\pi r + \ell = 880$. Therefore, we want to maximize $A = f(r) = 2r(880 - \pi r) = 1760r - 2\pi r^2$. The restriction on r is $0 \le r \le \frac{880}{\pi}$. To maximize A, we compute $f'(r) = 1760 - 4\pi r$. Setting $f'(r) = 0$ gives $r = \frac{1760}{4\pi} = \frac{440}{\pi} \approx 140$. Because $f(0) = f\left(\frac{880}{\pi}\right) = 0$, we see that the maximum rectangular area is enclosed if we take $r = \frac{440}{\pi}$ and $\ell = 880 - \pi\left(\frac{440}{\pi}\right) = 440$. So $r = 140$ and $\ell = 440$. The total area enclosed is $2r\ell + \pi r^2 = 2\left(\frac{440}{\pi}\right)(440) + \pi\left(\frac{440}{\pi}\right)^2 = \frac{2(440)^2}{\pi} + \frac{440^2}{\pi} = \frac{580{,}800}{\pi} \approx 184{,}874\ \text{ft}^2$.

32. Let x denote the number of motorcycle tires in each order. We want to minimize $C(x) = 400\left(\dfrac{40{,}000}{x}\right) + x = \dfrac{16{,}000{,}000}{x} + x$. We compute $C'(x) = -\dfrac{16{,}000{,}000}{x^2} + 1 = \dfrac{x^2 - 16{,}000{,}000}{x^2}$. Setting $C'(x) = 0$ gives $x = 4000$, a critical number of C. Because $C''(x) = \dfrac{32{,}000{,}000}{x^3} > 0$ for all $x > 0$, we see that the graph of C is concave upward and so $x = 4000$ gives an absolute minimum of C. So there should be 10 orders per year, each for 4000 tires.

33. Let x denote the number of bottles in each order. We want to minimize $C(x) = 200\left(\dfrac{2{,}000{,}000}{x}\right) + \dfrac{x}{2}(0.40) = \dfrac{400{,}000{,}000}{x} + 0.2x$. We compute $C'(x) = -\dfrac{400{,}000{,}000}{x^2} + 0.2$. Setting $C'(x) = 0$ gives $x^2 = \dfrac{400{,}000{,}000}{0.2} = 2{,}000{,}000{,}000$, or $x = 44{,}721$, a critical number of C. $C''(x) = \dfrac{800{,}000{,}000}{x^3} > 0$ for all $x > 0$, and we see that the graph of C is concave upward and so $x = 44{,}721$ gives an absolute minimum of C. Therefore, there should be $2{,}000{,}000/x \approx 45$ orders per year (since we can not have fractions of an order.) Each order should be for $2{,}000{,}000/45 \approx 44{,}445$ bottles.

34. We want to minimize the function $C(x) = \dfrac{500{,}000{,}000}{x} + 0.2x + 500{,}000$ on the interval $(0, 1000000)$. Differentiating $C(x)$, we have $C'(x) = -\dfrac{500{,}000{,}000}{x^2} + 0.2$. Setting $C'(x) = 0$ and solving the resulting equation, we find $0.2x^2 = 500{,}000{,}000$ and $x = \sqrt{2{,}500{,}000{,}000}$ or $x = 50{,}000$. Next, we find $C''(x) = \dfrac{1{,}000{,}000{,}000}{x^3} > 0$ for all x, and so the graph of C is concave upward on $(0, \infty)$. Thus, $x = 50{,}000$ gives rise to the absolute minimum of C. The company should produce 50,000 containers of cookies per production run.

35. a. Because the sales are assumed to be steady and D units are expected to be sold per year, the number of orders per year is D/x. Because is costs $\$K$ per order, the ordering cost is KD/x. The purchasing cost is pD (cost per item times number purchased). Finally, the holding cost is $\frac{1}{2}xh$ (the average number on hand times holding cost per item). Therefore, $C(x) = \dfrac{KD}{x} + pD + \dfrac{hx}{2}$.

b. $C'(x) = -\dfrac{KD}{x^2} + \dfrac{h}{2} = 0$ implies $\dfrac{KD}{x^2} = \dfrac{h}{2}$, so $x^2 = \dfrac{2KD}{h}$ and $x = \pm\sqrt{\dfrac{2KD}{h}}$. We reject the negative root. So $x = \sqrt{\dfrac{2KD}{h}}$ is the only critical number. Next, $C''(x) = \dfrac{2KD}{x^3} > 0$ for $x > 0$, so $C''\left(\sqrt{\dfrac{2KD}{h}}\right) > 0$ and the Second Derivative Test shows that $x = \sqrt{\dfrac{2KD}{h}}$ does give a relative minimum. Because C is concave upward, this is also the absolute minimum.

36. a. We use the result of Exercise 35 with $D = 960$, $K = 10$, $p = 80$, and $h = 12$. Thus, the EOQ is $x = \sqrt{\dfrac{2KD}{h}} = \sqrt{\dfrac{2(10)(960)}{12}} = 40$.

b. The number of orders to be placed each year is $\frac{960}{40} = 24$.

c. The interval between orders is $\frac{12}{24} = \frac{1}{2}$, or one-half month.

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1. a. $f(x_1) < f(x_2)$ **b.** $f(x_1) > f(x_2)$

2. a. increasing **b.** $f'(x) < 0$ **c.** constant

3. a. $f(x) \leq f(c)$ **b.** $f(x) \geq f(c)$

4. a. domain, $= 0$, exist **b.** critical number **c.** relative extremum

5. a. $f'(x)$ **b.** > 0 **c.** concavity **d.** relative maximum; relative extremum

6. $\pm\infty, \pm\infty$ **7.** 0, 0 **8.** b, b

9. a. $f(x) \leq f(c)$, absolute maximum value **b.** $f(x) \geq f(c)$, open interval

10. continuous, absolute, absolute

CHAPTER 4 Review Exercises page 333

1. a. $f(x) = \frac{1}{3}x^3 - x^2 + x - 6$, so $f'(x) = x^2 - 2x + 1 = (x-1)^2$. $f'(x) = 0$ gives $x = 1$, the critical number of f. Now $f'(x) > 0$ for all $x \neq 1$. Thus, f is increasing on $(-\infty, \infty)$.

b. Because $f'(x)$ does not change sign as we move across the critical number $x = 1$, the First Derivative Test implies that $x = 1$ does not give a relative extremum of f.

c. $f''(x) = 2(x-1)$. Setting $f''(x) = 0$ gives $x = 1$ as a candidate for an inflection point of f. Because $f''(x) < 0$ for $x < 1$, and $f''(x) > 0$ for $x > 1$, we see that f is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.

d. The results of part (c) imply that $\left(1, -\frac{17}{3}\right)$ is an inflection point.

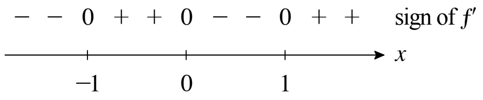
2. a. $f(x) = (x-2)^3$, so $f'(x) = 3(x-2)^2 > 0$ for all $x \neq 2$. Therefore, f is increasing on $(-\infty, \infty)$.

b. There is no relative extremum.

c. $f''(x) = 6(x-2)$. Because $f''(x) < 0$ if $x < 2$ and $f''(x) > 0$ if $x > 2$, we see that f is concave downward on $(-\infty, 2)$ and concave upward on $(2, \infty)$.

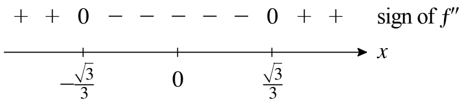
d. The results of part (c) show that $(2, 0)$ is an inflection point.

3. a. $f(x) = x^4 - 2x^2$, so $f'(x) = 4x^3 - 4x = 4x\left(x^2 - 1\right) = 4x(x+1)(x-1)$. The sign diagram of f' shows that f is decreasing on $(-\infty, -1)$ and $(0, 1)$ and increasing on $(-1, 0)$ and $(1, \infty)$.



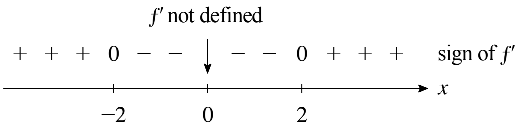
b. The results of part (a) and the First Derivative Test show that $(-1, -1)$ and $(1, -1)$ are relative minima and $(0, 0)$ is a relative maximum.

c. $f''(x) = 12x^2 - 4 = 4\left(3x^2 - 1\right) = 0$ if $x = \pm\frac{\sqrt{3}}{3}$. The sign diagram shows that f is concave upward on $\left(-\infty, -\frac{\sqrt{3}}{3}\right)$ and $\left(\frac{\sqrt{3}}{3}, \infty\right)$ and concave downward on $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.



d. The results of part (c) show that $\left(-\frac{\sqrt{3}}{3}, -\frac{5}{9}\right)$ and $\left(\frac{\sqrt{3}}{3}, -\frac{5}{9}\right)$ are inflection points.

4. a. $f(x) = x + \dfrac{4}{x}$, so $f'(x) = 1 - \dfrac{4}{x^2} = \dfrac{x^2 - 4}{x^2} = \dfrac{(x-2)(x+2)}{x^2}$. Setting $f'(x) = 0$ gives $x = -2$ and $x = 2$ as critical numbers of f. $f'(x)$ is undefined at $x = 0$ as well. The sign diagram for f' shows that f is increasing on $(-\infty, -2)$ and $(2, \infty)$ and decreasing on $(-2, 0)$ and $(0, 2)$.



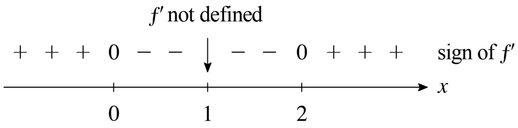
b. $f(-2) = -4$ is a relative maximum and $f(2) = 4$ is a relative minimum.

c. $f''(x) = \dfrac{8}{x^3}$. Because $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, we see that f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$.

d. There is no inflection point. Note that $x = 0$ is not in the domain of f and is therefore not a candidate for an inflection point.

5. a. $f(x) = \dfrac{x^2}{x-1}$, so $f'(x) = \dfrac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \dfrac{x^2-2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$.

The sign diagram of f' shows that f is increasing on $(-\infty, 0)$ and $(2, \infty)$ and decreasing on $(0, 1)$ and $(1, 2)$.



b. The results of part (a) show that $(0, 0)$ is a relative maximum and $(2, 4)$ is a relative minimum.

c. $f''(x) = \dfrac{(x-1)^2(2x-2) - x(x-2)2(x-1)}{(x-1)^4} = \dfrac{2(x-1)\left[(x-1)^2 - x(x-2)\right]}{(x-1)^4} = \dfrac{2}{(x-1)^3}$. Because $f''(x) < 0$ if $x < 1$ and $f''(x) > 0$ if $x > 1$, we see that f is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.

d. Because $x = 1$ is not in the domain of f, there is no inflection point.

6. a. $f(x) = \sqrt{x-1}$, so $f'(x) = \frac{1}{2}(x-1)^{-1/2} = \dfrac{1}{2\sqrt{x-1}}$. Because $f'(x) > 0$ for $x > 1$, we see that f is increasing on $(1, \infty)$.

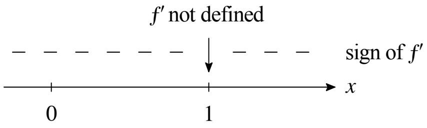
b. Because there is no critical numbers in $(1, \infty)$, f has no relative extremum.

c. $f''(x) = -\frac{1}{4}(x-1)^{-3/2} = -\dfrac{1}{4(x-1)^{3/2}} < 0$ if $x > 1$, and so f is concave downward on $(1, \infty)$.

d. There are no inflection points because $f''(x) \neq 0$ for all x in $(1, \infty)$.

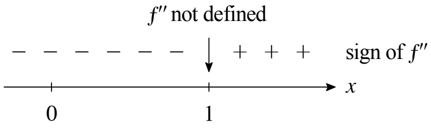
7. a. $f(x) = (1-x)^{1/3}$, so

$f'(x) = -\frac{1}{3}(1-x)^{-2/3} = -\dfrac{1}{3(1-x)^{2/3}}$. The sign diagram for f' shows that f is decreasing on $(-\infty, \infty)$.



b. There is no relative extremum.

c. $f''(x) = -\frac{2}{9}(1-x)^{-5/3} = -\dfrac{2}{9(1-x)^{5/3}}$. The sign diagram for f'' shows that f is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.



d. $x = 1$ is a candidate for an inflection point of f. Referring to the sign diagram for f'', we see that $(1, 0)$ is an inflection point.

8. a. $f(x) = x\sqrt{x-1} = x(x-1)^{1/2}$, so

$f'(x) = x\left(\frac{1}{2}\right)(x-1)^{-1/2} + (x-1)^{1/2} = \frac{1}{2}(x-1)^{-1/2}[x + 2(x-1)] = \dfrac{3x-2}{2(x-1)^{1/2}}$. Setting $f'(x) = 0$ gives $x = \frac{2}{3}$. But this point lies outside the domain of f, which is $[1, \infty)$. Thus, f has no critical number. Now, $f'(x) > 0$ for all $x \in (1, \infty)$ so f is increasing on $(1, \infty)$.

b. Because there is no critical numbers, f has no relative extremum.

c. $f''(x) = \frac{1}{2}\left[\frac{(x-1)^{1/2}(3) - (3x-2)\frac{1}{2}(x-1)^{-1/2}}{x-1}\right] = \frac{1}{2}\left[\frac{\frac{1}{2}(x-1)^{-1/2}[6(x-1)-(3x-2)]}{x-1}\right]$
$= \frac{3x-4}{4(x-1)^{3/2}}$.

$f''(x) = 0$ implies that $x = \frac{4}{3}$. $f''(x) < 0$ if $x < \frac{4}{3}$ and $f''(x) > 0$ if $x > \frac{4}{3}$, so f is concave downward on $\left(1, \frac{4}{3}\right)$ and concave upward on $\left(\frac{4}{3}, \infty\right)$.

d. From the results of part (c), we conclude that $\left(\frac{4}{3}, \frac{4\sqrt{3}}{9}\right)$ is an inflection point of f.

9. a. $f(x) = \frac{2x}{x+1}$, so $f'(x) = \frac{(x+1)(2) - 2x(1)}{(x+1)^2} = \frac{2}{(x+1)^2} > 0$ if $x \neq -1$. Therefore f is increasing on $(-\infty, -1)$ and $(-1, \infty)$.

b. Because there is no critical number, f has no relative extremum.

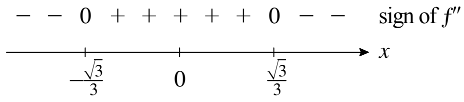
c. $f''(x) = -4(x+1)^{-3} = -\frac{4}{(x+1)^3}$. Because $f''(x) > 0$ if $x < -1$ and $f''(x) < 0$ if $x > -1$, we see that f is concave upward on $(-\infty, -1)$ and concave downward on $(-1, \infty)$.

d. There is no inflection point because $f''(x) \neq 0$ for all x in the domain of f.

10. a. $f(x) = -\frac{1}{1+x^2}$, so $f'(x) = \frac{2x}{\left(1+x^2\right)^2}$. Setting $f'(x) = 0$ gives $x = 0$ as the only critical number of f. For $x < 0$, $f'(x) < 0$ and for $x > 0$, $f'(x) > 0$. Therefore, f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

b. f has a relative minimum at $f(0) = -1$.

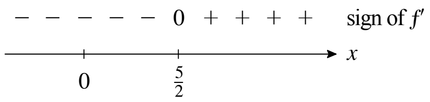
c. $f''(x) = \frac{\left(1+x^2\right)^2(2) - 2x(2)\left(1+x^2\right)(2x)}{\left(1+x^2\right)^4} = \frac{2\left(1+x^2\right)\left(1+x^2-4x^2\right)}{\left(1+x^2\right)^4} = -\frac{2\left(3x^2-1\right)}{\left(1+x^2\right)^3}$, and we see that $x = \pm\frac{\sqrt{3}}{3}$ are candidates for inflection points of f. The sign diagram for f'' shows that f is concave downward on $\left(-\infty, -\frac{\sqrt{3}}{3}\right)$ and $\left(\frac{\sqrt{3}}{3}, \infty\right)$ and concave upward on $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.

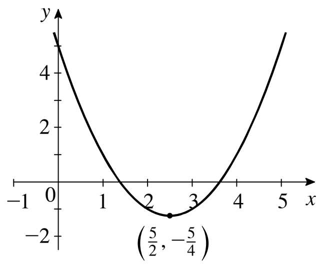


d. $\left(-\frac{\sqrt{3}}{3}, -\frac{3}{4}\right)$ and $\left(\frac{\sqrt{3}}{3}, -\frac{3}{4}\right)$ are inflection points of f.

11. $f(x) = x^2 - 5x + 5$.

1. The domain of f is $(-\infty, \infty)$.
2. Setting $x = 0$ gives 5 as the y-intercept.
3. $\lim_{x\to-\infty}\left(x^2 - 5x + 5\right) = \lim_{x\to\infty}\left(x^2 - 5x + 5\right) = \infty$.
4. There is no asymptote because f is a quadratic function.
5. $f'(x) = 2x - 5 = 0$ if $x = \frac{5}{2}$. The sign diagram shows that f is increasing on $\left(\frac{5}{2}, \infty\right)$ and decreasing on $\left(-\infty, \frac{5}{2}\right)$.

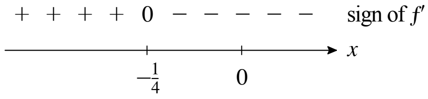




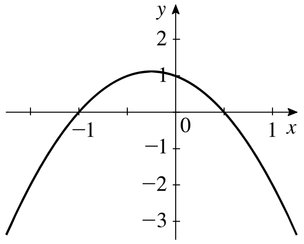
6. The First Derivative Test implies that $\left(\frac{5}{2}, -\frac{5}{4}\right)$ is a relative minimum.
7. $f''(x) = 2 > 0$ and so f is concave upward on $(-\infty, \infty)$.
8. There is no inflection point.

12. $f(x) = -2x^2 - x + 1$.

1. The domain of f is $(-\infty, \infty)$.
2. Setting $x = 0$ gives 1 as the y-intercept.
3. $\lim_{x\to-\infty} \left(-2x^2 - x + 1\right) = \lim_{x\to\infty} \left(-2x^2 - x + 1\right) = -\infty$.
4. There is no asymptote because f is a polynomial function.
5. $f'(x) = -4x - 1 = 0$ if $x = -\frac{1}{4}$. The sign diagram of f' shows that f is increasing on $\left(-\infty, -\frac{1}{4}\right)$ and decreasing on $\left(-\frac{1}{4}, \infty\right)$.

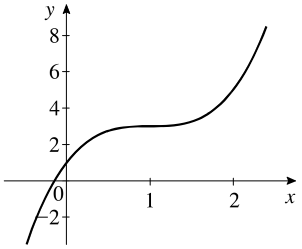


6. The results of step 5 show that $\left(-\frac{1}{4}, \frac{9}{8}\right)$ is a relative maximum.
7. $f''(x) = -4 < 0$ for all x in $(-\infty, \infty)$, and so f is concave downward on $(-\infty, \infty)$.
8. There is no inflection point.



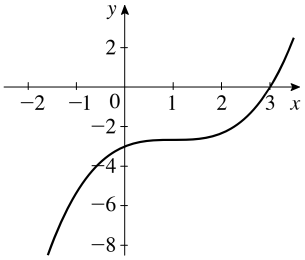
13. $g(x) = 2x^3 - 6x^2 + 6x + 1$.

1. The domain of g is $(-\infty, \infty)$.
2. Setting $x = 0$ gives 1 as the y-intercept.
3. $\lim_{x\to-\infty} g(x) = -\infty$ and $\lim_{x\to\infty} g(x) = \infty$.
4. There is no vertical or horizontal asymptote.
5. $g'(x) = 6x^2 - 12x + 6 = 6\left(x^2 - 2x + 1\right) = 6(x-1)^2$. Because $g'(x) > 0$ for all $x \neq 1$, we see that g is increasing on $(-\infty, 1)$ and $(1, \infty)$.
6. $g'(x)$ does not change sign as we move across the critical number $x = 1$, so there is no extremum.
7. $g''(x) = 12x - 12 = 12(x-1)$. Because $g''(x) < 0$ if $x < 1$ and $g''(x) > 0$ if $x > 1$, we see that g is concave upward on $(1, \infty)$ and concave downward on $(-\infty, 1)$.
8. The point $x = 1$ gives rise to the inflection point $(1, 3)$.



14. $g(x) = \frac{1}{3}x^3 - x^2 + x - 3$

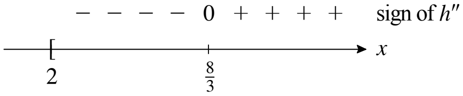
1. The domain of g is $(-\infty, \infty)$.
2. Setting $x = 0$ gives -3 as the y-intercept.
3. $\lim\limits_{x\to-\infty} \left(\frac{1}{3}x^3 - x^2 + x - 3\right) = -\infty$ and $\lim\limits_{x\to\infty} \left(\frac{1}{3}x^3 - x^2 + x - 3\right) = \infty$.
4. There is no asymptote because $g(x)$ is a polynomial.
5. $g'(x) = x^2 - 2x + 1 = (x-1)^2 = 0$ if $x = 1$, a critical number of g. Observe that $g'(x) > 0$ if $x \neq 1$, and so g is increasing on $(-\infty, 1)$ and $(1, \infty)$.
6. The results of step 5 show that there is no relative extremum.
7. $g'(x) = 2x - 2 = 2(x-1) = 0$ if $x = 1$. Observe that $g'(x) < 0$ if $x < 1$ and $g''(x) > 0$ if $x > 1$ and so g is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.
8. The results of step 7 show that $\left(1, -\frac{8}{3}\right)$ is an inflection point.



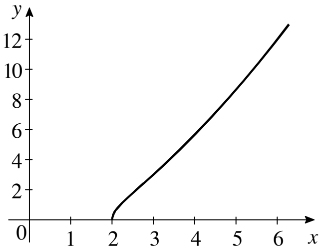
15. $h(x) = x\sqrt{x-2}$.

1. The domain of h is $[2, \infty)$.
2. There is no y-intercept. Setting $y = 0$ gives 2 as the x-intercept.
3. $\lim\limits_{x\to\infty} x\sqrt{x-2} = \infty$.
4. There is no asymptote.
5. $h'(x) = (x-2)^{1/2} + x\left(\frac{1}{2}\right)(x-2)^{-1/2} = \frac{1}{2}(x-2)^{-1/2}[2(x-2) + x] = \dfrac{3x-4}{2\sqrt{x-2}} > 0$ on $[2, \infty)$, and so h is increasing on $[2, \infty)$.
6. Because h has no critical number in $(2, \infty)$, there is no relative extremum.
7. $h''(x) = \dfrac{1}{2}\left[\dfrac{(x-2)^{1/2}(3) - (3x-4)\frac{1}{2}(x-2)^{-1/2}}{x-2}\right] = \dfrac{(x-2)^{-1/2}[6(x-2) - (3x-4)]}{4(x-2)} = \dfrac{3x-8}{4(x-2)^{3/2}}$.

   The sign diagram for h'' shows that h is concave downward on $\left(2, \frac{8}{3}\right)$ and concave upward on $\left(\frac{8}{3}, \infty\right)$.

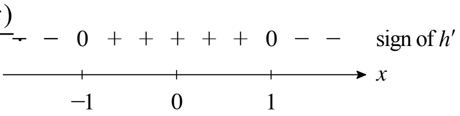


8. The results of step 7 tell us that $\left(\frac{8}{3}, \frac{8\sqrt{6}}{9}\right)$ is an inflection point.



16. $h(x) = \dfrac{2x}{1+x^2}$.

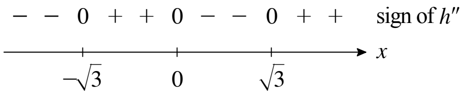
1. The domain of h is $(-\infty, \infty)$.
2. Setting $x = 0$ gives 0 as the y-intercept.
3. $\lim\limits_{x\to-\infty} \dfrac{2x}{1+x^2} = \lim\limits_{x\to\infty} \dfrac{2x}{1+x^2} = 0$.
4. The results of step 3 tell us that $y = 0$ is a horizontal asymptote.
5. $h'(x) = \dfrac{(1+x^2)(2) - 2x(2x)}{(1+x^2)^2} = \dfrac{2(1-x^2)}{(1+x^2)^2} = \dfrac{2(1-x)(1+x)}{(1+x^2)^2}$



The sign diagram of h' shows us that h is decreasing on $(-\infty, -1)$ and $(1, \infty)$ and increasing on $(-1, 1)$.

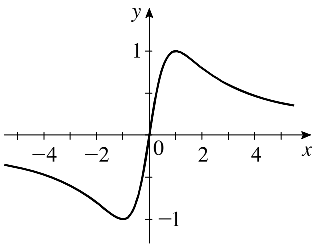
6. The results of step 6 show that $(-1, -1)$ is a relative minimum and $(1, 1)$ is a relative maximum.
7. $h''(x) = 2\left[\dfrac{(1+x^2)^2(-2x) - (1-x^2)2(1+x^2)(2x)}{(1+x^2)^4}\right] = \dfrac{4x(1+x^2)\left[-(1+x^2) - 2(1-x^2)\right]}{(1+x^2)^4}$

$$= \frac{4x(x^2-3)}{(1+x^2)^3}.$$



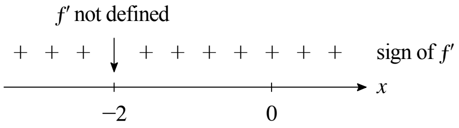
The sign diagram of h'' shows that h is concave downward on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$ and concave upward on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$.

8. The results of step 6 also tell us that $\left(-\sqrt{3}, -\frac{\sqrt{3}}{2}\right)$ and $\left(\sqrt{3}, \frac{\sqrt{3}}{2}\right)$ are inflection points.



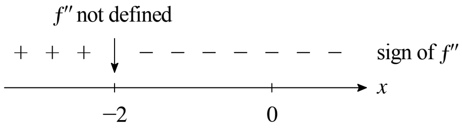
17. $f(x) = \dfrac{x-2}{x+2}$.

1. The domain of f is $(-\infty, -2) \cup (-2, \infty)$.
2. Setting $x = 0$ gives -1 as the y-intercept. Setting $y = 0$ gives 2 as the x-intercept.
3. $\lim\limits_{x\to-\infty} \dfrac{x-2}{x+2} = \lim\limits_{x\to\infty} \dfrac{x-2}{x+2} = 1$.
4. The results of step 3 tell us that $y = 1$ is a horizontal asymptote. Next, observe that the denominator of $f(x)$ is equal to zero at $x = -2$, but its numerator is not equal to zero there. Therefore, $x = -2$ is a vertical asymptote.
5. $f'(x) = \dfrac{(x+2)(1) - (x-2)(1)}{(x+2)^2} = \dfrac{4}{(x+2)^2}$. The sign diagram of f' tells us that f is increasing on $(-\infty, -2)$ and $(-2, \infty)$.

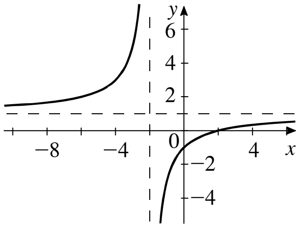


6. The results of step 5 tell us that there is no relative extremum.

7. $f''(x) = -\frac{8}{(x+2)^3}$. The sign diagram of f'' shows that f is concave upward on $(-\infty, -2)$ and concave downward on $(-2, \infty)$.



8. There is no inflection point.



18. $f(x) = x - \frac{1}{x}$.

1. The domain of f is $(-\infty, 0) \cup (0, \infty)$.

2. There is no y-intercept. Setting $y = 0$ gives $\frac{x^2 - 1}{x} = \frac{(x+1)(x-1)}{x} = 0$, and so the x-intercepts are -1 and 1.

3. $\lim_{x \to -\infty} \left(x - \frac{1}{x} \right) = -\infty$ and $\lim_{x \to \infty} \left(x - \frac{1}{x} \right) = \infty$.

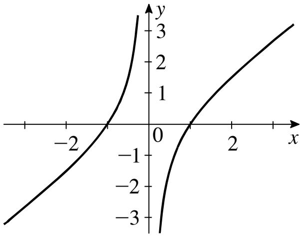
4. There is no horizontal asymptote. From $f(x) = \frac{x^2 - 1}{x}$, we see that the denominator of $f(x)$ is equal to zero at $x = 0$. Because the numerator is not equal to zero there, we conclude that $x = 0$ is a vertical asymptote.

5. $f'(x) = 1 + \frac{1}{x^2} = \frac{x^2 + 1}{x^2} > 0$ for all $x \neq 0$. Therefore, f is increasing on $(-\infty, 0)$ and $(0, \infty)$.

6. The results of step 5 show that f has no relative extremum.

7. $f''(x) = -\frac{2}{x^3}$. Observe that $f''(x) > 0$ if $x < 0$ and $f''(x) < 0$ if $x > 0$. Therefore, f is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$.

8. There is no inflection point.



19. $\lim_{x \to -\infty} \frac{1}{2x + 3} = \lim_{x \to \infty} \frac{1}{2x + 3} = 0$ and so $y = 0$ is a horizontal asymptote. Because the denominator is equal to zero at $x = -\frac{3}{2}$ but the numerator is not equal to zero there, we see that $x = -\frac{3}{2}$ is a vertical asymptote.

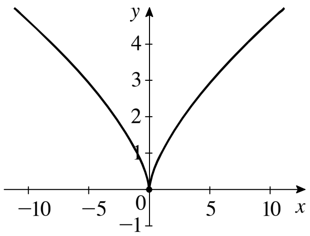
20. $\lim_{x \to -\infty} \frac{2x}{x + 1} = \lim_{x \to \infty} \frac{2x}{x + 1} = 2$ and so $y = 2$ is a horizontal asymptote. Because the denominator is equal to zero at $x = -1$, but the numerator is not equal to zero there, we see that $x = -1$ is a vertical asymptote.

21. $\lim_{x \to -\infty} \frac{5x}{x^2 - 2x - 8} = \lim_{x \to \infty} \frac{5x}{x^2 - 2x - 8} = 0$, so $y = 0$ is a horizontal asymptote. Next, note that the denominator is zero if $x^2 - 2x - 8 = (x - 4)(x + 2) = 0$, that is, if $x = -2$ or $x = 4$. Because the numerator is not equal to zero at these points, we see that $x = -2$ and $x = 4$ are vertical asymptotes.

22. $\lim\limits_{x\to-\infty}\dfrac{x^2+x}{x^2-x}=\lim\limits_{x\to\infty}\dfrac{x^2+x}{x^2-x}=1$, we see that $y=1$ is a horizontal asymptote. Next, observe that the denominator is equal to zero at $x=0$ or $x=1$. Because the numerator is not equal to zero at $x=1$, we see that $x=1$ is a vertical asymptote.

23. $f(x)=2x^2+3x-2$, so $f'(x)=4x+3$. Setting $f'(x)=0$ gives $x=-\frac{3}{4}$ as a critical number of f. Next, $f''(x)=4>0$ for all x, so f is concave upward on $(-\infty,\infty)$. Therefore, $f\left(-\frac{3}{4}\right)=-\frac{25}{8}$ is an absolute minimum of f. There is no absolute maximum.

24. $g(x)=x^{2/3}$, so $g'(x)=\frac{2}{3}x^{-1/3}=\dfrac{2}{3x^{1/3}}$, so $x=0$ is a critical number. Because $g'(x)<0$ if $x<0$ and $g'(x)>0$ if $x>0$, we see that $(0,0)$ is a relative minimum. The graph of g shows that $(0,0)$ is an absolute minimum.



25. $g(t)=\sqrt{25-t^2}=\left(25-t^2\right)^{1/2}$. Differentiating $g(t)$, we have $g'(t)=\frac{1}{2}\left(25-t^2\right)^{-1/2}(-2t)=-\dfrac{t}{\sqrt{25-t^2}}$. Setting $g'(t)=0$ gives $t=0$ as a critical number of g. The domain of g is given by solving the inequality $25-t^2\geq 0$ or $(5-t)(5+t)\geq 0$ which implies that $t\in[-5,5]$. From the table, we conclude that $g(0)=5$ is the absolute maximum of g and $g(-5)=0$ and $g(5)=0$ is the absolute minimum value of g.

t	-5	0	5
$g(t)$	0	5	0

26. $f(x)=\frac{1}{3}x^3-x^2+x+1$, so $f'(x)=x^2-2x+1=(x-1)^2$. Therefore, $x=1$ is a critical number of f. From the table, we see that $f(0)=1$ is the absolute minimum value and $f(2)=\frac{5}{3}$ is the absolute maximum value of f.

x	0	1	2
$f(x)$	1	$\frac{4}{3}$	$\frac{5}{3}$

27. $h(t)=t^3-6t^2$, so $h'(t)=3t^2-12t=3t(t-4)=0$ if $t=0$ or $t=4$, critical numbers of h. But only $t=4$ lies in $(2,5)$. From the table, we see that h has an absolute minimum at $(4,-32)$ and an absolute maximum at $(2,-16)$.

t	2	4	5
$h(t)$	-16	-32	-25

28. $g(x)=\dfrac{x}{x^2+1}$, so $g'(x)=\dfrac{\left(x^2+1\right)(1)-x(2x)}{\left(x^2+1\right)^2}=\dfrac{1-x^2}{\left(x^2+1\right)^2}=0$ if $x=\pm1$. But only the critical number $x=1$ lies in $(0,5)$. From the table, we see that $(0,0)$ is an absolute minimum and $\left(1,\frac{1}{2}\right)$ is an absolute maximum of g.

x	0	1	5
$g(x)$	0	$\frac{1}{2}$	$\frac{5}{26}$

29. $f(x)=x-\dfrac{1}{x}$ on $[1,3]$, so $f'(x)=1+\dfrac{1}{x^2}$. Because $f'(x)$ is never zero, f has no critical number. Calculating $f(x)$ at the endpoints, we see that $f(1)=0$ is the absolute minimum value and $f(3)=\frac{8}{3}$ is the absolute maximum value.

30. $h(t) = 8t - \frac{1}{t^2}$ on $[1, 3]$, so $h'(t) = 8 + \frac{2}{t^3} = \frac{8t^3 + 2}{t^3} = 0$ if $t = -\frac{1}{4^{1/3}}$, but this number does not lie in $(1, 3)$. $\frac{8t^3 + 2}{t^3}$ is undefined at $t = 0$, but this value also lies outside $(1, 3)$. Evaluating $h(t)$ at the endpoints, we see that h has an absolute minimum at $(1, 7)$ and an absolute maximum at $\left(3, \frac{215}{9}\right)$.

31. $f(s) = s\sqrt{1 - s^2}$ on $[-1, 1]$. The function f is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Next, $f'(s) = \left(1 - s^2\right)^{1/2} + s\left(\frac{1}{2}\right)\left(1 - s^2\right)^{-1/2}(-2s) = \frac{1 - 2s^2}{\sqrt{1 - s^2}}$. Setting $f'(s) = 0$, we find that $s = \pm\frac{\sqrt{2}}{2}$ are critical numbers of f. From the table, we see that $f\left(-\frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$ is the absolute minimum value and $f\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}$ is the absolute maximum value of f.

x	-1	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$f(x)$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0

32. $f(x) = \frac{x^2}{x - 1}$. Observe that $\lim_{x \to 1^-} \frac{x^2}{x - 1} = -\infty$ and $\lim_{x \to 1^+} \frac{x^2}{x - 1} = \infty$. Therefore, there is no absolute extremum.

33. a. The sign of R_1' is negative and the sign of R_2' is positive on $(0, T)$. The sign of R_1'' is negative and the sign of R_2'' is positive on $(0, T)$.

b. The revenue of the neighborhood bookstore is decreasing at an increasing rate, while the revenue of the new bookstore is increasing at an increasing rate.

34. The rumor spreads initially with increasing speed. The rate at which the rumor is spread reaches a maximum at the time corresponding to the t-coordinate of the point P on the curve. Thereafter, the speed at which the rumor is spread decreases.

35. We want to maximize $P(x) = -x^2 + 8x + 20$. Now, $P'(x) = -2x + 8 = 0$ if $x = 4$, a critical number of P. Because $P''(x) = -2 < 0$, the graph of P is concave downward. Therefore, the critical number $x = 4$ yields an absolute maximum. So, to maximize profit, the company should spend \$4000 per month on advertising.

36. a. $S'(t) = \frac{d}{dt}\left(0.195t^2 + 0.32t + 23.7\right) = 0.39t + 0.32 > 0$ on $[0, 7]$, so sales were increasing through the years in question.

b. $S''(t) = 0.39 > 0$ on $[0, 7]$, so sales continued to accelerate through the years.

37. a. $f'(t) = \frac{d}{dt}\left(0.0117t^3 + 0.0037t^2 + 0.7563t + 4.1\right) = 0.0351t^2 + 0.0074t + 0.7563 \geq 0.7563$ for all t in the interval $[0, 9]$. This shows that f is increasing on $(0, 9)$, which tells us that the projected amount of AMT will keep on increasing over the years in question.

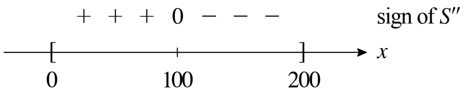
b. $f''(t) = \frac{d}{dt}\left(0.0351t^2 + 0.0074t + 0.7563\right) = 0.0702t + 0.0074 \geq 0.0074$. This shows that f' is increasing on $(0, 9)$. Our result tells us that not only is the amount of AMT paid increasing over the period in question, but it is actually accelerating.

38. **a.** The number of measles deaths in 1999 is given by $N(0) = 506$, or 506,000. The number of measles deaths in 2005 is given by $N(6) = -2.42\left(6^3\right) + 24.5\left(6^2\right) - 123.3(6) + 506 \approx 125.48$, or approximately 125,480.

b. $N'(t) = \dfrac{d}{dt}\left(-2.42t^3 + 24.5t^2 - 123.3t + 506\right) = -7.26t^2 + 49t - 123.3$. Because $(49)^2 - 4(-7.26)(-123.3) = -1179.6 < 0$, we see that $N'(t)$ has no zero. Because $N'(0) = -123.3 < 0$, we conclude that $N'(t) < 0$ on $(0, 6)$. This shows that N is decreasing on $(0, 6)$, so the number of measles deaths was dropping from 1999 through 2005.

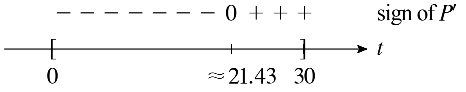
c. $N''(t) = -14.52t + 49 = 0$ implies that $t \approx 3.37$, so the number of measles deaths was decreasing most rapidly in April 2002. The rate is given by $N'(3.37) = -7.26(3.37)^2 + 49(3.37) - 123.3 \approx -40.62$, or approximately -41 deaths/yr.

39. $S(x) = -0.002x^3 + 0.6x^2 + x + 500$, so $S'(x) = -0.006x^2 + 1.2x + 1$ and $S''(x) = -0.012x + 1.2$. $x = 100$ is a candidate for an inflection point of S. The sign diagram for S'' shows that $(100, 4600)$ is an inflection point of S.



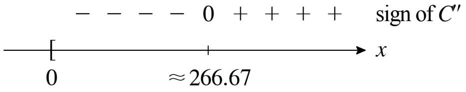
40. **a.** $P(t) = 0.00093t^3 - 0.018t^2 - 0.51t + 25$, so $P'(t) = 0.00279t^2 - 0.036t - 0.51$. Setting $P'(t) = 0$ and solving the resulting equation, we have $t = \dfrac{0.036 \pm \sqrt{(-0.036)^2 - 4(0.00279)(-0.51)}}{2(0.00279)} \approx -8.53$ or 21.43.

From the sign diagram, we see that P is decreasing on $(0, 21.43)$ and increasing on $(21.43, 30)$.



b. The percentage of men 65 and older in the workforce decreased from 1970 through about the middle of 1991, and then increased through the year 2000.

41. $C(x) = 0.0001x^3 - 0.08x^2 + 40x + 5000$, so $C'(x) = 0.0003x^2 - 0.16x + 40$ and $C''(x) = 0.0006x - 0.16$. Thus, $x = 266.67$ is a candidate for an inflection point of C. The sign diagram for C'' shows that C has an inflection point at $(266.67, 11874.08)$.



42. **a.** $S(t) = 6.8(t + 1.03)^{0.49}$, so $S'(t) = 6.8(0.49)(t + 1.03)^{-0.51} = \dfrac{3.332}{(t + 1.03)^{0.51}} > 0$ on $(0, 4)$, so S is increasing on $(0, 4)$. This tells us that the sales are increasing from 2003 through 2007.

b. $S''(t) = 3.332(-0.51)(t + 1.03)^{-1.51} = -\dfrac{1.69932}{(t + 1.03)^{1.51}} < 0$ on $(0, 4)$. This tells us that the graph of S is concave downward on $(0, 4)$, and that the sales are increasing but at a decreasing rate.

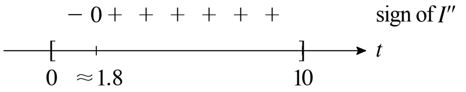
43. **a.** $f(0) \approx 12.98$, so the proportion in 2005 was approximately 13.0%. The projected proportion in 2015 is given by $f(10) \approx 22.21$, or approximately 22.2%.

b. $f'(t) = \dfrac{\left(59 - t^{1/2}\right)(150)\left(\frac{1}{2}t^{-1/2}\right) - \left(150t^{1/2} + 766\right)\left(-\frac{1}{2}t^{-1/2}\right)}{\left(59 - t^{1/2}\right)^2} = \dfrac{4.808}{\sqrt{t}\left(59 - t^{1/2}\right)^2} > 0$ for $0 < t \leq 10$, so f is increasing on $(0, 10)$. This says that the percentage of small and lower-midsize vehicles will be growing over the period from 2005 to 2015.

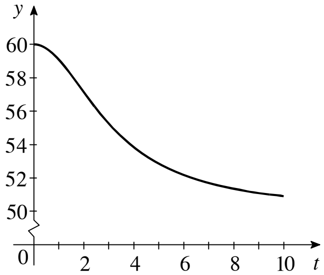
44. a. $I(t) = \dfrac{50t^2 + 600}{t^2 + 10}$, so $I'(t) = \dfrac{(t^2 + 10)(100t) - (50t^2 + 600)(2t)}{(t^2 + 10)^2} = -\dfrac{200t}{(t^2 + 10)^2} < 0$ on $(0, 10)$, and so I is decreasing on $(0, 10)$.

b. $I''(t) = -200\left[\dfrac{(t^2 + 10)^2(1) - t(2)(t^2 + 10)(2t)}{(t^2 + 10)^4}\right] = \dfrac{-200(t^2 + 10)\left[(t^2 + 10) - 4t^2\right]}{(t^2 + 10)^4} = -\dfrac{200(10 - 3t^2)}{(t^2 + 10)^3}$.

The sign diagram of I'' for $t > 0$ shows that I is concave downward on $\left(0, \sqrt{\frac{10}{3}}\right)$ and concave upward on $\left(\sqrt{\frac{10}{3}}, \infty\right)$.



c.



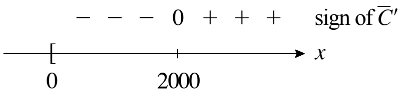
d. The rate of decline in the environmental quality of the wildlife was increasing for the first 1.8 years. After that time the rate of decline decreased.

45. The revenue is $R(x) = px = x(-0.0005x^2 + 60) = -0.0005x^3 + 60x$. Therefore, the total profit is $P(x) = R(x) - C(x) = -0.0005x^3 + 0.001x^2 + 42x - 4000$. $P'(x) = -0.0015x^2 + 0.002x + 42$, and setting $P'(x) = 0$ gives $3x^2 - 4x - 84{,}000 = 0$. Solving for x, we find $x = \dfrac{4 \pm \sqrt{16 - 4(3)(84{,}000)}}{2(3)} = \dfrac{4 \pm 1004}{6} = 168$ or -167. We reject the negative root. Next, $P''(x) = -0.003x + 0.002$ and $P''(168) = -0.003(168) + 0.002 = -0.502 < 0$. By the Second Derivative Test, $x = 168$ gives a relative maximum. Therefore, the required level of production is 168 DVDs.

46. $P(x) = -0.04x^2 + 240x - 10{,}000$, so $P'(x) = -0.08x + 240 = 0$ if $x = 3000$. The graph of P is a parabola that opens downward and so $x = 3000$ gives rise to the absolute maximum of P. Thus, to maximize profits, the company should produce 3000 cameras per month.

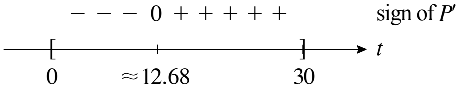
47. a. $C(x) = 0.001x^2 + 100x + 4000$, so $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{0.001x^2 + 100x + 4000}{x} = 0.001x + 100 + \dfrac{4000}{x}$.

b. $\overline{C}'(x) = 0.001 - \dfrac{4000}{x^2} = \dfrac{0.001x^2 - 4000}{x^2} = \dfrac{0.001(x^2 - 4{,}000{,}000)}{x^2}$. Setting $\overline{C}'(x) = 0$ gives $x = \pm 2000$. We reject the negative root. The sign diagram of $\overline{C}'$shows that $x = 2000$ gives rise to a relative minimum of $\overline{C}$. Because $\overline{C}''(x) = \dfrac{8000}{x^3} > 0$ if $x > 0$, we see that $\overline{C}$ is concave upward on $(0, \infty)$, and so $x = 2000$ yields an absolute minimum. The required production level is 2000 units.



48. $N(t) = -2t^3 + 12t^2 + 2t$. We wish to find the inflection point of the function N. $N'(t) = -6t^2 + 24t + 2$ and $N''(t) = -12t + 24 = -12(t - 2) = 0$ if $t = 2$. Furthermore, $N''(t) > 0$ when $t < 2$ and $N''(t) < 0$ when $t > 2$. Therefore, $t = 2$ gives an inflection point of N. The average worker is performing at peak efficiency at 10 a.m.

49. **a.** $P(t) = -0.0002t^3 + 0.018t^2 - 0.36t + 10$, so $P'(t) = -0.0006t^2 + 0.036t - 0.36$. Setting $P'(t) = 0$ gives $-0.0006t^2 + 0.036t - 0.36 = 0$, or $t^2 - 60t + 600 = 0$. Thus, $t = \dfrac{60 \pm \sqrt{60^2 - 4(1)(600)}}{2} \approx 12.7$ or 47.3. We reject the root 47.3 because it lies outside $[0, 30]$. The sign diagram for P' shows that P is decreasing on $(0, 12.7)$ and increasing on $(12.7, 30)$.



b. The absolute minimum of P occurs at $t = 12.7$, and $P(12.7) \approx 7.9$.

c. The percentage of women 65 and older in the workforce was decreasing from 1970 to September 1982 and increasing from September 1982 to 2000. It reached a minimum value of 7.9% in September 1982.

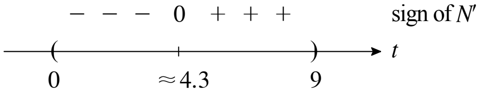
50. $f(x) = 15\left(0.08333x^2 + 1.91667x + 1\right)^{-1}$, so by the Chain Rule,

$$f'(x) = -15\left(0.08333x^2 + 1.91667x + 1\right)^{-2} \frac{d}{dx}\left(0.08333x^2 + 1.91667x + 1\right) = \frac{-15(0.16666x + 1.91667)}{\left(0.08333x^2 + 1.91667x + 1\right)^2} < 0$$

for all x in $(0, 11)$. Therefore, f is decreasing on $(0, 11)$. Our result shows that as the age of the driver increases from 16 to 27, the predicted crash fatalities drop.

51. $R'(x) = k\dfrac{d}{dx}x(M - x) = k[(M - x) + x(-1)] = k(M - 2x)$. Setting $R'(x) = 0$ gives $M - 2x = 0$, or $x = \frac{1}{2}M$, a critical number of R. Because $R''(x) = -2k < 0$, we see that $x = \frac{1}{2}M$ gives a maximum; that is, R is greatest when half the population is infected.

52. For $0 < t < 9$, $N'(t) = 0.72t - 3.10$, so $N'(t) = 0$ implies $t = \dfrac{3.10}{0.72} \approx 4.3$. From the sign diagram and the fact that $N(t) = 42.46$ for $9 \leq t \leq 11$, we see that the number of medical school applicants was decreasing from 1997–1998 to 2001–2002, increasing from 2001–2002 to 2006–2007, and constant from 2006–2007 to 2008–2009.



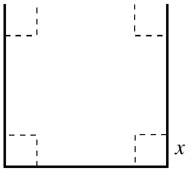
53. **a.** $f'(x) = -2x$ if $x \neq 0$, $f'(-1) = 2$, and $f'(1) = -2$, so $f'(x)$ changes sign from positive to negative as we move across $x = 0$.

b. f does not have a relative maximum at $x = 0$ because $f(0) = 2$ but a neighborhood of $x = 0$, for example $\left(-\frac{1}{2}, \frac{1}{2}\right)$, contains numbers with values larger than 2. This does not contradict the First Derivative Test because f is not continuous at $x = 0$.

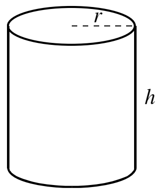
54. The volume is $V = f(x) = x(10 - 2x)^2$ in.^{3} for $0 \leq x \leq 5$. To maximize V, we compute

$$f'(x) = 12x^2 - 80x + 100 = 4\left(3x^2 - 20x + 25\right)$$
$$= 4(3x - 5)(x - 5).$$

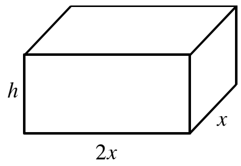
Setting $f'(x) = 0$ gives $x = \frac{5}{3}$ and 5 as critical numbers of f. From the table, we see that the box has a maximum volume of 74.07 in.^{3}.



x	0	$\frac{5}{3}$	5
$f(x)$	0	$\frac{2000}{27} \approx 74.07$	0



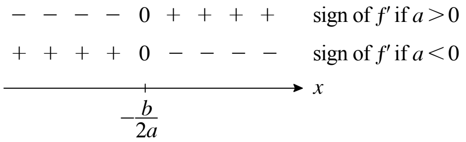
55. Suppose the radius is r and the height is h. Then the capacity is $\pi r^2 h$, and this must be equal to 32π ft^3; that is, $\pi r^2 h = 32\pi$. Let the cost per square foot for the sides be \$$c$. Then the cost of construction is $C = 2\pi rhc + 2\left(\pi r^2\right)(2c) = 2\pi crh + 4\pi cr^2$. But $h = \dfrac{32\pi}{\pi r^2} = \dfrac{32}{r^2}$, so $C = f(r) = \dfrac{64c\pi}{r} + 4\pi cr^2$, giving $C' = f'(r) = -\dfrac{64\pi c}{r^2} + 8\pi cr = \dfrac{-64\pi c + 8\pi cr^3}{r^2} = \dfrac{8\pi c\left(-8 + r^3\right)}{r^2}$. Setting $f'(r) = 0$ gives $r^3 = 8$, or $r = 2$. Next, $f''(r) = \dfrac{128\pi c}{r^3} + 8\pi c$, and so $f''(2) > 0$. Therefore, $r = 2$ minimizes f. The required dimensions are $r = 2$ and $h = \frac{32}{4} = 8$. That is, the radius is 2 ft and the height is 8 ft.



56. $C(x) = 30(2)(2x)(x) + 20(2)(2xh + xh) = 120x^2 + 120xh$. But $x(2x)h = 4$, so $h = \dfrac{2}{x^2}$. Therefore, $C(x) = 120x^2 + 120x\left(\dfrac{2}{x^2}\right) = 120x^2 + \dfrac{240}{x}$, so $C'(x) = 240x - \dfrac{240}{x^2}$. Setting $C'(x) = 0$ gives $240x - \dfrac{240}{x^2} = 0$, or $x^3 = 1$. Therefore, $x = 1$. $C''(x) = 240 + \dfrac{480}{x^3}$, and in particular, $C''(1) > 0$. Therefore, the cost is minimized by taking $x = 1$. The required dimensions are 1 ft × 2 ft × 2 ft.

57. Let x denote the number of cases in each order. Then the average number of cases of beer in storage during the year is $\frac{1}{2}x$. The storage cost in dollars is $2\left(\frac{1}{2}x\right) = x$. Next, we see that the number of orders required is $\dfrac{800{,}000}{x}$, and so the ordering cost is $\dfrac{500(800{,}000)}{x} = \dfrac{400{,}000{,}000}{x}$. Thus, the total cost incurred by the company per year is given by $C(x) = x + \dfrac{400{,}000{,}000}{x}$. We want to minimize C in the interval $(0, \infty)$, so we calculate $C'(x) = 1 - \dfrac{400{,}000{,}000}{x^2}$. Setting $C'(x) = 0$ gives $x^2 = 400{,}000{,}000$, or $x = 20{,}000$ (we reject $x = -20{,}000$). Next, $C''(x) = \dfrac{800{,}000{,}000}{x^3} > 0$ for all x, so C is concave upward. Thus, $x = 20{,}000$ gives rise to the absolute minimum of C. The company should order 20,000 cases of beer per order.

58. $f(x) = ax^2 + bx + c$, so $f'(x) = 2ax + b = 2a\left(x + \dfrac{b}{2a}\right)$. Then f' is continuous everywhere and has a zero at $x = -\dfrac{b}{2a}$. From the sign diagrams of f' for $a > 0$ and $a < 0$, we conclude that if $a > 0$, f is decreasing on $\left(-\infty, -\dfrac{b}{2a}\right)$ and increasing on $\left(-\dfrac{b}{2a}, \infty\right)$, and if $a < 0$, f is increasing on $\left(-\infty, -\dfrac{b}{2a}\right)$ and decreasing on $\left(-\dfrac{b}{2a}, \infty\right)$.



59. $f(x) = x^2 + ax + b$, so $f'(x) = 2x + a$. We require that $f'(2) = 0$, so $(2)(2) + a = 0$, and $a = -4$. Next, $f(2) = 7$ implies that $f(2) = 2^2 + (-4)(2) + b = 7$, so $b = 11$. Thus, $f(x) = x^2 - 4x + 11$. Because the graph of f is a parabola that opens upward, $(2, 7)$ is a relative minimum.

60. $f(x) = x^4 + 2x^3 + cx^2 + 2x + 2$, so $f'(x) = 4x^3 + 6x^2 + 2cx + 2$ and $f''(x) = 12x^2 + 12x + 2c$. For f to be concave upward everywhere, we require that $f''(x) \geq 0$ for all x in $(-\infty, \infty)$. This is true provided the discriminant $b^2 - 4ac \leq 0$, or $144 - 4(12)(2c) \leq 0$, or $c \geq \frac{3}{2}$.

61. Because $(a, f(a))$ is an inflection point of f, $f''(a) = 0$. This shows that a is a critical number of f'. Next, f changes concavity at $(a, f(a))$. If the concavity changes from concave downward to concave upward [that is, $f''(x) < 0$ for $x < a$ and $f''(x) > 0$ for $x > a$], then f' has a relative minimum at a. On the other hand, if the concavity changes from concave upward to concave downward, [$f''(x) > 0$ for $x < a$ and $f''(x) < 0$ for $x > a$], then f' has a relative maximum at a. In either case, f' has a relative extremum at a.

62. **a.** $f'(x) = 3x^2$ for $x \neq 0$. We see that $f'(x) > 0$ for $x < 0$ as well as for $x > 0$, so $f'(x)$ does not change sign at $x = 0$.

b. $f(0) = 2$ is larger than $f(x)$ for x near 0, so f has a relative maximum at $x = 0$. This does not contradict the First Derivative Test because f is not continuous at $x = 0$.

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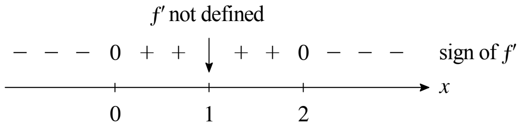
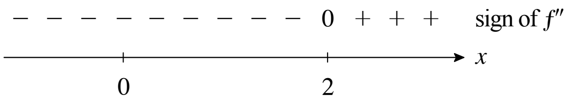
1. $f(x) = \dfrac{x^2}{1-x}$, so $f'(x) = \dfrac{(1-x)(2x) - x^2(-1)}{(1-x)^2} = \dfrac{2x - 2x^2 + x^2}{(1-x)^2} = \dfrac{x(2-x)}{(1-x)^2}$; f' is not defined at 1 and has zeros at 0 and 2. The sign diagram of f shows that f is decreasing on $(-\infty, 0)$ and $(2, \infty)$ and increasing on $(0, 1)$ and $(1, 2)$.



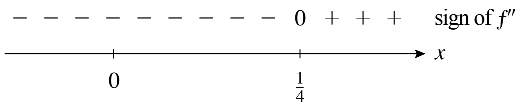
2. $f(x) = 2x^2 - 12x^{1/3}$, so $f'(x) = 4x - 4x^{-2/3} = 4x^{-2/3}\left(x^{5/3} - 1\right) = \dfrac{4\left(x^{5/3} - 1\right)}{x^{2/3}}$. f' is discontinuous at $x = 0$ and has a zero where $x^{5/3} = 1$ or $x = 1$.

Therefore, f has critical numbers at 0 and 1. From the sign diagram for f', we see that $x = 1$ gives a relative minimum. Because $f(1) = 2 - 12 = -10$, the relative minimum is $(1, -10)$. There is no relative maximum.



3. $f(x) = \frac{1}{3}x^3 - \frac{1}{4}x^2 - \frac{1}{2}x + 1$, so $f'(x) = x^2 - \frac{1}{2}x - \frac{1}{2}$ and $f''(x) = 2x - \frac{1}{2} = 0$ gives $x = \frac{1}{4}$.

The sign diagram of f'' shows that f is concave downward on $\left(-\infty, \frac{1}{4}\right)$ and concave upward on $\left(\frac{1}{4}, \infty\right)$. Because $f\left(\frac{1}{4}\right) = \frac{1}{3}\left(\frac{1}{4}\right)^3 - \frac{1}{4}\left(\frac{1}{4}\right)^2 - \frac{1}{2}\left(\frac{1}{4}\right) + 1 = \frac{83}{96}$, the inflection point is $\left(\frac{1}{4}, \frac{83}{96}\right)$.

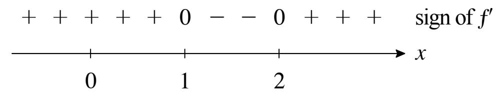


4. $f(x) = 2x^3 - 9x^2 + 12x - 1$.

1. The domain of f is $(-\infty, \infty)$.
2. Setting $y = f(x) = 0$ gives -1 as the y-intercept of f.
3. $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$.

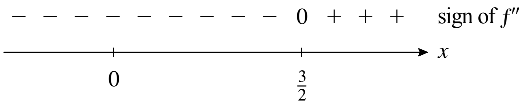
4. There is no asymptote.

5. $f'(x) = 6x^2 - 18x + 12 = 6\left(x^2 - 3x + 2\right) = 6(x-2)(x-1)$.



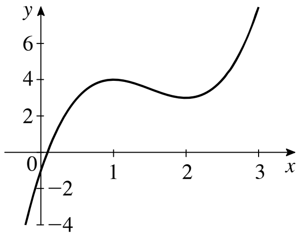
The sign diagram of f' shows that f is increasing on $(-\infty, 1)$ and $(2, \infty)$ and decreasing on $(1, 2)$.

6. We see that $(1, 4)$ is a relative maximum and $(2, 3)$ is a relative minimum.



7. $f''(x) = 12x - 18 = 6(2x - 3)$. The sign diagram of f'' shows that f is concave downward on $\left(-\infty, \frac{3}{2}\right)$ and concave upward on $\left(\frac{3}{2}, \infty\right)$.

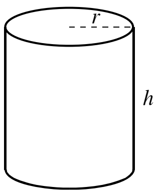
8. $f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + 12\left(\frac{3}{2}\right) - 1 = \frac{7}{2}$, so $\left(\frac{3}{2}, \frac{7}{2}\right)$ is an inflection point of f.



5. $f(x) = 2x^3 + 3x^2 - 1$ is continuous on the closed interval $[-2, 3]$. $f'(x) = 6x^2 + 6x = 6x(x+1)$, so the critical numbers of f are -1 and 0. From the table, we see that the absolute maximum value of f is 80 and the absolute minimum value is -5.

x	−2	−1	0	3
y	−5	0	−1	80

6. The amount of material used (the surface area) is $A = \pi r^2 + 2\pi rh$. But $V = \pi r^2 h = 1$, and so $h = \dfrac{1}{\pi r^2}$. Therefore, $A = \pi r^2 + 2\pi r\left(\dfrac{1}{\pi r^2}\right) = \pi r^2 + \dfrac{2}{r}$, so $A' = 2\pi r - \dfrac{2}{r^2} = 0$ implies $2\pi r = \dfrac{2}{r^2}$, $r^3 = \dfrac{2}{r^2}$, $r^3 = \dfrac{1}{\pi}$, and $r = \dfrac{1}{\sqrt[3]{\pi}}$. Because $A'' = 2\pi + \dfrac{4}{r^3} > 0$ for $r > 0$, we see that $r = \dfrac{1}{\sqrt[3]{\pi}}$ does give an absolute minimum. Also, $h = \dfrac{1}{\pi r^2} = \dfrac{1}{\pi} \cdot \pi^{2/3} = \dfrac{1}{\pi^{1/3}} = \dfrac{1}{\sqrt[3]{\pi}}$. Therefore, the radius and height should each be $\dfrac{1}{\sqrt[3]{\pi}}$ ft.



CHAPTER 4 Explore & Discuss

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1. This is false. Consider $f(x) = x^3$, which is continuous at c. Now f is increasing at $x = 0$, because if we pick any interval containing 0, then f is increasing on that interval. This fact can be established rigorously, but we will accept a “geometric proof”. (Simply sketch the graph of f.) But $f'(x) = 3x^2$, so that $f'(c) = f'(0) = 0$.

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1. $P'(x) = R'(x) - C'(x)$.

2. **a.** P is increasing at $x = a$ if $P'(a) = R'(a) - C'(a) > 0$, or $R'(a) > C'(a)$.
   b. P is decreasing at $x = a$ if $R'(a) < C'(a)$.
   c. P is constant at $x = a$ if $R'(a) = C'(a)$.

3. **a.** The profit is increasing when the level of production is a if the revenue is increasing faster than the cost at a.
   b. The profit is decreasing when the level of production is a if the revenue is decreasing faster than the cost at a.
   c. The profit is flat if the revenue is increasing at the same pace as the cost.

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1. $\overline{C}'(x) = \dfrac{xC'(x) - C(x)}{x^2} = \dfrac{C'(x) - \frac{C(x)}{x}}{x} = \dfrac{C'(x) - \overline{C}(x)}{x}$.

2. Because $x > 0$, we see that $C'(x) < \overline{C}(x)$ implies $\overline{C}'(x) < 0$, and so $\overline{C}'(x) < 0$. Thus, $\overline{C}$ is decreasing for values of x satisfying the condition $C'(x) < \overline{C}(x)$. Similarly, we see that $\overline{C}$ is increasing for values of x for which $C'(x) > \overline{C}(x)$. Finally, $\overline{C}$ is constant for values of x satisfying $C'(x) = \overline{C}(x)$.

3. The results of part (b) tell us that the marginal average cost is decreasing for values of x for which the marginal cost is less than the average cost , increasing for values of x for which the marginal cost is greater than the average cost, and constant for values of x for which the marginal cost is equal to the average cost.

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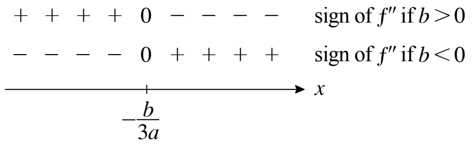
1. No. $f(x) = x^3$ has an inflection point, but no extremum, at (0, 0).

2. True. Consider $f(x) = ax^3 + bx^2 + cx + d$ for $a \neq 0$. Without loss of generality, assume $a > 0$. Then $f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b = 6a\left(x + \frac{b}{3a}\right)$. Observe that $f''(0) = 0$ implies $x = -\frac{b}{3a}$. From the sign diagrams for f'' in cases $b > 0$ and $b < 0$, we see that $\left(-\frac{b}{3a}, f\left(-\frac{b}{3a}\right)\right)$ is the only inflection point of f.



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$f'(c) = 0$ tells us that $x = c$ is a critical number of f. Because $f''(x) > 0$ for all x in (a, b), we see that $f''(c) > 0$, and so by the Second Derivative Test, the point $(c, f(c))$ is a relative minimum of f. If property 1 is replaced by the property that $f''(x) < 0$ for all x in (a, b), then the Second Derivative Test implies that $(c, f(c))$ is a relative maximum of f.

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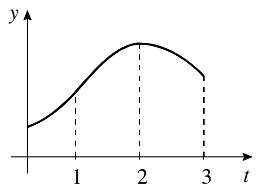
1. On (0, 1), $f'(t) > 0$ and $f''(t) > 0$; on (1, 2), $f'(t) > 0$ and $f''(t) < 0$; and on (2, 3), $f'(t) < 0$ and $f''(t) < 0$.

2.



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1. At the level of production x_0, $P'(x_0) = 0$. This means that $P'(x_0) = R'(x_0) - C'(x_0) = 0$, so $R'(x_0) = C'(x_0)$. Furthermore, since x_0 gives a relative maximum, $P''(x_0) < 0$, and this translates into the condition $P''(x_0) = R''(x_0) - C''(x_0) < 0$, or $R''(x_0) < C''(x_0)$.

2. The condition $R'(x_0) = C'(x_0)$ says that at the maximal level of production, the marginal revenue equals the marginal cost. This makes sense because at that level of production, we can expect that the rate of change of revenue should not exceed or be smaller than the rate of growth of the total cost.
   The condition $R''(x_0) < C''(x_0)$ says that at the maximal level of production, the rate of change of the marginal revenue should be smaller than that of the marginal cost function. This makes sense because if the opposite were true, that is, $R'(x_0) > C''(x_0)$, then increased production would increase revenue more than cost, leading to increased profit.

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1. The average cost is $\overline{C}(x) = \dfrac{C(x)}{x}$. Thus, $\overline{C}'(x) = \dfrac{xC'(x) - C(x)}{x^2}$. Setting $\overline{C}'(x) = 0$ gives $x = \dfrac{C(x)}{C'(x)}$ as the only critical number of $\overline{C}$. Next,
$$\overline{C}''(x) = \frac{x^2\left[C'(x) + xC''(x) - C'(x)\right] - \left[xC'(x) - C(x)\right]2x}{x^4} = \frac{C''(x)}{x} - \frac{2}{x^3}\left[xC'(x) - C(x)\right].$$

2. At the critical number, $xC'(x) - C(x) = 0$, and so $\overline{C}''(x) = \dfrac{C''(x)}{x} > 0$ because C is concave upward. Therefore $x = \dfrac{C(x)}{C'(x)}$ does give the minimum value. Rewriting, we have $C'(x) = \dfrac{C(x)}{x} = \overline{C}(x)$, and the proof is complete.

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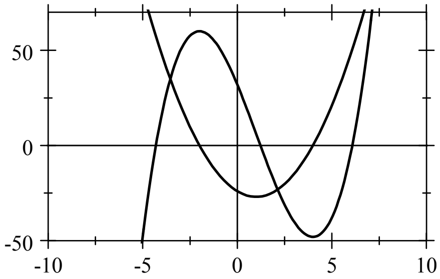
1.



2. This is to be expected because f is increasing over an interval where $f'(x) > 0$ and decreasing over an interval where $f'(x) < 0$.

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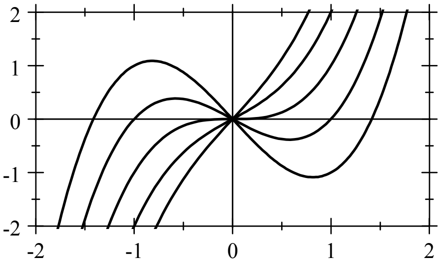
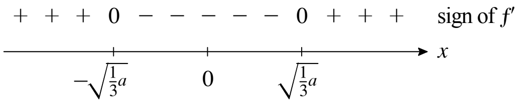
1.



2. f is increasing on $(-\infty, \infty)$ if and only if $a < 0$.

3. If $a \leq 0$, then $f'(x) = 3x^2 - a \geq 0$ for all x, so f is increasing on $(-\infty, \infty)$. If $a > 0$, then

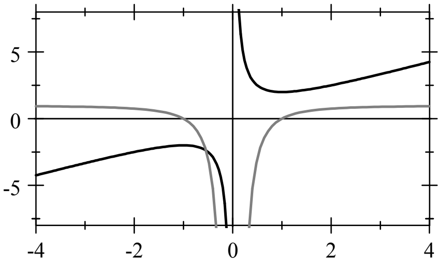
$f'(x) = 3x^2 - a = 0$ if $x = \pm\sqrt{\frac{1}{3}a}$, critical numbers of f.



From the sign diagram of f', we see that in this case f is decreasing on $\left(-\sqrt{\frac{1}{3}a}, \sqrt{\frac{1}{3}a}\right)$.

Page 262

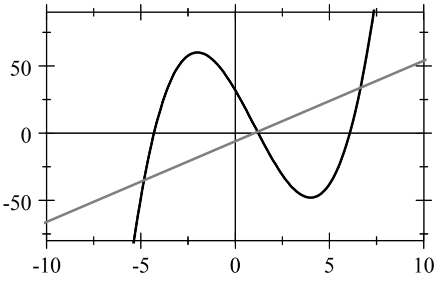
1. The graphs of f and f' are shown in the figure.



2. From the graph of f', we see that $x = \pm 1$ are critical numbers of f. The sign of $f'(x)$ changes from positive to negative as we move across $x = -1$, showing that the point $(-1, -2)$ is a relative maximum. The sign of $f'(x)$ changes from negative to positive as we move across $x = 1$, and so $(1, 2)$ is a relative minimum. These conclusions are the same as those arrived at in Example 8, as expected.

Page 274

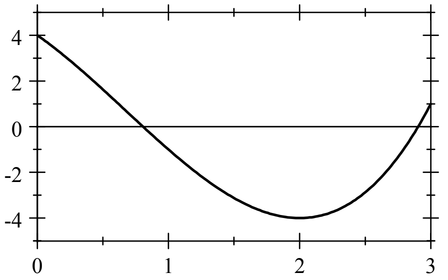
1.



2. Yes.

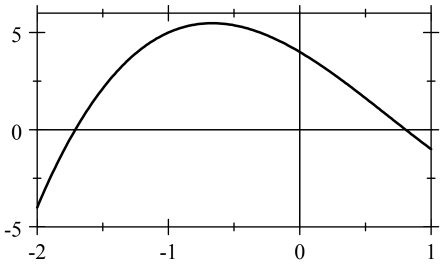
Page 308

1.



The absolute minimum is $f(0) = 4$ and the absolute minimum is $f(2) = -4$.

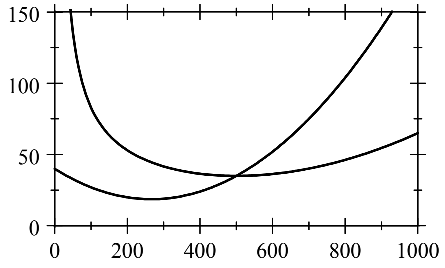
2.



From the graph, it appears that the absolute minimum is $f(-2) = -4$ and the absolute maximum is $f(-1.5) = 2.125$.
To verify this analytically, we calculate $f'(x) = 3x^2 - 4x - 4 = (3x + 2)(x - 2) = 0$, giving $x = -\frac{3}{2}$ or 2, of which only $-\frac{3}{2}$ lies in $[-2, 1]$. We calculate $f(-2) = -4$, $f\left(-\frac{3}{2}\right) = \frac{17}{8}$, and $f(1) = -1$, verifying our results.

Page 311

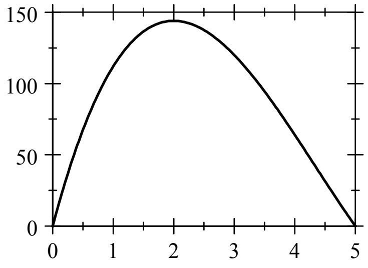
1.



2. The point of intersection of the graphs of $\overline{C}$ and C' is $(500, 35)$. This verifies the proof found in the Explore & Discuss exercises on page 310.

Page 323

1.



The value of $f(x)$ is zero at $x = 0$, increases to a maximum value somewhere in the interval $[0, 5]$, then decreases until it again takes on the value zero at $x = 5$. If $x = 0$, the volume $f(0)$ of the box is zero because there is really no box for $x = 0$. If $x = 5$, we have cut clear across the cardboard and once again there is no box, so the volume is zero. For $0 < x < 5$, a square is removed from each corner and the volume of the resulting box increases, then decreases, as the length of the square increases. So there is a definite size of squares to be removed such that the volume of the resulting box is as large as possible.

2. Using ZOOM and TRACE, we find that the absolute maximum value is $f(2) = 144$, as found in Example 2.

5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

5.1 Exponential Functions

Concept Questions page 342

1. $f(x) = b^x$ with $b > 0$ and $b \neq 1$.

2. a. $y = b^x$, $b > 0$, $b \neq 1$ has domain $(-\infty, \infty)$ and range $(0, \infty)$.

b. The y-intercept is 1.

c. The function is continuous on $(-\infty, \infty)$.

d. The function is increasing on $(-\infty, \infty)$ if $b > 1$ and decreasing on $(-\infty, \infty)$ if $b < 1$.

Exercises page 342

1. a. $4^{-3} \times 4^5 = 4^{-3+5} = 4^2 = 16.$

b. $3^{-3} \times 3^6 = 3^{6-3} = 3^3 = 27.$

2. a. $\left(2^{-1}\right)^3 = 2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}.$

b. $\left(3^{-2}\right)^3 = 3^{-6} = \dfrac{1}{3^6} = \dfrac{1}{729}.$

3. a. $9\,(9)^{-1/2} = \dfrac{9}{9^{1/2}} = \dfrac{9}{3} = 3.$

b. $5\,(5)^{-1/2} = 5^{1/2} = \sqrt{5}.$

4. a. $\left[\left(-\tfrac{1}{2}\right)^3\right]^{-2} = \left(-\tfrac{1}{2}\right)^{-6} = \dfrac{(-1)^{-6}}{2^{-6}} = 2^6 = 64.$

b. $\left[\left(-\tfrac{1}{3}\right)^2\right]^{-3} = \left(-\dfrac{1}{3}\right)^{-6} = \dfrac{(-1)^{-6}}{3^{-6}} = 3^6 = 729.$

5. a. $\dfrac{(-3)^4\,(-3)^5}{(-3)^8} = (-3)^{4+5-8} = (-3)^1 = -3.$

b. $\dfrac{\left(2^{-4}\right)\left(2^6\right)}{2^{-1}} = 2^{-4+6+1} = 2^3 = 8.$

6. a. $$3^{1/4} \times 9^{-5/8} = 3^{1/4}\left(3^2\right)^{-5/8} = 3^{1/4} \times 3^{-5/4}$$
$$= 3^{(1/4)-(5/4)} = 3^{-1} = \tfrac{1}{3}.$$

b. $$2^{3/4} \times 4^{-3/2} = 2^{3/4}\left(2^2\right)^{-3/2} = 2^{3/4} \times 2^{-3}$$
$$= 2^{(3/4)-3} = 2^{-9/4} = \frac{1}{2^{9/4}}.$$

7. a. $\dfrac{5^{3.3} \cdot 5^{-1.6}}{5^{-0.3}} = \dfrac{5^{3.3-1.6}}{5^{-0.3}} = 5^{1.7+(0.3)} = 5^2 = 25.$

b. $\dfrac{4^{2.7} \cdot 4^{-1.3}}{4^{-0.4}} = 4^{2.7-1.3+0.4} = 4^{1.8} \approx 12.126.$

8. a. $\left(\tfrac{1}{16}\right)^{-1/4}\left(\tfrac{27}{64}\right)^{-1/3} = (16)^{1/4}\left(\tfrac{64}{27}\right)^{1/3} = 2\left(\tfrac{4}{3}\right) = \tfrac{8}{3}.$

b. $\tfrac{8}{27}^{-1/3}\left(\tfrac{81}{256}\right)^{-1/4} = \left(\tfrac{27}{8}\right)^{1/3}\left(\tfrac{256}{81}\right)^{1/4} = \tfrac{3}{2} \cdot \tfrac{4}{3} = 2.$

9. a. $\left(64x^9\right)^{1/3} = 64^{1/3}\left(x^{9/3}\right) = 4x^3.$

b. $$\left(25x^3y^4\right)^{1/2} = \left(25^{1/2}\right)\left(x^{3/2}\right)\left(y^{4/2}\right) = 5x^{3/2}y^2$$
$$= 5xy^2\sqrt{x}.$$

10. a. $\left(2x^3\right)\left(-4x^{-2}\right) = -8x^{3-2} = -8x.$

b. $\left(4x^{-2}\right)\left(-3x^5\right) = -12x^{-2+5} = -12x^3.$

11. a. $\frac{6a^{-4}}{3a^{-3}} = 2a^{-4+3} = 2a^{-1} = \frac{2}{a}.$ **b.** $\frac{4b^{-4}}{12b^{-6}} = \frac{1}{3}b^{-4+6} = \frac{1}{3}b^2.$

12. a. $y^{-3/2}y^{5/3} = y^{(-3/2)+(5/3)} = y^{1/6}.$ **b.** $x^{-3/5}x^{8/3} = x^{(-3/5)+(8/3)} = x^{31/15}.$

13. a. $\left(2x^3y^2\right)^3 = 2^3 \times x^{3(3)} \times y^{2(3)} = 8x^9y^6.$ **b.** $\left(4x^2y^2z^3\right)^2 = 4^2 \cdot x^{2(2)} \cdot y^{2(2)} \cdot z^{3(2)} = 16x^4y^4z^6.$

14. a. $\left(x^{r/s}\right)^{s/r} = x^{(r/s)(s/r)} = x.$ **b.** $\left(x^{-b/a}\right)^{-a/b} = x^{(-b/a)(-a/b)} = x.$

15. a. $\frac{5^0}{\left(2^{-3}x^{-3}y^2\right)^2} = \frac{1}{2^{-3(2)}x^{-3(2)}y^{2(2)}} = \frac{2^6x^6}{y^4} = \frac{64x^6}{y^4}.$ **b.** $\frac{(x+y)(x-y)}{(x-y)^0} = (x+y)(x-y).$

16. a. $\frac{\left(a^m \cdot a^{-n}\right)^{-2}}{\left(a^{m+n}\right)^2} = \frac{a^{-2m} \cdot a^{2n}}{a^{2(m+n)}} = a^{-2m+2n-2(m+n)} = \frac{1}{a^{4m}}.$ **b.** $\left(\frac{x^{2n-2}y^{2n}}{x^{5n+1}y^{-n}}\right)^{1/3} = \left(\frac{y^{3n}}{x^{3n+3}}\right)^{1/3} = \frac{y^n}{x^{n+1}}.$

17. $6^{2x} = 6^6$ if and only if $2x = 6$ or $x = 3$.

18. $5^{-x} = 5^3$ if and only if $-x = 3$ or $x = -3$.

19. $3^{3x-4} = 3^5$ if and only if $3x - 4 = 5$, $3x = 9$, or $x = 3$.

20. $10^{2x-1} = 10^{x+3}$ if and only if $2x - 1 = x + 3$, or $x = 4$.

21. $(2.1)^{x+2} = (2.1)^5$ if and only if $x + 2 = 5$, or $x = 3$.

22. $(-1.3)^{x-2} = (-1.3)^{2x+1}$ if and only if $x - 2 = 2x + 1$, or $x = -3$.

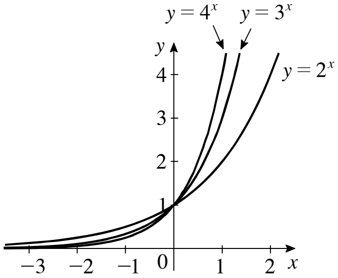
23. $8^x = \left(\frac{1}{32}\right)^{x-2}$, $\left(2^3\right)^x = (32)^{2-x} = \left(2^5\right)^{2-x}$, so $2^{3x} = 2^{5(2-x)}$, $3x = 10 - 5x$, $8x = 10$, or $x = \frac{5}{4}$.

24. $3^{x-x^2} = \frac{1}{9^x} = \left(3^2\right)^{-x} = 3^{-2x}$. This is true if and only if $x - x^2 = -2x$, $x^2 - 3x = x(x-3) = 0$, so $x = 0$ or 3.

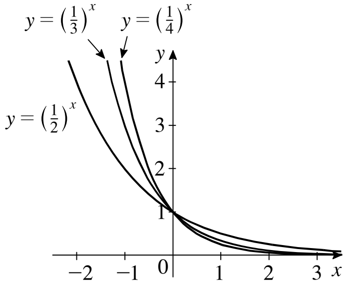
25. Let $y = 3^x$. Then the given equation is equivalent to $y^2 - 12y + 27 = 0$, or $(y-9)(y-3) = 0$, giving $y = 3$ or 9. So $3^x = 3$ or $3^x = 9$, and therefore, $x = 1$ or $x = 2$.

26. $2^{2x} - 4 \cdot 2^x + 4 = 0$, $(2^x)^2 - 4(2^x) + 4 = 0$. Let $y = 2^x$, so $y^2 - 4y + 4 = (y-2)^2 = 0$, or $y = 2$. Thus, $2^x = 2$, or $x = 1$.

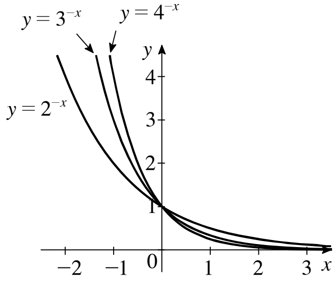
27. $y = 2^x$, $y = 3^x$, and $y = 4^x$.



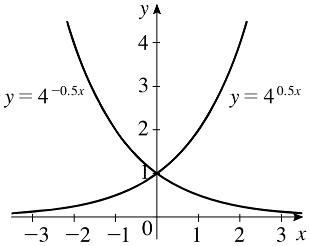
28. $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{3}\right)^x$, and $y = \left(\frac{1}{4}\right)^x$.



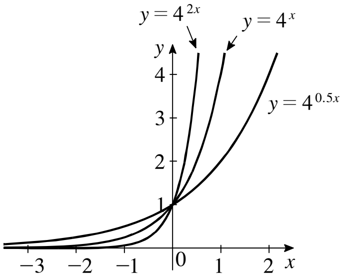
29. $y = 2^{-x}$, $y = 3^{-x}$, and $y = 4^{-x}$.



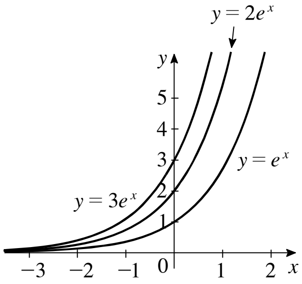
30. $y = 4^{0.5x}$ and $y = 4^{-0.5x}$.



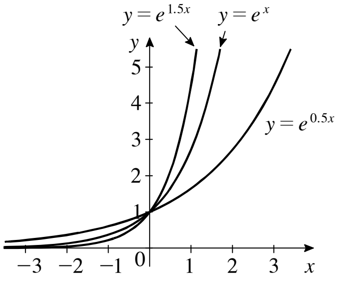
31. $y = 4^{0.5x}$, $y = 4^x$, and $y = 4^{2x}$.



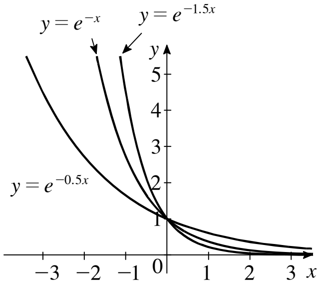
32. $y = e^x$, $y = 2e^x$, and $y = 3e^x$.



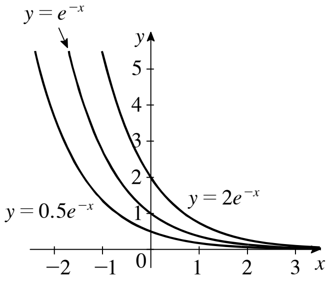
33. $y = e^{0.5x}$, $y = e^x$, $y = e^{1.5x}$.



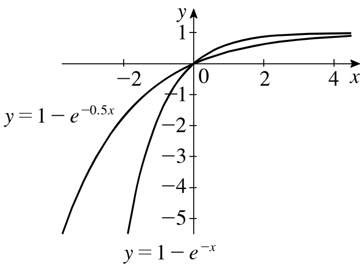
34. $y = e^{-0.5x}$, $y = e^{-x}$, and $y = e^{-1.5x}$.



35. $y = 0.5e^{-x}$, $y = e^{-x}$, and $y = 2e^{-x}$.



36. $y = 1 - e^{-x}$ and $y = 1 - e^{-0.5x}$.



37. Because $f(0) = A = 100$ and $f(1) = 120$, we have $100e^k = 120$, and so $e^k = \frac{12}{10} = \frac{6}{5}$. Therefore, $f(x) = 100e^{kx} = 100\left(e^k\right) = 100\left(\frac{6}{5}\right)^x$.

38. Because $f(1)=5$, $Ae^{-k}=5$ and $e^{-k}=\frac{5}{A}$. Next, $f(2)=7$ and so $2Ae^{-2k}=2A\left(e^{-k}\right)^2=2A\left(\frac{5}{A}\right)^2=7$, $2A\left(\frac{25}{A^2}\right)=7$, $\frac{50}{A}=7$, and so $A=\frac{50}{7}$. Finally, $f(3)=3Ae^{-3k}=3A\left(e^{-k}\right)^3=3\left(\frac{50}{7}\right)\left(\frac{5}{\frac{50}{7}}\right)^3=7.35$.

39. $f(0)=20$ implies that $\frac{1000}{1+B}=20$, so $1000=20+20B$, or $B=\frac{980}{20}=49$. Therefore, $f(t)=\frac{1000}{1+49e^{-kt}}$. Next, $f(2)=30$, so $\frac{1000}{1+49e^{-2t}}=30$. We have $1+49e^{-2k}=\frac{1000}{30}=\frac{100}{3}$, $49e^{-2k}=\frac{100}{3}-1=\frac{97}{3}$, $e^{-2k}=\frac{97}{147}$, and finally $e^{-k}=\left(\frac{97}{147}\right)^{1/2}$. Therefore, $f(t)=\frac{1000}{1+49\left(\frac{97}{147}\right)^{t/2}}$, so $f(5)=\frac{1000}{1+49\left(\frac{97}{147}\right)^{5/2}}\approx 54.6$.

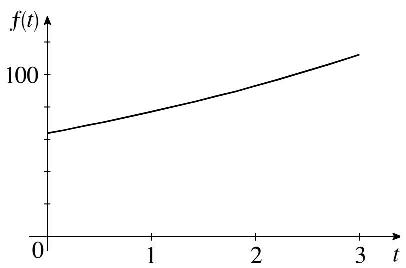
40. a. The average number of viewers in the 2011 season was $f(1)=32.744e^{-0.252(1)}\approx 25.450$, or approximately 25.450 million.

b. The average number of viewers in the 2014 season was $f(4)=32.744e^{-0.252(4)}=11.950$, or approximately 11.950 million.

41. a. $f(t)=64e^{0.188t}$.

t	0	1	2	3
$f(t)$	64	77.2	93.2	112.5

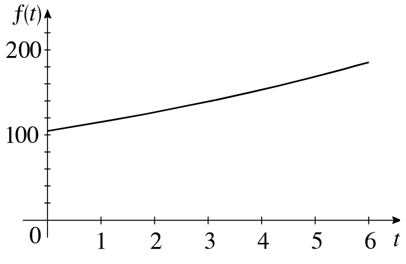
b.



42. a. $f(t)=105e^{0.095t}$.

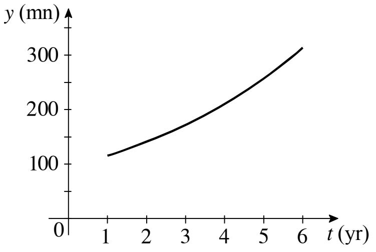
t	1	2	3	4	5	6
$f(t)$	115.5	127.0	139.6	153.5	168.8	185.7

b.



43. a. The number of internet users in 2005 was $f(1)=115.423$, or 115,423,000. In 2006, it was $f(2)=94.5e^{0.2(2)}\approx 140.977$, or 140,977,000. The number of internet users in 2010 was $f(6)=94.5e^{1.2}\approx 313.751$, or 313,751,000.

b.



44. $N(t) = \dfrac{385.474}{1 + 2.521e^{-0.214t}}$. The number of cellphone subscribers in 2000 was $N(0) = \dfrac{385.474}{1 + 2.521} \approx 109.48$, or approximately 109.5 million. The number in 2012 was $N(12) = \dfrac{385.4}{1 + 2.521e^{-0.214(12)}} \approx 322.96$, or approximately 323.0 million.

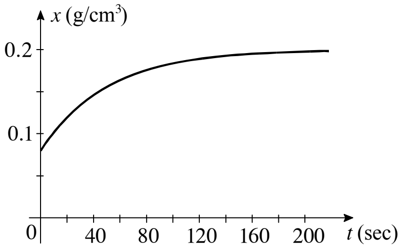
45. $N(t) = \dfrac{35.5}{1 + 6.89e^{-0.8674t}}$, so $N(6) = \dfrac{35.5}{1 + 6.89e^{-0.8674(6)}} \approx 34.2056$, or 34.21 million.

46. a. The initial concentration is given by
$x(0) = 0.08 + 0.12\left(1 - e^{-0.02 \cdot 0}\right) = 0.08$, or 0.08 g/cm^3.

b. The concentration after 20 seconds is given by
$x(20) = 0.08 + 0.12\left(1 - e^{-0.02 \cdot 20}\right) = 0.11956$, or 0.1196 g/cm^3.

c. The concentration in the long run is given by
$\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} \left[0.08 + 0.12\left(1 - e^{-0.02t}\right)\right] = 0.2$, or 0.2 g/cm^3.

d.



47. a. The initial concentration is given by $C(0) = 0.3(0) - 18\left(1 - e^{-0/60}\right)$, or 0 g/cm^3.

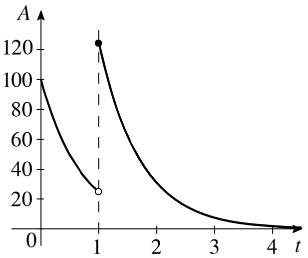
b. The concentration after 10 seconds is given by $C(10) = 0.3(10) - 18\left(1 - e^{-10/60}\right) = 0.23667$, or 0.2367 g/cm^3.

c. The concentration after 30 seconds is given by $C(30) = 18e^{-30/60} - 12e^{-(30-20)/60} = 0.75977$, or 0.7598 g/cm^3.

d. The concentration of the drug in the long run is given by $\lim\limits_{t\to\infty} C(t) = \lim\limits_{t\to\infty} \left(18e^{-t/60} - 12e^{-(t-20)/60}\right) = 0$.

48. a. The amount of drug in Jane's body immediately after the second dose is $A(1) = 100\left(1 + e^{1.4}\right)e^{-1.4(1)} = 100\left(e^{-1.4} + 1\right)$, or approximately 124.66 mg. The amount of drug in Jane's body after 2 days is $A(2) = 100\left(1 + e^{1.4}\right)e^{-1.4(2)} \approx 30.741$, or approximately 30.74 mg. The amount of drug in Jane's body in the long run is given by $\lim\limits_{t\to\infty} A(t) = \lim\limits_{t\to\infty} \left[100\left(1 + e^{1.4}\right)e^{-1.4t}\right] = 0$, or 0 mg.

b.



49. False. Take $a = b = x = 2$. Then the left-hand side is $(2 + 2)^2 = 16$, but the right-hand side is $2^2 + 2^2 = 8$.

50. True. $f(x) = e^x$ is an increasing function and so if $x < y$, then $f(x) < f(y)$, or $e^x < e^y$.

51. True. If $0 < b < 1$, then $f(x) = b^x$ is a decreasing function of x and so if $x < y$, then $f(x) > f(y)$; that is, $b^x > b^y$.

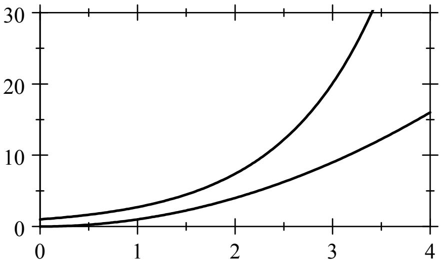
52. False. Take $k = x = -1$. Then $k < 0$ and $x < 0$, but $e^{kx} = e^1 > 1$.

53. True. If $k > 0$, then $f(x) = e^{kx} = \left(e^k\right)^x = b^x$ (where $b = e^k > 1$) and so f is increasing. If $k < 0$, then $f(x) = \left(e^k\right)^x = b^x$ (where $b = e^k < 1$) and so f is decreasing.

54. True. The functions $g(x) = x$ and $h(x) = 1 + e^x$ are both continuous on $(-\infty, \infty)$. Furthermore, $h(x) > 1 \neq 0$, so the quotient $f = g/h$ is continuous on $(-\infty, \infty)$.

Using Technology page 345

1.



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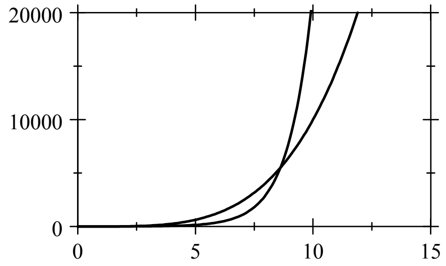

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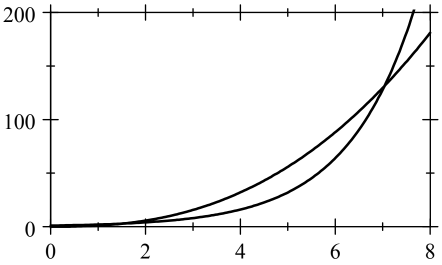
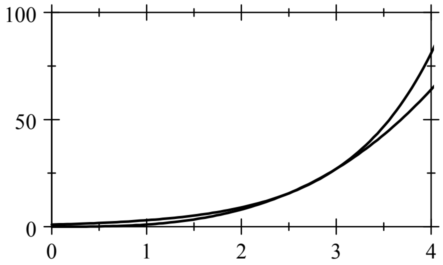


4.



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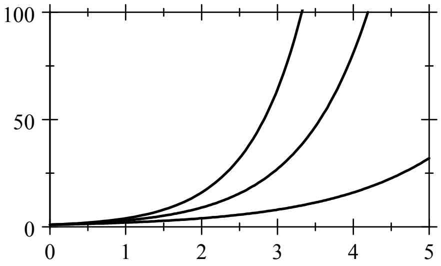

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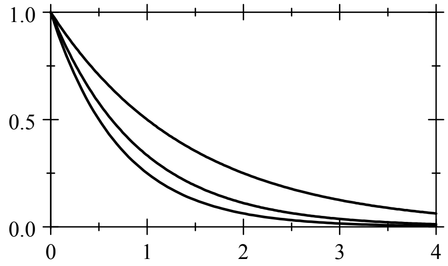

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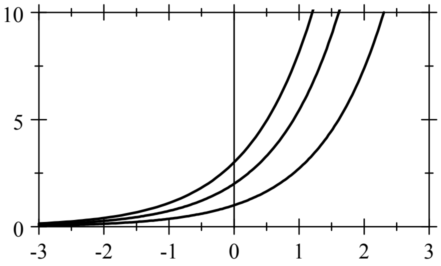

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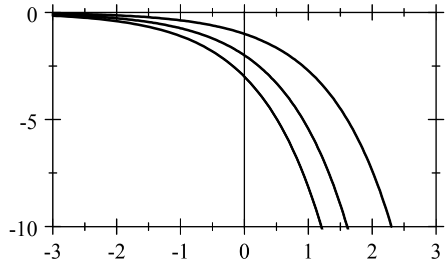

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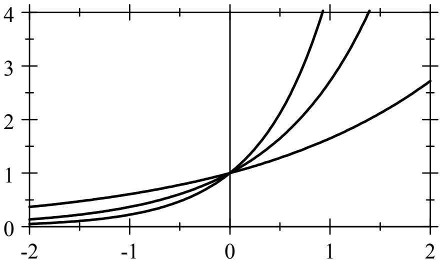

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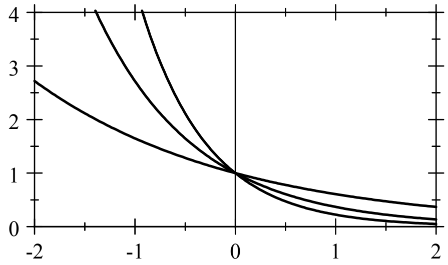
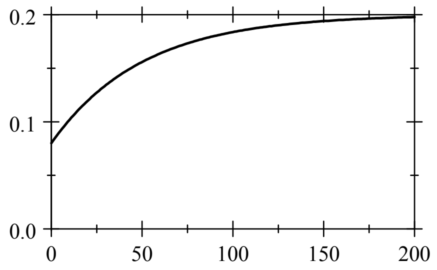


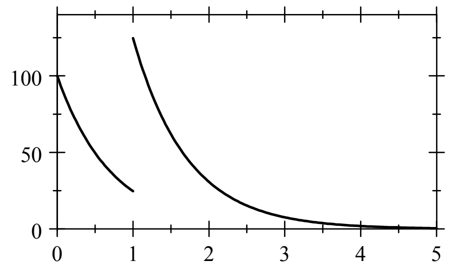
11. a.



b. 0.08 g/cm^3. **c.** 0.12 g/cm^3.

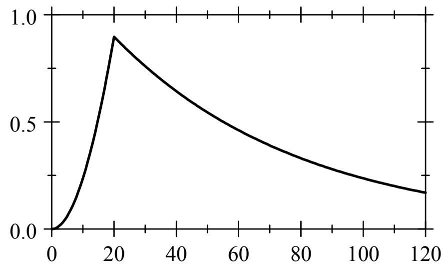
d. 0.2 g/cm^3

12. a.



b. The graph confirms the results of Exercise 48.

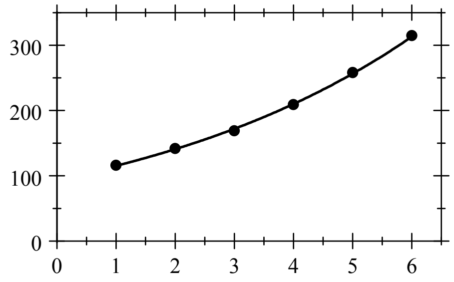
13. a.



b. 20 seconds. **c.** 35.1 seconds.

14. a. Using **ExpReg** we find $f(x) = 94.48(1.221^x) = 94.48e^{0.1997x}$.

b.



5.2 Logarithmic Functions

Concept Questions page 351

1. **a.** $y = \log_b x$ if and only if $x = b^y$.
   b. $f(x) = \log_b x$, $b > 0$, $b \neq 1$. Its domain is $(0, \infty)$.

2. **a.** $\log_b x$ has domain $(0, \infty)$ and range $(-\infty, \infty)$.
   b. Its x-intercept is 1.
   c. It is continuous on $(0, \infty)$.
   d. It is increasing on $(0, \infty)$ if $b > 1$ and decreasing on $(0, \infty)$ if $b < 1$.

3. **a.** $e^{\ln x} = x$. **b.** $\ln e^x = x$.

4. No. The domain of f is $(-\infty, \infty)$, whereas the domain of g is $(0, \infty)$.

Exercises page 351

1. $\log_2 64 = 6$.
2. $\log_3 243 = 5$.
3. $\log_4 \frac{1}{16} = -2$.
4. $\log_5 \frac{1}{125} = -3$.
5. $\log_{1/3} \frac{1}{3} = 1$.
6. $\log_{1/2} 16 = -4$.
7. $\log_{32} 16 = \frac{4}{5}$.
8. $\log_{81} 27 = \frac{3}{4}$.
9. $\log_{10} 0.001 = -3$.
10. $\log_{16} 0.5 = -\frac{1}{4}$.

11. $\log 12 = \log 4 \times 3 = \log 4 + \log 3 = 0.6021 + 0.4771 = 1.0792.$

12. $\log \frac{3}{4} = \log 3 - \log 4 = 0.4771 - 0.6021 = -0.125.$

13. $\log 16 = \log 4^2 = 2 \log 4 = 2\,(0.6021) = 1.2042.$

14. $\log \sqrt{3} = \log 3^{1/2} = \frac{1}{2} \log 3 = \frac{1}{2}\,(0.4771) = 0.2386.$

15. $\log 48 = \log \left(3 \cdot 4^2\right) = \log 3 + 2 \log 4 = 0.4771 + 2\,(0.6021) = 1.6813.$

16. $\log \frac{1}{300} = \log 1 - \log 300 = -\log 300 = -\log (3 \cdot 100) = -(\log 3 + \log 100) = -(\log 3 + 2 \log 10)$

$= -(\log 3 + 2) \approx -2.4771.$

17. $2 \ln a + 3 \ln b = \ln a^2 b^3.$

18. $\frac{1}{2} \ln x + 2 \ln y - 3 \ln z = \ln \dfrac{x^{1/2} y^2}{3z} = \ln \dfrac{\sqrt{x} y^2}{3z}.$

19. $\ln 3 + \frac{1}{2} \ln x + \ln y - \frac{1}{3} \ln z = \ln \dfrac{3\sqrt{x} y}{\sqrt[3]{z}}.$

20. $\ln 2 + \frac{1}{2} \ln (x+1) - 2 \ln \left(1 + \sqrt{x}\right) = \ln \dfrac{2\,(x+1)^{1/2}}{\left(1+\sqrt{x}\right)^2}.$

21. $\log x\,(x+1)^4 = \log x + \log (x+1)^4 = \log x + 4 \log (x+1).$

22. $\log x \left(x^2+1\right)^{-1/2} = \log x - \frac{1}{2} \log \left(x^2+1\right).$

23. $\log \dfrac{\sqrt{x+1}}{x^2+1} = \log (x+1)^{1/2} - \log \left(x^2+1\right) = \frac{1}{2} \log (x+1) - \log \left(x^2+1\right).$

24. $\ln \dfrac{e^x}{1+e^x} = x - \ln (1+e^x).$

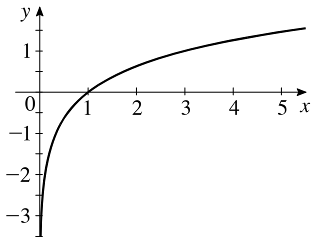
25. $\ln x e^{-x^2} = \ln x - x^2.$

26. $\ln x\,(x+1)\,(x+2) = \ln x + \ln (x+1) + \ln (x+2).$

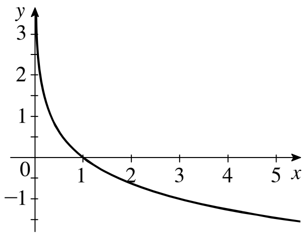
27. $\ln \left(\dfrac{x^{1/2}}{x^2\sqrt{1+x^2}}\right) = \ln x^{1/2} - \ln x^2 - \ln \left(1+x^2\right)^{1/2} = \frac{1}{2} \ln x - 2 \ln x - \frac{1}{2} \ln \left(1+x^2\right) = -\frac{3}{2} \ln x - \frac{1}{2} \ln \left(1+x^2\right).$

28. $\ln \dfrac{x^2}{\sqrt{x}\,(1+x)^2} = 2 \ln x - \frac{1}{2} \ln x - 2 \ln (1+x) = \frac{3}{2} \ln x - 2 \ln (1+x).$

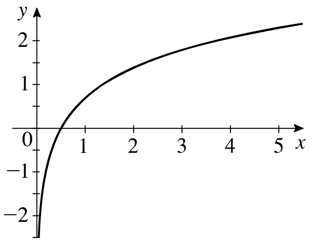
29. $y = \log_3 x$.



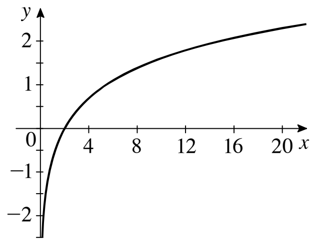
30. $y = \log_{1/3} x$.



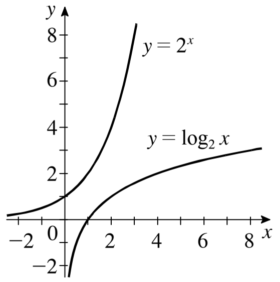
31. $y = \ln 2x$.



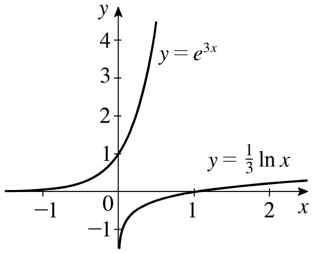
32. $y = \ln \frac{1}{2}x$.



33. $y = 2^x$ and $y = \log_2 x$.



34. $y = e^{3x}$ and $y = \frac{1}{3} \ln x$.



35. $e^{0.4t} = 8$, so $0.4t \ln e = \ln 8$ and thus $0.4t = \ln 8$ because $\ln e = 1$. Therefore, $t = \dfrac{\ln 8}{0.4} \approx 5.1986$.

36. $\frac{1}{3}e^{-3t} = 0.9$, $e^{-3t} = 2.7$. Taking the logarithm, we have $-3t \ln e = \ln 2.7$, so $t = -\dfrac{\ln 2.7}{3} \approx -0.3311$.

37. $5e^{-2t} = 6$, so $e^{-2t} = \frac{6}{5} = 1.2$. Taking the logarithm, we have $-2t \ln e = \ln 1.2$, so $t = -\dfrac{\ln 1.2}{2} \approx -0.0912$.

38. $4e^{t-1} = 4$, so $e^{t-1} = 1$, $\ln e^{t-1} = \ln 1$, $(t-1)\ln e = 0$, and $t = 1$.

39. $2e^{-0.2t} - 4 = 6$, so $2e^{-0.2t} = 10$. Taking the logarithm on both sides of this last equation, we have $\ln e^{-0.2t} = \ln 5$, $-0.2t \ln e = \ln 5$, $-0.2t = \ln 5$, and and $t = -\frac{\ln 5}{0.2} \approx -8.0472$.

40. $12 - e^{0.4t} = 3$, $e^{0.4t} = 9$, $\ln e^{0.4t} = \ln 9$, $0.4t \ln e = \ln 9$, and $0.4t = \ln 9$, so $t = \frac{\ln 9}{0.4} \approx 5.4931$.

41. $\dfrac{50}{1 + 4e^{0.2t}} = 20$, so $1 + 4e^{0.2t} = \dfrac{50}{20} = 2.5$, $4e^{0.2t} = 1.5$, $e^{0.2t} = \dfrac{1.5}{4} = 0.375$, $\ln e^{0.2t} = \ln 0.375$, and $0.2t = \ln 0.375$. Thus, $t = \dfrac{\ln 0.375}{0.2} \approx -4.9041$.

42. $\dfrac{200}{1 + 3e^{-0.3t}} = 100$, so $1 + 3e^{-0.3t} = \frac{200}{100} = 2$, $3e^{-0.3t} = 1$, $e^{-0.3t} = \frac{1}{3}$, and $\ln e^{-0.3t} = \ln \frac{1}{3} = \ln 1 - \ln 3 = -\ln 3$. Thus, $-0.3t \ln e = -\ln 3$, so $0.3t = \ln 3$. Therefore, $t = \dfrac{\ln 3}{0.3} \approx 3.6620$.

43. Taking logarithms of both sides, we obtain $\ln A = \ln Be^{-t/2}$, $\ln A = \ln B + \ln e^{-t/2}$, and $\ln A - \ln B = -\dfrac{t}{2} \ln e$, so $\ln \dfrac{A}{B} = -\dfrac{t}{2}$ and $t = -2 \ln \dfrac{A}{B} = 2 \ln \dfrac{B}{A}$.

44. $\dfrac{A}{1+Be^{t/2}}=C$, $A=C+BCe^{t/2}$, $A-C=BCe^{t/2}$, $\dfrac{A-C}{BC}=e^{t/2}$, and $\dfrac{t}{2}=\ln\dfrac{A-C}{BC}$, so $t=2\ln\left(\dfrac{A-C}{BC}\right)$.

45. $f(1)=2$, so $a+b(0)=2$. Thus, $a=2$. Therefore, $f(x)=2+b\ln x$. We calculate $f(2)=4$, so $2+b\ln 2=4$. Solving for b, we obtain $b=\dfrac{2}{\ln 2}\approx 2.8854$, so $f(x)=2+2.8854\ln x$.

46. a. The average life expectancy in 1907 was $W(1)=49.9$ years.

b. The average life expectancy in 2027 will be $W(7)=49.9+17.1\ln 7\approx 83.2$ years.

47. $p(x)=19.4\ln x+18$. For a child weighing 92 lb, we find $p(92)=19.4\ln 92+18\approx 105.7$, or approximately 106 millimeters of mercury.

48. a. $5=\log\dfrac{I}{I_0}$, so $\dfrac{I}{I_0}=10^5$ and $I=10^5 I_0=100{,}000I_0$.

b. $8=\log\dfrac{I}{I_0}$, from which we find $I=10^8 I_0$. Thus, it is 1000 times greater.

c. $7.0=\log\dfrac{I}{I_0}$ gives $I=10^{7.0}I_0$. So it is $\dfrac{10^{7.0}}{10^5}=10^2$, or 100 times greater than one with magnitude 5.

49. a. $30=10\log\dfrac{I}{I_0}$, so $3=\log\dfrac{I}{I_0}$, and $\dfrac{I}{I_0}=10^3=1000$. Thus, $I=1000I_0$.

b. When $D=80$, $I=10^8 I_0$ and when $D=30$, $I=10^3 I_0$. Therefore, an 80-decibel sound is $10^8/10^3=10^5=100{,}000$ times louder than a 30-decibel sound.

c. It is $10^{15}/10^8=10^7=10{,}000{,}000$ times louder.

50. We solve the equation $29.92e^{-0.2x}=20$, obtaining $e^{-0.2x}=\dfrac{20}{29.92}=0.6684$, $-0.2x=\ln 0.6684$, and $x=-\dfrac{\ln 0.6684}{0.2}\approx 2.01$. Thus, the balloonist's altitude is 2.01 miles.

51. a. The temperature when it was first poured is given by $T(0)=70+100e^0=170$, or 170°F.

b. We solve the equation $70+100e^{-0.0446t}=120$; $100e^{-0.0446t}=50$, obtaining $e^{-0.0446t}=\frac{50}{100}=\frac{1}{2}$, $\ln e^{-0.0446t}=\ln\frac{1}{2}=\ln 1-\ln 2=-\ln 2$, $-0.0446t=-\ln 2$, and so $t=\dfrac{\ln 2}{0.0446}\approx 15.54$. Thus, it will take approximately 15.54 minutes.

52. We solve the equation $\dfrac{160}{1+240e^{-0.2t}}=80$ for t, obtaining $1+240e^{-0.2t}=\dfrac{160}{80}$, $240e^{-0.2t}=2-1=1$, $e^{-0.2t}=\dfrac{1}{240}$, $-0.2t=\ln\dfrac{1}{240}$, and $t=-\dfrac{1}{0.2}\ln\dfrac{1}{240}\approx 27.40$, or approximately 27.4 years old.

53. When $f(t)=40$, we have $\dfrac{46.5}{1+2.324e^{-0.05113t}}=40$, so $1+2.324e^{-0.05113t}=\dfrac{46.5}{40}$, $2.324e^{-0.05113t}=\dfrac{46.5}{40}-1=0.1625$, $e^{-0.05113t}=\dfrac{0.1625}{2.324}$, $-0.05113t=\ln\left(\dfrac{0.1625}{2.324}\right)$, and $t\approx 52.03$. Thus, the percentage of obese adults will reach 40% around 2022.

54. We solve the equation $200\left(1-0.956e^{-0.18t}\right)=140$ for t, obtaining $1-0.956e^{-0.18t}=\frac{140}{200}=0.7$, $-0.956e^{-0.18t}=0.7-1=-0.3$, $e^{-0.18t}=\frac{0.3}{0.956}$, $-0.18t=\ln\left(\frac{0.3}{0.956}\right)$, and finally $t=-\dfrac{\ln\left(\frac{0.3}{0.956}\right)}{0.18}\approx 6.43875$. Thus, it is approximately 6.4 years old.

55. a. We solve the equation $0.08+0.12e^{-0.02t}=0.18$, obtaining $0.12e^{-0.02t}=0.1$, $e^{-0.02t}=\frac{0.1}{0.12}=\frac{1}{1.2}$, $\ln e^{-0.02t}=\ln\frac{1}{1.2}=\ln 1-\ln 1.2=-\ln 1.2$, $-0.02t=-\ln 1.2$, and $t=\frac{\ln 1.2}{0.02}\approx 9.116$, or approximately 9.1 seconds.

b. We solve the equation $0.08+0.12e^{-0.02t}=0.16$, obtaining $0.12e^{-0.02t}=0.08$, $e^{-0.02t}=\frac{0.08}{0.12}=\frac{2}{3}$, $-0.02t=\ln\frac{2}{3}$, and $t=-\frac{1}{0.02}\ln\frac{2}{3}\approx 20.2733$, or approximately 20.3 seconds.

56. a. We solve the equation $0.08\left(1-e^{-0.02t}\right)=0.02$, obtaining $1-e^{-0.02t}=\frac{0.02}{0.08}=\frac{1}{4}$, $-e^{-0.02t}=\frac{1}{4}-1=-\frac{3}{4}$, $e^{-0.02t}=\frac{3}{4}$, $\ln e^{-0.02t}=\ln\frac{3}{4}$, $-0.02t=\ln\frac{3}{4}$, and so $t\approx 14.38$, or 14.38 seconds.

b. $1-e^{-0.02t}=\frac{0.04}{0.08}$, so $-e^{-0.02t}=\frac{1}{2}-1=-\frac{1}{2}$, $e^{-0.02t}=0.5$, and $t=-\frac{\ln 0.5}{0.02}\approx 34.66$, or 34.66 seconds.

57. With $T_0=70$, $T_1=98.6$, and $T=80$, we have $80=70+(98.6-70)(0.97)^t$, so $28.6(0.97)^t=10$ and $(0.97)^t=0.34965$. Taking logarithms, we have $\ln(0.97)^t=\ln 0.34965$, or $t=\dfrac{\ln 0.34965}{\ln 0.97}\approx 34.50$. Thus, he was killed $34\frac{1}{2}$ hours earlier, at 1:30 p.m.

58. a. Solving the given demand equation $p=50\ln\dfrac{50}{x}$ for x in terms of p, we find $\ln\left(\dfrac{50}{x}\right)=\dfrac{p}{50}$, so $\dfrac{50}{x}=e^{p/50}$ and $x=f(p)=50e^{-p/50}=50e^{-0.02p}$ for $p>0$. Next, we find $f'(p)=-e^{-0.02p}$, and so $E(p)=-\dfrac{pf'(p)}{f(p)}=\dfrac{pe^{-0.02p}}{50e^{-0.02p}}=\dfrac{p}{50}$. Now $E(p)<1$ if and only if $\dfrac{p}{50}<1$ or $p<50$, and similarly $E(p)=1$ when $p=50$ and $f(p)>1$ if $p>50$. Thus, demand is inelastic if $0<p<50$, unitary if $p=50$, and elastic if $p>50$.

b. Since demand is unitary at $p=50$, we see that at that price, a slight increase in the unit price will not affect revenue.

59. False. Take $x=e$. Then $(\ln e)^3=1^3=1\neq 3\ln e=3$.

60. False. Take $a=b=1$. Then $\ln(a+b)=\ln(1+1)=\ln 2\neq\ln a+\ln b=\ln 1+\ln 1=0$.

61. True. $e^{\ln b}=b$ and $\ln e^b=b$ as well.

62. False. Take $a=2e$ and $b=e$. Then $\ln a-\ln b=\ln 2e-\ln e=\ln 2+\ln e-\ln e=\ln 2$, But $\ln(a-b)=\ln(2e-e)=\ln e=1$.

63. True. $g(x)=\ln x$ is continuous and greater than zero on $(1,\infty)$. Therefore, $f(x)=\dfrac{1}{\ln x}$ is continuous on $(1,\infty)$.

64. True. If $a=\log_2 3$, then $3=2^a$ and $\ln 3=\ln 2^a=a\ln 2$, so $a=\dfrac{\ln 3}{\ln 2}$. Similarly, if $b=\log_3 2$, then $2=3^b$, $\ln 2=b\ln 3$, and $b=\dfrac{\ln 2}{\ln 3}$. Therefore, $ab=\left(\log_2 3\right)\left(\log_3 2\right)=\dfrac{\ln 3}{\ln 2}\cdot\dfrac{\ln 2}{\ln 3}=1$.

65. a. Taking logarithms of both sides gives $\ln 2^x = \ln e^{kx}$, so $x \ln 2 = kx (\ln e) = kx$. Thus, $x (\ln 2 - k) = 0$ for all x, and this implies that $k = \ln 2$.

b. Proceeding as in part (a), we find that $k = \ln b$.

66. a. Let $p = \log_b m$ and $q = \log_b n$, so that $m = b^p$ and $n = b^q$. Then $mn = b^p b^q = b^{p+q}$ and by definition, $p + q = \log_b mn$; that is, $\log_b mn = \log_b m + \log_b n$.

b. $\dfrac{m}{n} = \dfrac{b^p}{b^q} = b^{p-q}$, so by definition, $p - q = \log_b \dfrac{m}{n}$; that is, $\log_b \dfrac{m}{n} = \log_b m - \log_b n$.

67. Let $\log_b m = p$. Then $m = b^p$. Therefore, $m^n = (b^p)^n = b^{np}$, and so $\log_b m^n = \log_b b^{np} = np \log_b b = np$ (since $\log_b b = 1$) $= n \log_b m$.

68. a. By definition, $\log_b 1 = 0$ means $1 = b^0 = 1$.

b. By definition, $\log_b b = 1$ means $b = b^1 = b$.

5.3 Compound Interest

Concept Questions page 365

1. a. When simple interest is computed, the interest earned is based on the original principal. When compound interest is computed, the interest earned is periodically added to the principal and thereafter earns interest at the same rate.

b. The simple interest formula is $A = P (1 + rt)$ and the compound interest formula is $A = P \left(1 + \dfrac{r}{m}\right)^{mt}$.

2. a. The effective rate of interest is the simple interest that would produce the same amount in 1 year as the nominal rate compounded m times per year.

b. $r_{\text{eff}} = \left(1 + \dfrac{r}{m}\right)^m - 1$.

3. $P = A \left(1 + \dfrac{r}{m}\right)^{-mt}$.

4. $A = Pe^{rt}$.

Exercises page 365

1. $A = 2500 \left(1 + \dfrac{0.04}{2}\right)^{20} = 3714.87$, or \$3714.87.

2. $A = 12{,}000 \left(1 + \dfrac{0.05}{4}\right)^{40} = 19{,}723.43$, or \$19,723.43.

3. $A = 150{,}000 \left(1 + \dfrac{0.06}{12}\right)^{48} = 190{,}573.37$, or \$190,573.37.

4. $A = 150{,}000 \left(1 + \dfrac{0.04}{12}\right)^{1095} = 169{,}123.42$, or \$169,123.42.

5. **a.** Using the formula $r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$ with $r = 0.06$ and $m = 2$, we have $r_{\text{eff}} = \left(1 + \frac{0.06}{2}\right)^2 - 1 = 0.0609$, or 6.09%/yr.

   b. Using the formula $r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$ with $r = 0.05$ and $m = 4$, we have $r_{\text{eff}} = \left(1 + \frac{0.05}{4}\right)^4 - 1 = 0.05095$, or 5.095%/yr.

6. **a.** Using the formula $r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$ with $r = 0.045$ and $m = 12$, we have $r_{\text{eff}} = \left(1 + \frac{0.045}{12}\right)^{12} - 1 = 0.04594$, or 4.6%/yr.

   b. The effective rate is given by $R = \left(1 + \frac{0.045}{365}\right)^{365} - 1 = 0.04602$, or 4.602%/yr.

7. **a.** The present value is given by $P = 40{,}000\left(1 + \frac{0.05}{2}\right)^{-8} = 32{,}829.86$, or \$32,829.86.

   b. The present value is given by $P = 40{,}000\left(1 + \frac{0.05}{4}\right)^{-16} = 32{,}789.85$, or \$32,789.85.

8. **a.** The present value is given by $P = 40{,}000\left(1 + \frac{0.04}{12}\right)^{-48} = 34{,}094.82$, or \$34,094.82.

   b. The present value is given by $P = 40{,}000\left(1 + \frac{0.06}{365}\right)^{-(365)(4)} = 31{,}465.74$, or \$31,465.74.

9. $A = 5000e^{0.05(4)} \approx 6107.01$, or \$6107.01.

10. $A = 25000\,(1 + 0.04)^6 \approx 31{,}632.98$, or approximately \$31,632.98. The interest earned is \$6632.98.

11. We use Formula (8) with $A = 10{,}000$, $m = 365$, $r = 0.04$, and $t = 2$. The required deposit is $P = 10{,}000\left(1 + \frac{0.04}{365}\right)^{-365(2)} \approx 9231.20$, or \$9231.20.

12. We use Formula (8) with $A = 15{,}000$, $m = 365$, $r = 0.05$, and $t = 3$. The initial deposit is $P = 15{,}000\left(1 + \frac{0.05}{365}\right)^{-365(3)} \approx 12{,}910.752$, or \$12,910.75.

13. We use Formula (11) with $A = 20{,}000$, $r = 0.06$, and $t = 3$. Jack should deposit $P = 20{,}000e^{-(0.06)(3)} \approx 16{,}705.404$, or \$16,705.40.

14. We use Formula (11) with $A = 12{,}000$, $r = 0.06$, and $t = 2$. Diego's deposit is $P = 12{,}000e^{-(0.06)(2)} \approx 10{,}643.045$, or \$10,643.05.

15. $P = Ae^{-rt} = 59{,}673e^{-(0.06)5} \approx 44{,}206.85$, or approximately \$44,206.85.

16. We use Formula (6) with $A = 7500$, $P = 5000$, $m = 4$, and $t = 3$. Thus, $7500 = 5000\left(1 + \frac{r}{4}\right)^{12}$, so $\left(1 + \frac{r}{4}\right)^{12} = \frac{7500}{5000} = \frac{3}{2}$, $\ln\left(1 + \frac{r}{4}\right)^{12} = \ln 1.5$, $12\ln\left(1 + \frac{r}{4}\right) = \ln 1.5$, $\ln\left(1 + \frac{r}{4}\right) = \frac{\ln 1.5}{12} \approx 0.0337888$, $1 + \frac{r}{4} \approx e^{0.0337888} \approx 1.034366$, $\frac{r}{4} \approx 0.034366$, and $r \approx 0.137464$. The required annual interest rate is 13.75%.

17. We use Formula (6) with $A = 7500$, $P = 5000$, $m = 12$, and $t = 3$. Thus, $7500 = 5000\left(1 + \frac{r}{12}\right)^{36}$, $\left(1 + \frac{r}{12}\right)^{36} = \frac{7500}{5000} = \frac{3}{2}$, $\ln\left(1 + \frac{r}{12}\right)^{36} = \ln 1.5$, $36\ln\left(1 + \frac{r}{12}\right) = \ln 1.5$, $\ln\left(1 + \frac{r}{12}\right) = \frac{\ln 1.5}{36} = 0.0112629$, $1 + \frac{r}{12} = e^{0.0112629} = 1.011327$, $\frac{r}{12} = 0.011327$, and $r = 0.13592$. The annual interest rate is 13.59%.

18. We use Formula (6) with $A = 8000$, $P = 5000$, $m = 2$, and $t = 4$. Thus, $8000 = 5000\left(1 + \frac{r}{2}\right)^{8}$, $\left(1 + \frac{r}{2}\right)^{8} = \frac{8000}{5000} = 1.6$, $\ln\left(1 + \frac{r}{2}\right)^{8} = \ln 1.6$, $8\ln\left(1 + \frac{r}{2}\right) = \ln 1.6$, $\ln\left(1 + \frac{r}{2}\right) = \frac{\ln 1.6}{8} = 0.05875$, $1 + \frac{r}{2} = e^{0.05875} = 1.06051$; $\frac{r}{2} = 0.06051$, and so $r = 0.1210$. The required annual interest rate is 12.1%.

19. We use Formula (6) with $A = 5500$, $P = 5000$, $m = 12$, and $t = \frac{1}{2}$. Thus, $5500 = 5000\left(1 + \frac{r}{12}\right)^{6}$, and so $\left(1 + \frac{r}{12}\right)^{6} = \frac{5500}{5000} = 1.1$. Proceeding as in the previous exercise, we find $r = 0.1921$, so the required annual interest rate is 19.21%.

20. We use Formula (6) with $A = 4000$, $P = 2000$, $m = 1$, and $t = 5$. Thus, $4000 = 2000(1 + r)^{5}$, $(1 + r)^{5} = 2$, $5\ln(1 + r) = \ln 2$, $\ln(1 + r) = \frac{\ln 2}{5} \approx 0.138629$, $1 + r \approx e^{0.138629} \approx 1.148698$, and so $r \approx 0.1487$. The required annual interest rate is 14.87%.

21. We use Formula (6) with $A = 6000$, $P = 2000$, $m = 12$, and $t = 5$. Thus, $6000 = 2000\left(1 + \frac{r}{12}\right)^{60}$. Thus, $\left(1 + \frac{r}{12}\right)^{60} = 3$, $60\ln\left(1 + \frac{r}{12}\right) = \ln 3$, $\ln\left(1 + \frac{r}{12}\right) = \frac{\ln 3}{60}$, $1 + \frac{r}{12} = e^{(\ln 3)/60}$, $\frac{r}{12} = e^{(\ln 3)/60} - 1$, and $r = 12\left(e^{(\ln 3)/60} - 1\right) \approx 0.2217$, so the required interest rate is 22.17% per year.

22. We use Formula (6) with $A = 15000$, $P = 12000$, $m = 12$, and $r = 0.05$. Thus, $15000 = 12000\left(1 + \frac{0.05}{12}\right)^{12t}$, $\left(1 + \frac{0.05}{12}\right)^{12t} = \frac{15{,}000}{12{,}000} = 1.25$, $12t\ln\left(1 + \frac{0.05}{12}\right) = \ln 1.25$, and so $t = \dfrac{\ln 1.25}{12\ln\left(1 + \frac{0.05}{12}\right)} \approx 4.47$. Thus, it will take approximately 4.5 years.

23. We use Formula (6) with $A = 6500$, $P = 5000$, $m = 12$, and $r = 0.06$. Thus, $6500 = 5000\left(1 + \frac{0.06}{12}\right)^{12t}$, $(1.005)^{12t} = \frac{6500}{5000} = 1.3$, $12t\ln(1.005) = \ln 1.3$, and so $t = \dfrac{\ln 1.3}{12\ln 1.005} \approx 4.384$. It will take approximately 4.4 years.

24. We use Formula (6) with $A = 4000$, $P = 2000$, $m = 12$, and $r = 0.06$. Thus, $4000 = 2000\left(1 + \frac{0.06}{12}\right)^{12t}$, $\left(1 + \frac{0.06}{12}\right)^{12t} = 2$, $12t\ln\left(1 + \frac{0.06}{12}\right) = \ln 2$, and so $t = \dfrac{\ln 2}{12\ln\left(1 + \frac{0.06}{12}\right)} \approx 11.58$. It will take approximately 11.6 years.

25. We use Formula (6) with $A = 15000$, $P = 5000$, $m = 365$, and $r = 0.04$. Thus, $15{,}000 = 5000\left(1 + \dfrac{0.04}{365}\right)^{365t}$, from which we find $t = \dfrac{\ln\left(\frac{15{,}000}{5000}\right)}{365\ln\left(1 + \frac{0.04}{365}\right)} \approx 27.47$. Thus, it will take approximately 27.5 years.

26. We use Formula (10) with $A = 6000$, $P = 5000$, and $t = 3$. Thus, $6000 = 5000e^{3r}$, $e^{3r} = \frac{6000}{5000} = 1.2$, $3r = \ln 1.2$, and $r = \frac{1}{3}\ln 1.2 \approx 0.06077$. The annual interest rate is 6.08%.

27. We use Formula (10) with $A = 8000$, $P = 4000$, and $t = 5$. Thus, $8000 = 4000e^{5r}$, $e^{5r} = \frac{8000}{4000} = 2$, $5r = \ln 2$, and $r = \dfrac{\ln 2}{5} \approx 0.13863$. The annual interest rate is 13.86%.

28. We use Formula (10) with $A = 7000$, $P = 6000$, and $r = 0.075$. Thus, $7000 = 6000e^{0.075t}$, $e^{0.075t} = \frac{7000}{6000} = \frac{7}{6}$, $0.075t \ln e = \ln \frac{7}{6}$, and so $t = \dfrac{\ln \frac{7}{6}}{0.075} \approx 2.055$. It will take 2.06 years.

29. We use Formula (10) with $A = 16{,}000$, $P = 8000$, and $r = 0.05$. Thus, $16{,}000 = 8000e^{0.05t}$, and we find that $t = \frac{\ln 2}{0.05} \approx 13.863$. It will take 13.9 years.

30. The Estradas can expect to pay $180{,}000\,(1 + 0.04)^4$, or approximately \$210,575.

31. The utility company will have to increase its generating capacity by a factor of $(1.08)^{10} \approx 2.16$.

32. The investment will be worth $A = 1.5\left(1 + \dfrac{0.025}{2}\right)^{20} \approx 1.9231$, or approximately \$1.9231 million.

33. After 1 year, Maria's investment is worth $(1.2)\,(10{,}000)$ dollars and after 2 years, it is worth $(1.1)\,(1.2)\,(10{,}000)$ dollars.
After 1 year, Laura's investment is worth $(1.1)\,(10{,}000)$ dollars and after 2 years, it is worth $(1.2)\,(1.1)\,(10{,}000)$ dollars.
So after 2 years, both investments are worth the same amount, namely \$13,200.

34. The value of Alan's stock portfolio after 1 year is $(1.2)\,P$ dollars, where P is the original amount invested. Its value after 2 years is $(1.1)\,(1.2)\,P$; after 3 years, it is $(0.9)\,(1.1)\,(1.2)\,P$; and finally after 4 years, it is $(0.8)\,(0.9)\,(1.1)\,(1.2)\,P$ or $0.9504P$ dollars. Thus, the value of Alan's stock portfolio after 4 years is less than the its initial value.

35. Suppose Jack's portfolio is worth $\$P$ initially. After 1 year, it is worth $(0.8)\,P$ dollars, and after 2 years, it is worth $(1.2)\,(0.8)\,P$ or $0.96P$ dollars. This shows that after the second year, Jack's investment has not recouped all of its losses from the first year.

36. Suppose Arabella's stock portfolio is worth $\$P$ initially. Then after 1 year, it is worth $(0.8)\,P$ dollars. Let r denote the annual rate (compounded annually) which the portfolio must earn in the second year in order to regain its original value at the end of the third year. Then $(1 + r)^2\,(0.8)\,P = P$, so $(1 + r)^2 = \frac{1}{0.8}$, $1 + r = \sqrt{\frac{1}{0.8}} \approx 1.1180$, and $r \approx 0.1180$. The required rate is thus approximately 11.8% per year.

37. We use Formula 3 with $P = 15{,}000$, $r = 0.078$, $m = 12$, and $t = 4$, giving the worth of Jodie's account as $A = 15{,}000\left(1 + \frac{0.078}{12}\right)^{(12)(4)} \approx 20{,}471.641$, or approximately \$20,471.64.

38. If the money earns interest at the rate of 6% compounded annually, he receives $A = (1.06)^{21}\,(10{,}000) \approx 33{,}995.64$, or \$33,995.64. If the money earns interest at the rate of 6% compounded quarterly, he receives $A = \left(1 + \frac{0.06}{4}\right)^{4(21)}(10{,}000) \approx 34{,}925.90$, or \$34,925.90. If the money earns interest at the rate of 6% compounded monthly, he receives $A = \left(1 + \frac{0.06}{12}\right)^{12(21)}(10{,}000) \approx 35{,}143.71$, or \$35,143.71.

39. We use Formula 3 with $P = 10{,}000$, $r = 0.0682$, $m = 4$, and $t = \frac{11}{2}$, giving the worth of Chris' account as $A = 10{,}000\left(1 + \frac{0.0682}{4}\right)^{(4)(11/2)} \approx 14{,}505.433$, or approximately \$14,505.43.

40. a. The accumulated amount before taxes is $A = 25{,}000\left(1 + \frac{0.06}{1}\right)^{10} \approx 44{,}771.19$. After taxes, it is worth \$33,235.26.

b. The accumulated tax-free amount is $A = 25{,}000\,(1 + 0.0432)^{10} \approx 38{,}160.65$, or \$38,160.65.

41. He can expect the minimum revenue for 2016 to be $240{,}000\,(1.2)\,(1.3)\,(1.25)^3 \approx 731{,}250$, or \$731,250.

42. The projected online sales for 2009 are $141.4\,(1.243)\,(1.14)\,(1.305)\,(1.176)\,(1.105) \approx 339.79$, or approximately \$339.79 billion.

43. We want the value of a 2013 dollar at the beginning of 2009. Denoting this value by x, we have $(1.027)\,(1.015)\,(1.030)\,(1.017)\,x = 1$, so $x \approx 0.916$. Thus, the purchasing power is approximately 92 cents.

44. a. If they invest the money at 4.6% compounded quarterly, they should set aside $P = 120{,}000\left(1 + \frac{0.046}{4}\right)^{-28} \approx 87{,}123.7$, or \$87,123.70.

b. If they invest the money at 4.6% compounded continuously, they should set aside $P = 120{,}000e^{-(0.046)(7)} \approx 86{,}963.8$, or \$86, 963.80.

45. He needs $65{,}000e^{0.03(10)} \approx 87{,}740.82$ or approximately \$87,740.82 annually.

46. Bernie originally invested $P = 22{,}289.22\left(1 + \dfrac{0.03}{4}\right)^{-20} \approx 19{,}195.25$, or \$19,195.25.

47. The present value of the \$8000 loan due in 3 years is given by $P = 8000\left(1 + \dfrac{0.08}{2}\right)^{-6} \approx 6322.52$, or \$6322.52.

The present value of the \$15,000 loan due in 6 years is given by $P = 15{,}000\left(1 + \dfrac{0.08}{2}\right)^{-12} \approx 9368.96$, or \$9368.96.

Therefore, the amount the proprietors of the inn will be required to pay at the end of 5 years is given by $A = 15{,}691.48\left(1 + \dfrac{0.08}{2}\right)^{10} \approx 23{,}227.22$, or \$23,227.22.

48. a. If inflation over the next 15 years is 3%, then the first year of Eleni's pension will be worth $P_3 = 40{,}000e^{-0.03(15)} = 25{,}505.13$, or \$25,505.13.

b. If inflation over the next 15 years is 4%, then the first year of Eleni's pension will be worth $P_4 = 40{,}000e^{-0.04(15)} = 21{,}952.47$, or \$21,952.47.

c. If inflation over the next 15 years is 6%, then the first year of Eleni's pension will be worth $P_6 = 40{,}000e^{-0.06(15)} = 16{,}262.79$, or \$16,262.79.

49. $P(t) = V(t)\,e^{-rt} = 80{,}000e^{\sqrt{t}/2}e^{-rt} = 80{,}000e^{(\sqrt{t}/2)-0.09t}$. Thus, $P(4) = 80{,}000e^{1-0.09(4)} \approx 151{,}718.47$, or approximately \$151,718.

50. **a.** Using Formula (11) with $A = V(t)$ and $r = 0.08$, we find $P(t) = V(t)\,e^{-0.08t} = 500{,}000e^{-0.08t+0.5t^{0.4}}$, $0 \le t \le 8$.

b.

t	4	5	6
$P(t)$	\$867,104	\$868,211	\$861, 301

From the table, we see that the present value of the mall seems to decrease after a certain period of growth. We see that sometime between $t = 5$ and $t = 6$, the present value of the mall attains its highest market value, at least \$868,211.

51. Suppose \$1 is invested in each investment. The accumulated amount in investment A is $\left(1 + \frac{0.08}{2}\right)^8 \approx 1.36857$ and the accumulated amount in investment B is $e^{0.0775(4)} \approx 1.36343$. Thus, investment A has a higher rate of return.

52. We solve the equation $366{,}000 = 300{,}000(1 + r)^6$, or $(1 + r)^6 = 1.22$, obtaining $1 + r = (1.22)^{1/6} \approx 1.0337$, and so $r \approx 0.037$, or 3.37%.

53. The effective annual rate of return on his investment is found by solving the equation $(1 + r)^2 = \frac{32{,}100}{25{,}250}$. We find $1 + r = \left(\frac{32{,}100}{25{,}250}\right)^{1/2}$, so $1 + r \approx 1.1275$, and $r \approx 0.1275$, or 12.75%.

54. We solve the equation $3.6 = 1.4e^{6r}$, finding $e^{6r} = \frac{3.6}{1.4}$, $6r \ln e \approx \ln \frac{3.6}{1.4}$, $6r \approx 0.944462$, and so $r \approx 0.1574$, or approximately 15.7%.

55. $r_{\text{eff}} = \lim_{m\to\infty} \left(1 + \dfrac{r}{m}\right)^m - 1 = e^r - 1$.

56. **a.** $r_{\text{eff}} = \left(1 + \frac{0.1}{4}\right)^4 - 1 \approx 0.1038$, or 10.38%.

b. $r_{\text{eff}} = \left(1 + \frac{0.1}{12}\right)^{12} - 1 \approx 0.1047$, or 10.47%.

c. $r_{\text{eff}} = e^{0.1} - 1 \approx 0.1052$, or 10.52%.

57. The effective rate of interest at Bank A is given by $R = \left(1 + \frac{0.07}{4}\right)^4 - 1 = 0.07186$, or 7.186%. The effective rate at Bank B is given by $R = e^r - 1 = e^{0.07125} - 1 = 0.07385$, or 7.385%. We conclude that Bank B has the higher effective rate of interest.

58. $\left(1 + \frac{r}{12}\right)^{12} - 1 = r_{\text{eff}}$, $\left(1 + \frac{r}{12}\right)^{12} = 1.06$, $1 + \frac{r}{12} = (1.06)^{1/12}$, $\frac{r}{12} = (1.06)^{1/12} - 1$, and so $r = 12\left[(1.06)^{1/12} - 1\right] = 0.05841$, or 5.84%.

59. By definition, $A = P(1 + r_{\text{eff}})^t$, so $(1 + r_{\text{eff}})^t = \frac{A}{P}$, $1 + r_{\text{eff}} = \left(\frac{A}{P}\right)^{1/t}$, and $r_{\text{eff}} = \left(\frac{A}{P}\right)^{1/t} - 1$.

60. According to the result of Exercise 59, $r_{\text{eff}} = \left(\frac{A}{P}\right)^{1/t} - 1$. Here $P = 40{,}000$, $A = 60{,}000$, and $t = 5$, so the required effective rate is $r_{\text{eff}} = \left(\frac{60{,}000}{40{,}000}\right)^{1/5} - 1 \approx 0.0845$, or 8.45%.

61. Using the formula $r_{\text{eff}} = \left(\frac{A}{P}\right)^{1/t} - 1$ with $A = 5070.42$, $P = 5000$, and $t = \frac{245}{365}$, we have $r_{\text{eff}} = \left(\frac{5070.42}{5000}\right)^{1/(245/365)} - 1 = \left(\frac{5070.42}{5000}\right)^{(365/245)} - 1 \approx 0.0211$, or 2.11%.

62. $\lim_{i\to 0} R\left[\frac{(1+i)^n - 1}{i}\right] = R \lim_{i\to 0}\left[\frac{(1+i)^n - 1}{i}\right]$. Consider the function $f(x) = x^n$. Then by definition of the derivative, $f'(1) = \lim_{h\to 0} \frac{(1+h)^n - 1}{h}$. With the variable h taken to be i, we see that $\lim_{i\to 0} R\left[\frac{(1+i)^n - 1}{i}\right] = Rf'(1) = Rnx^{n-1}\big|_{x=1} = nR$. Thus, if the interest rate is 0, then after n payments of R dollars each, the future value of the annuity will be nR dollars as expected.

63. True. With $m = 1$, the effective rate is $r_{\text{eff}} = \left(1 + \frac{r}{1}\right)^1 - 1 = r$.

64. False. If Susan had gotten annual increases of 5% over 5 years, her salary would have been $A = 50{,}000\,(1 + 0.05)^5 \approx 63{,}814.08$, or approximately \$63,814 and not \$60,000 after 5 years.

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1. \$5872.78

2. \$475.47

3. 8.95%/yr

4. 11.158%

5. \$29,743.30

6. \$94,038.74

5.4 Differentiation of Exponential Functions

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1. a. $f'(x) = e^x$

b. $g'(x) = e^{f(x)} \cdot f'(x)$

2. a. $f'(x) = ke^{kx}$

b. If $k > 0$, then $f'(x) > 0$ and f is increasing on $(-\infty, \infty)$. If $k < 0$, then $f'(x) < 0$ and f is decreasing on $(-\infty, \infty)$.

Exercises page 376

1. $f(x) = e^{3x}$, so $f'(x) = 3e^{3x}$

2. $f(x) = 3e^x$, so $f'(x) = 3e^x$.

3. $g(t) = e^{-t}$, so $g'(t) = -e^{-t}$

4. $f(x) = e^{-2x}$, so $f'(x) = -2e^{-2x}$.

5. $f(x) = e^x + x^2$, so $f'(x) = e^x + 2x$.

6. $f(x) = 2e^x - x^2$, so $f'(x) = 2e^x - 2x = 2(e^x - x)$.

7. $f(x) = x^3e^x$, so $f'(x) = x^3e^x + e^x\left(3x^2\right) = x^2e^x(x+3)$.

8. $f(u) = u^2e^{-u}$, so $f'(u) = 2ue^{-u} + u^2e^{-u}(-1) = u(2-u)e^{-u}$.

9. $f(x) = \frac{e^x}{x}$, so $f'(x) = \frac{x(e^x) - e^x(1)}{x^2} = \frac{e^x(x-1)}{x^2}$.

10. $f(x) = \frac{x}{e^x}$, so $f'(x) = \frac{e^x(1) - xe^x}{e^{2x}} = \frac{1-x}{e^x}$.

11. $f(x) = 3\left(e^x + e^{-x}\right)$, so $f'(x) = 3\left(e^x - e^{-x}\right)$.

12. $f(x) = \dfrac{e^x + e^{-x}}{2}$, so $f'(x) = \dfrac{e^x - e^{-x}}{2}$.

13. $f(w) = \dfrac{e^w + 2}{e^w} = 1 + \dfrac{2}{e^w} = 1 + 2e^{-w}$, so $f'(w) = -2e^{-w} = -\dfrac{2}{e^w}$.

14. $f(x) = \dfrac{e^x}{e^x + 1}$, so $f'(x) = \dfrac{(e^x + 1)\,e^x - e^x\,(e^x)}{(e^x + 1)^2} = \dfrac{e^x}{(e^x + 1)^2}$.

15. $f(x) = 2e^{3x-1}$, so $f'(x) = 2e^{3x-1}(3) = 6e^{3x-1}$.

16. $f(t) = 4e^{3t+2}$, so $f'(t) = 4e^{3t+2}(3) = 12e^{3t+2}$.

17. $h(x) = e^{-x^2}$, so $h'(x) = e^{-x^2}(-2x) = -2xe^{-x^2}$.

18. $f(x) = e^{x^2-1}$, so $f'(x) = e^{x^2-1}(2x) = 2xe^{x^2-1}$.

19. $f(x) = 3e^{-1/x}$, so $f'(x) = 3e^{-1/x} \cdot \dfrac{d}{dx}\left(-\dfrac{1}{x}\right) = 3e^{-1/x}\left(\dfrac{1}{x^2}\right) = \dfrac{3e^{-1/x}}{x^2}$.

20. $f(x) = e^{1/(2x)}$, so $f'(x) = e^{1/2x} \cdot \dfrac{d}{dx}\left(\dfrac{1}{2x}\right) = \frac{1}{2}e^{1/(2x)}\left(-x^{-2}\right) = -\dfrac{e^{1/(2x)}}{2x^2}$.

21. $f(x) = (e^x + 1)^{25}$, so $f'(x) = 25\,(e^x + 1)^{24}\,e^x = 25e^x\,(e^x + 1)^{24}$.

22. $f(x) = \left(4 - e^{-3x}\right)^3$, so $f'(x) = 3\left(4 - e^{-3x}\right)^2\left(-e^{-3x}\right)(-3) = 9e^{-3x}\left(4 - e^{-3x}\right)^2$.

23. $f(x) = e^{\sqrt{x}}$, so $f'(x) = e^{\sqrt{x}}\dfrac{d}{dx}\left(x^{1/2}\right) = e^{\sqrt{x}}\frac{1}{2}x^{-1/2} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$.

24. $f(t) = -e^{-\sqrt{2t}}$, so $f'(t) = -e^{-\sqrt{2t}}\dfrac{d}{dt}\left(-\sqrt{2t}\right) = e^{-\sqrt{2t}}\left(\frac{1}{2}\right)(2t)^{-1/2}(2) = \dfrac{e^{-\sqrt{2t}}}{\sqrt{2t}}$.

25. $f(x) = (x - 1)\,e^{3x+2}$, so $f'(x) = (x - 1)(3)\,e^{3x+2} + e^{3x+2} = e^{3x+2}(3x - 3 + 1) = e^{3x+2}(3x - 2)$.

26. $f(s) = \left(s^2 + 1\right)e^{-s^2}$, so $f'(s) = 2se^{-s^2} + \left(s^2 + 1\right)e^{-s^2}(-2s) = -2s^3e^{-s^2}$.

27. $f(x) = \dfrac{e^x - 1}{e^x + 1}$, so $f'(x) = \dfrac{(e^x + 1)(e^x) - (e^x - 1)(e^x)}{(e^x + 1)^2} = \dfrac{e^x\,(e^x + 1 - e^x + 1)}{(e^x + 1)^2} = \dfrac{2e^x}{(e^x + 1)^2}$.

28. $g(t) = \dfrac{e^{-t}}{1 + t^2}$, so

$$g'(t) = \frac{\left(1 + t^2\right)\left(-e^{-t}\right) - \left(e^{-t}\right)(2t)}{\left(1 + t^2\right)^2} = \frac{e^{-t}\left(-1 - t^2 - 2t\right)}{\left(1 + t^2\right)^2} = \frac{-e^{-t}\left(t^2 + 2t + 1\right)}{\left(1 + t^2\right)^2} = \frac{-e^{-t}\,(t + 1)^2}{\left(1 + t^2\right)^2}.$$

29. $f(x) = e^{-4x} + e^{3x}$, so $f'(x) = -4e^{-4x} + 3e^{3x}$ and $f''(x) = 16e^{-4x} + 9e^{3x}$.

30. $f(t) = 3e^{-2t} - 5e^{-t}$, so $f'(t) = -6e^{-2t} + 5e^{-t}$ and $f''(t) = 12e^{-2t} - 5e^{-t}$.

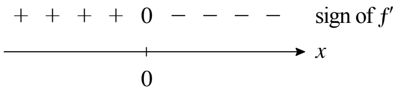
31. $f(x) = 2xe^{3x}$, so $f'(x) = 2e^{3x} + 2xe^{3x}(3) = 2(3x+1)e^{3x}$ and $f''(x) = 6e^{3x} + 2(3x+1)e^{3x}(3) = 6(3x+2)e^{3x}$.

32. $f(t) = t^2e^{-2t}$, so $f'(t) = 2te^{-2t} + t^2e^{-2t}(-2) = 2t(1-t)e^{-2t}$ and $f''(t) = (2-4t)e^{-2t} + 2t(1-t)e^{-2t}(-2) = 2\left(2t^2 - 4t + 1\right)e^{-2t}$.

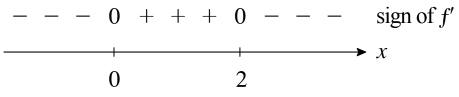
33. $y = f(x) = e^{2x-3}$, so $f'(x) = 2e^{2x-3}$. To find the slope of the tangent line to the graph of f at $x = \frac{3}{2}$, we compute $f'\left(\frac{3}{2}\right) = 2e^{3-3} = 2$. Next, using the point-slope form of the equation of a line, we find that $y - 1 = 2\left(x - \frac{3}{2}\right) = 2x - 3$, or $y = 2x - 2$.

34. $y = e^{-x^2}$. The slope of the tangent line at any point is $y' = e^{-x^2}(-2x) = -2xe^{-x^2}$. The slope of the tangent line when $x = 1$ is $m = -2e^{-1}$. Therefore, an equation of the tangent line is $y - \frac{1}{e} = -\frac{2}{e}(x-1)$, or $y = -\frac{2}{e}x + \frac{3}{e}$.

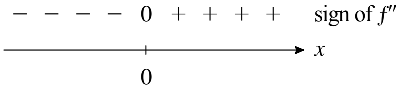
35. $f(x) = e^{-x^2/2}$, so $f'(x) = e^{-x^2/2}(-x) = -xe^{-x^2/2}$. Setting $f'(x) = 0$, gives $x = 0$ as the only critical point of f. From the sign diagram, we conclude that f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.



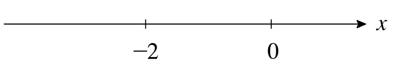
36. $f(x) = x^2e^{-x}$, so $f'(x) = 2xe^{-x} + x^2e^{-x}(-1) = x(2-x)e^{-x}$. Observe that $f'(x) = 0$ if $x = 0$ or 2. The sign diagram of f' shows that f is increasing on $(0, 2)$ and decreasing on $(-\infty, 0)$ and $(2, \infty)$.



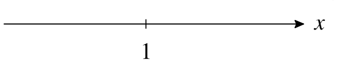
37. $f(x) = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$, so $f'(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$ and $f''(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$. Setting $f''(x) = 0$ gives $e^x = e^{-x}$ or $e^{2x} = 1$, and so $x = 0$. From the sign diagram for f'',we conclude that f is concave upward on $(0, \infty)$ and concave downward on $(-\infty, 0)$.



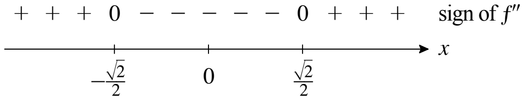
38. $f(x) = xe^x$, so $f'(x) = e^x + xe^x = (x+1)e^x$ and $f''(x) = (x+1)e^x + e^x = (x+2)e^x$. Setting $f''(x) = 0$ gives $x = -2$. The sign diagram of f''shows that f is concave downward on $(-\infty, -2)$ and concave upward on $(-2, \infty)$.



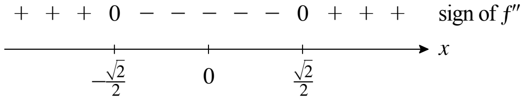
39. $f(x) = xe^{-2x}$, so $f'(x) = e^{-2x} + xe^{-2x}(-2) = (1-2x)e^{-2x}$ and $f''(x) = -2e^{-2x} + (1-2x)e^{-2x}(-2) = 4(x-1)e^{-2x}$. Observe that $f''(x) = 0$ if $x = 1$. The sign diagram of f'' shows that $\left(1, e^{-2}\right)$ is an inflection point.



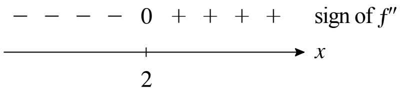
40. $f(x) = 2e^{-x^2} = 2\left(e^{-x^2}\right)$, so $f'(x) = 2(-2x)e^{-x^2} = -4xe^{-x^2}$ and $f''(x) = -4x(-2x)e^{-x^2} - 4e^{-x^2} = -4e^{-x^2}\left(-2x^2+1\right) = 4e^{-x^2}\left(2x^2-1\right)$. Setting $f''(x) = 0$ gives $2x^2 = 1$, $x^2 = \frac{1}{2}$, and so $x = \pm\frac{\sqrt{2}}{2}$. The sign diagram for f'' shows that $\left(-\frac{\sqrt{2}}{2}, 2e^{-1/2}\right)$ and $\left(\frac{\sqrt{2}}{2}, 2e^{-1/2}\right)$ are inflection points.



41. $f(x) = e^{-x^2}$, so $f'(x) = -2xe^{-x^2}$ and $f''(x) = -2e^{-x^2} - 2xe^{-x^2}(-2x) = -2e^{-x^2}\left(1-2x^2\right) = 0$ implies $x = \pm\frac{\sqrt{2}}{2}$. The sign diagram of f'' shows that the graph of f has inflection points at $\left(-\frac{\sqrt{2}}{2}, e^{-1/2}\right)$ and $\left(\frac{\sqrt{2}}{2}, e^{-1/2}\right)$. The slope of the tangent line at $\left(-\frac{\sqrt{2}}{2}, e^{-1/2}\right)$ is $f'\left(-\frac{\sqrt{2}}{2}\right) = \sqrt{2}e^{-1/2}$, and the tangent line has equation $y - e^{-1/2} = \sqrt{2}e^{-1/2}\left(x + \frac{\sqrt{2}}{2}\right)$, which can be simplified to $y = e^{-1/2}\left(\sqrt{2}x + 2\right)$. The slope of the tangent line at $\left(\frac{\sqrt{2}}{2}, e^{-1/2}\right)$ is $f'\left(\frac{\sqrt{2}}{2}\right) = -\sqrt{2}e^{-1/2}$, and this tangent line has equation $y - e^{-1/2} = -\sqrt{2}e^{-1/2}\left(x - \frac{\sqrt{2}}{2}\right)$ or $y = e^{-1/2}\left(-\sqrt{2}x + 2\right)$.



42. $f(x) = xe^{-x}$, so $f'(x) = e^{-x} + xe^{-x}(-1) = (1-x)e^{-x}$ and $f''(x) = -e^{-x} + (1-x)e^{-x}(-1) = (x-2)e^{-x} = 0$ implies $x - 2 = 0$ or $x = 2$. The sign diagram of f'' shows that the graph of f has an inflection point at $\left(2, 2e^{-2}\right)$. The slope of the tangent line at that point is $f'(2) = -e^{-2}$. The tangent line has equation $y - 2e^{-2} = -e^{-2}(x-2) = -e^{-2}x + 4e^{-2}$ or $y = -\frac{1}{e^2}x + \frac{4}{e^2}$.



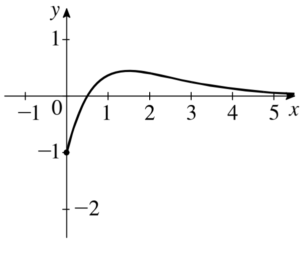
43. $f(x) = e^{-x^2}$, so $f'(x) = -2xe^{-x^2} = 0$ if $x = 0$, the only critical point of f. From the table, we see that f has an absolute minimum value of e^{-1} attained at $x = -1$ and $x = 1$. It has an absolute maximum at (0, 1).

x	-1	0	1
$f(x)$	e^{-1}	1	e^{-1}

44. $h(x) = e^{x^2-4}$, so $h'(x) = 2xe^{x^2-4}$. Setting $h'(x) = 0$ gives $x = 0$ as the only critical point of h. We see that $h(0) = e^{-4}$ is the absolute minimum and $h(-2) = h(2) = 1$ are the absolute maxima of h.

x	-2	0	2
$h(x)$	1	e^{-4}	1

45. $g(x) = (2x-1)e^{-x}$, so $g'(x) = 2e^{-x} + (2x-1)e^{-x}(-1) = (3-2x)e^{-x} = 0$ if $x = \frac{3}{2}$. The graph of g shows that $\left(\frac{3}{2}, 2e^{-3/2}\right)$ is an absolute maximum, and $(0, -1)$ is an absolute minimum.



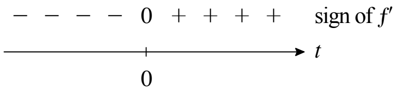
46. $f(x) = xe^{-x^2}$, so

$f'(x) = e^{-x^2} + xe^{-x^2}(-2x) = \left(1 - 2x^2\right)e^{-x^2} = 0$ if $x = \pm\frac{\sqrt{2}}{2}$. From the table, we see that f has an absolute minimum at (0, 0) and an absolute maximum at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}e^{-1/2}\right)$.

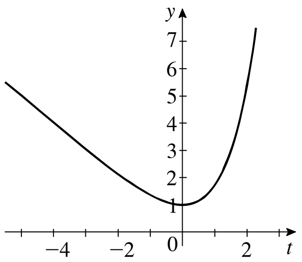
x	0	$\frac{\sqrt{2}}{2}$	2
$f(x)$	0	$\frac{\sqrt{2}}{2}e^{-1/2}$	$2e^{-4}$

47. $f(t) = e^t - t$. We first gather the following information on f.

1. The domain of f is $(-\infty, \infty)$.
2. Setting $t = 0$ gives 1 as the y-intercept.
3. $\lim_{t\to-\infty}\left(e^t - t\right) = \infty$ and $\lim_{t\to\infty}\left(e^t - t\right) = \infty$.
4. There is no asymptote.
5. $f'(t) = e^t - 1$ if $t = 0$, a critical point of f. From the sign diagram for f', we see that f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

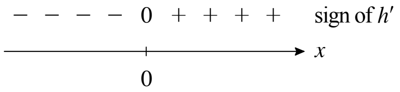


6. From the results of part 5, we see that (0, 1) is a relative minimum of f.
7. $f''(t) = e^t > 0$ for all t, so the graph of f is concave upward on $(-\infty, \infty)$.
8. There is no inflection point.

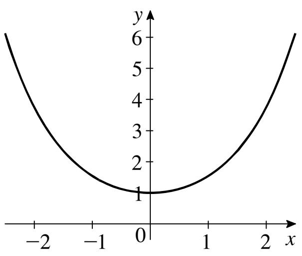


48. $h(x) = \dfrac{e^x + e^{-x}}{2}$. We first gather the following information on h.

1. The domain of h is $(-\infty, \infty)$.
2. Setting $x = 0$ gives 1 as the y-intercept.
3. $\lim_{x\to-\infty} h(x) = \lim_{x\to\infty} h(x) = \infty$.
4. There is no asymptote.
5. $h'(x) = \frac{1}{2}\left(e^x - e^{-x}\right) = 0$ implies $e^x = e^{-x}$, $e^{2x} = 1$, and $x = 0$, a critical point of h. The sign diagram of h' shows that h is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

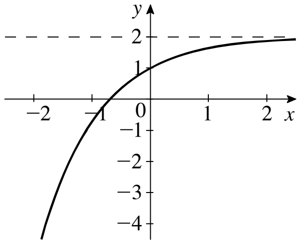


6. The results of part 5 show that (0, 1) is a relative minimum of h.
7. $h''(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$ is always positive, so the graph of h is concave upward everywhere.
8. The results of part 7 show that h has no inflection point.



49. $f(x) = 2 - e^{-x}$. We first gather the following information on f.

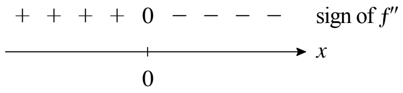
1. The domain of f is $(-\infty, \infty)$.
2. Setting $x = 0$ gives 1 as the y-intercept.
3. $\lim_{x\to-\infty} \left(2 - e^{-x}\right) = -\infty$ and $\lim_{x\to\infty} \left(2 - e^{-x}\right) = 2$.
4. From the results of part 3, we see that $y = 2$ is a horizontal asymptote of f.
5. $f'(x) = e^{-x} > 0$ for all x in $(-\infty, \infty)$, so f is increasing on $(-\infty, \infty)$.
6. Because there is no critical point, f has no relative extremum.
7. $f''(x) = -e^{-x} < 0$ for all x in $(-\infty, \infty)$ and so the graph of f is concave downward on $(-\infty, \infty)$.
8. There is no inflection point.



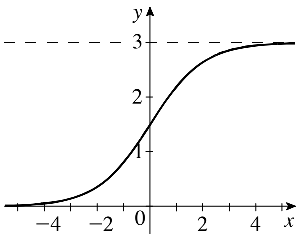
50. $f(x) = \dfrac{3}{1 + e^{-x}}$. We first gather the following information on f.

1. The domain of f is $(-\infty, \infty)$.
2. Letting $x = 0$ gives $\frac{3}{2}$ as the y-intercept.
3. $\lim_{x\to-\infty} \dfrac{3}{1 + e^{-x}} = 0$ and $\lim_{x\to\infty} \dfrac{3}{1 + e^{-x}} = 3$.
4. From the results of part 3, we see that $y = 0$ and $y = 3$ are horizontal asymptotes of f.
5. $f'(x) = 3\dfrac{d}{dx}\left(1 + e^{-x}\right)^{-1} = -3\left(1 + e^{-x}\right)^{-2}\left(e^{-x}\right)(-1) = \dfrac{3e^{-x}}{\left(1 + e^{-x}\right)^2}$. Observe that $f'(x) > 0$ for all x in $(-\infty, \infty)$, so f is increasing on $(-\infty, \infty)$.
6. f has no relative extremum since there is no critical point.
7. $$f''(x) = \frac{\left(1 + e^{-x}\right)^2\left(-3e^{-x}\right) - 3e^{-x}(2)\left(1 + e^{-x}\right)\left(-e^{-x}\right)}{\left(1 + e^{-x}\right)^4} = \frac{3e^{-x}\left(1 + e^{-x}\right)\left[2e^{-x} - \left(1 + e^{-x}\right)\right]}{\left(1 + e^{-x}\right)^4}$$ $$= \frac{3e^{-x}\left(e^{-x} - 1\right)}{\left(1 + e^{-x}\right)^3}.$$

   Observe that $f''(x) = 0$ if $e^{-x} = 1$, or $x = 0$. The sign diagram of f'' shows that f is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$.



8. The results of part 7 show that $\left(0, \frac{3}{2}\right)$ is an inflection point of f.



51. $x^2 + y^3 = 2e^{2y}$, so $2x + 3y^2y' = 4e^{2y}y'$ and $y'\left(4e^{2y} - 3y^2\right) = 2x$. Thus, $\dfrac{dy}{dx} = \dfrac{2x}{4e^{2y} - 3y^2}$.

52. $xy^2+\sqrt{x}e^y=10$, so $y^2+2xy\frac{dy}{dx}+\frac{1}{2}x^{-1/2}e^y+x^{1/2}e^y\frac{dy}{dx}=0$. Multiplying by $x^{1/2}$, we have $x^{1/2}y^2+2x^{3/2}y\frac{dy}{dx}+\frac{1}{2}e^y+xe^y\frac{dy}{dx}=0$, $\left(2x^{3/2}y+xe^y\right)\frac{dy}{dx}=-\left(\frac{1}{2}e^y+x^{1/2}y^2\right)$, and finally $\frac{dy}{dx}=-\frac{e^y+2\sqrt{x}y^2}{2x\left(\sqrt{x}y+e^y\right)}$.

53. $x=y+e^y$, so $1=y'+e^yy'=(1+e^y)\,y'$. Differentiating again, $0=\left(e^yy'\right)y'+(1+e^y)\,y''$, and so $\frac{d^2y}{dx^2}=-\frac{e^y\,(dy/dx)^2}{1+e^y}$. From our first differentiation, we have $\frac{dy}{dx}=\frac{1}{1+e^y}$, so we can write $\frac{d^2y}{dx^2}=-\frac{e^y}{(1+e^y)^3}$.

54. $e^x-e^y=y-x$, so $e^x-e^yy'=y'-1$ and $(1+e^y)\,y'=e^x+1$. Differentiating again, $\left(e^yy'\right)y'+(1+e^y)\,y''=e^x$, so $y''=\frac{e^x-e^y\left(y'\right)^2}{1+e^y}$; that is, $\frac{d^2y}{dx^2}=\frac{e^x-e^y\,(dy/dx)^2}{1+e^y}$.

55. $xy+e^y=e$, so $y+x\frac{dy}{dx}+e^y\frac{dy}{dx}=0$. When $x=0$ and $y=1$, we have $1+0\cdot\frac{dy}{dx}+e^1\frac{dy}{dx}=0$, so $e\frac{dy}{dx}=-1$ and $\frac{dy}{dx}=-\frac{1}{e}$.

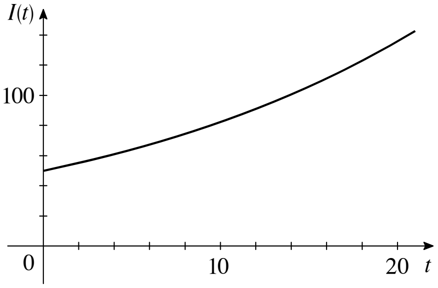
56. $x+y-e^{x-y}=1$, so $1+y'-e^{x-y}\left(1-y'\right)=0$. Substituting $x=y=1$ into this equation yields $1+y'-e^0\left(1-y'\right)=0$, so $1+y'-\left(1-y'\right)=0$ and $y'=0$. An equation of the tangent line is thus $y-1=0\,(x-1)$ or $y=1$.

57. a. The number of video viewers in 2012 was $N(5)=135e^{0.067(5)}$ or approximately 188.7 million.

b. $N'(t)=135\,(0.067)\,e^{0.067t}=9.045e^{0.067t}$, so the number of viewers was changing at the rate of $N'(5)=9.045e^{0.067(5)}$ or approximately 12.6 million viewers per year in 2012.

58. a. In 1990, the index was $I(0)=50e^{0.05(0)}=50$. In 2011, it was $I(21)=50e^{0.05(21)}\approx 143$.

b.



c. $I'(t)=50\,(0.05)\,e^{0.05t}$, so the index was changing at the rate of $I'(10)\approx 4$, or 4 per year.

59. We find $f'(t)=\frac{d}{dt}\left(20.5e^{0.74t}\right)=20.5\left(0.74e^{0.74t}\right)=15.17e^{0.74t}$, so $f'(2)=15.17e^{0.74(2)}\approx 66.64$. Thus, the value of stolen drugs is increasing at the rate of \$66.64 million per year at the beginning of 2008.

60. a. $G(t)=1.58e^{-0.213t}$. The projected annual average population growth rate in 2020 will be $G(3)\approx 0.834$, or approximately 0.83%/decade.

b. $G'(t)=-0.33654e^{-0.213t}$, so the projected annual average population growth rate will be changing at the rate of $G'(3)=-0.1776$, that is, it is decreasing at approximately 0.18%/decade/decade.

61. a. $S(t) = 20{,}000\left(1 + e^{-0.5t}\right)$, so $S'(t) = 20{,}000\left(-0.5e^{-0.5t}\right) = -10{,}000e^{-0.5t}$. Thus, $S'(1) = -10{,}000e^{-0.5} \approx -6065$, or $-\$6065/\text{day}/\text{day}$; $S'(2) = -10{,}000e^{-1} \approx -3679$, or $-\$3679/\text{day}/\text{day}$; $S'(3) = -10{,}000\left(e^{-1.5}\right) \approx -2231$, or $-\$2231/\text{day}/\text{day}$; and $S'(4) = -10{,}000e^{-2} \approx -1353$, or $-\$1353/\text{day}/\text{day}$.

b. $S(t) = 20{,}000\left(1 + e^{-0.5t}\right) = 27{,}400$, so $1 + e^{-0.5t} = \frac{27{,}400}{20{,}000}$, $e^{-0.5t} = \frac{274}{200} - 1$, $-0.5t = \ln\left(\frac{274}{200} - 1\right)$, and so

$$t = \frac{\ln\left(\frac{274}{200} - 1\right)}{-0.5} \approx 2, \text{ or 2 days.}$$

62. a. $A(t) = 0.23te^{-0.4t}$, so $A\left(\frac{1}{2}\right) = 0.23\left(\frac{1}{2}\right)e^{-0.2} \approx 0.094$ and $A(8) = 0.23(8)e^{-3.2} \approx 0.075$.

b. $A'(t) = 0.23\left[t(-0.4)e^{-0.4t} + e^{-0.4t}\right] = 0.23e^{-0.4t}(-0.4t + 1)$. Thus, $A'\left(\frac{1}{2}\right) = 0.23e^{-0.2}(0.8) = 0.151$ and $A'(8) = 0.23e^{-3.2}(-2.2) \approx -0.021$.

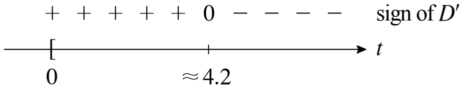
63. $N(t) = 5.3e^{0.095t^2 - 0.85t}$.

a. $N'(t) = 5.3e^{0.095t^2 - 0.85t}(0.19t - 0.85)$. Because $N'(t)$ is negative for $0 \le t \le 4$, we see that $N(t)$ is decreasing over that interval.

b. To find the rate at which the number of polio cases was decreasing at the beginning of 1959, we compute $N'(0) = 5.3e^{0.095(0^2) - 0.85(0)}(0.85) \approx 5.3(-0.85) = -4.505$, or 4505 cases per year per year (t is measured in thousands). To find the rate at which the number of polio cases was decreasing at the beginning of 1962, we compute $N'(3) = 5.3e^{0.095(9) - 0.85(3)}(0.57 - 0.85) \approx (-0.28)(0.9731) \approx -0.273$, or 273 cases per year per year.

64. a. $D'(t) = \dfrac{d}{dt}\left(6.9te^{-0.24t}\right) = 6.9(1 - 0.24t)e^{-0.24t}$. Setting $D'(t) = 0$ gives $6.9(1 - 0.24t) = 0$, $-0.24t = -1$, or $t \approx 4.2$. The sign diagram for $D'(t)$ shows that D is increasing on $(0, 4.2)$ and decreasing on $(4.2, 40)$.

Thus, the difference in size between the autistic brain and the normal brain increases between the time of birth and 4.2 years of age and decreases thereafter.

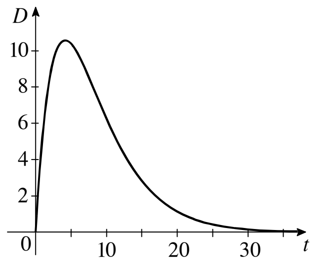


b. It is greatest at age 4.2. The difference is $D(4.2) \approx 10.6$, or 10.6%.

c.
$$D''(t) = 6.9\left[-0.24e^{-0.24t} + (1 - 0.24t)(-0.24)e^{-0.24t}\right]$$
$$= 6.9(-0.48 + 0.0576t)e^{-0.24t} = 0.$$

Setting $D''(t) = 0$ gives $-0.48 + 0.0576t = 0$, or $t \approx 8.33$. Because $D''(t) < 0$ if $0 \le t < 8.33$ and $D''(t) > 0$ if $t > 8.33$, we see that $(8.33, 7.78)$ is an inflection point of D. So the difference between the size of the autistic brain and the normal brain is decreasing at the fastest rate at age 8.3.

d.



65. **a.** The frequency for a 70 year old is given by $f(1) = 0.71e^{0.71(1)} \approx 1.43$ (%), and for a 90 year old it is $f(5) = 0.71e^{0.71(5)} \approx 23.51$ (%).

b. $f'(t) = \frac{d}{dt}\left(0.71e^{0.7t}\right) = 0.7\left(0.71e^{0.7t}\right) = 0.497e^{0.7t}$, which is positive for $0 < t < 5$, and so f is increasing on $(0, 5)$. This says that the frequency of Alzheimer's disease increases with age in the age range under consideration.

c. $f''(t) = \frac{d}{dt}\left(0.497e^{0.7t}\right) = 0.3479e^{0.7t}$, which is positive for $0 < t < 5$, and so f is concave upward on $(0, 5)$. This says that the frequency of Alzheimer's disease is increasing at an increasing rate in the age range under consideration.

66. **a.** The population aged 75 and over in 2010 was $f(0) = \frac{72.15}{1 + 2.7975e^{0}} \approx 19.00$, or approximately 19 million.

b.
$$\begin{aligned} f'(t) &= 72.15\frac{d}{dt}\left(1 + 2.7975e^{-0.02145t}\right)^{-1} \\ &= 72.15(-1)\left(1 + 2.7975e^{-0.02145t}\right)^{-2}(2.7975)(-0.02145)e^{-0.02145t} \\ &= \frac{4.32946e^{-0.02145t}}{\left(1 + 2.7975e^{-0.02145t}\right)^{2}} \end{aligned}$$

Thus, the population aged 75 and over is expected to be growing at the rate of $f'(20) \approx 0.35$, or 350,000 per year in 2030.

c. The population aged 75 and over in 2030 is expected to be $f(20) = \frac{72.15}{1 + 2.7975e^{-0.02145(20)}} \approx 25.57$, or 25.57 million.

67. **a.** $R(x) = px = 100xe^{-0.0001x}$.

b. $R'(x) = 100e^{-0.0001x} + 100xe^{-0.0001x}(-0.0001) = 100(1 - 0.0001x)e^{-0.0001x}$.

c. $R'(10{,}000) = 100[1 - 0.0001(10{,}000)]e^{-0.001} = 0$, or \$0/pair.

68. **a.** $N(t) = 130.7e^{-0.1155t^2} + 50$. The number of deaths in 1950 was $N(0) = 130.7 + 50 = 180.7$, or approximately 181 per 100,000 people.

b.
$$\begin{aligned} N'(t) &= (130.7)(-0.1155)(2t)e^{-0.1155t^2} \\ &= -30.1917te^{-0.1155t^2}. \end{aligned}$$

The rates of change of the number of deaths per 100,000 people per decade are given in the table.

Year	1950	1960	1970	1980
Rate	0	−27	−38	−32

c. $N''(t) = -30.1917\left[e^{-0.1155t^2} + t(-0.1155)(2t)e^{-0.1155t^2}\right] = -30.1917\left(1 - 0.231t^2\right)e^{-0.1155t^2}$. Setting $N''(t) = 0$ gives $t \approx \pm 2.08$. So $t \approx 2$ gives an inflection point, and we conclude that the decline was greatest around 1970.

d. The number is given by $N(6) \approx 52.04$, or approximately 52.

69. The demand equation is $p(x) = 100e^{-0.0002x} + 150$. Next, $p'(x) = 100(-0.0002)e^{-0.0002x} = -0.02e^{-0.0002x}$.

a. To find the rate of change of the price per bottle when $x = 1000$, we compute $p'(1000) = -0.02e^{-0.0002(1000)} = -0.02e^{-0.2} \approx -0.0164$, or -1.64 cents per bottle. To find the rate of change of the price per bottle when $x = 2000$, we compute $p'(2000) = -0.02e^{-0.0002(2000)} = -0.02e^{-0.4} \approx -0.0134$, or -1.34 cents per bottle.

b. The price per bottle when $x = 1000$ is given by $p(1000) = 100e^{-0.0002(1000)} + 150 \approx 231.87$, or \$231.87/bottle. The price per bottle when $x = 2000$ is given by $p(2000) = 100e^{-0.0002(2000)} + 150 \approx 217.03$, or \$217.03/bottle.

70. a. $p = 240\left(1 - \dfrac{3}{3 + e^{-0.0005x}}\right) = 240\left[1 - 3\left(3 + e^{-0.0005x}\right)^{-1}\right]$,

so $p' = 720\left(3 + e^{-0.0005x}\right)^{-2}\left(-0.0005e^{-0.0005x}\right)$. Thus,

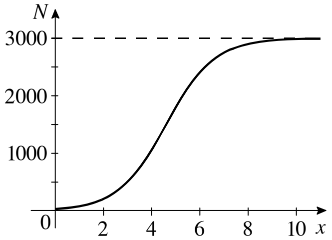
$p'(1000) = 720\left(3 + e^{-0.0005 \cdot 1000}\right)^{-2}\left(-0.0005e^{-0.0005 \cdot 1000}\right) = -\dfrac{0.36(0.606531)}{(3 + 0.606531)^2} \approx -0.0168$, or -1.68 cents per case per case.

b. $p(1000) = 240\left(1 - \dfrac{3}{3.606531}\right) \approx 40.36$, or \$40.36/case.

71. a. $N(0) = \dfrac{3000}{1 + 99} = 30$.

b. $N'(x) = 3000\dfrac{d}{dx}\left(1 + 99e^{-x}\right)^{-1} = -3000\left(1 + 99e^{-x}\right)^{-2}\left(-99e^{-x}\right) = \dfrac{297{,}000e^{-x}}{\left(1 + 99e^{-x}\right)^2}$. Because $N'(x) > 0$ for all x in $(0, \infty)$, we see that N is increasing on $(0, \infty)$.

c. From the graph of N, we see that the total number of students who contracted influenza during that particular epidemic is approximately $\lim_{x \to \infty} \dfrac{3000}{1 + 99e^{-x}} = 3000$.



72. a. The number of units sold 24 months after introduction was $N(24) = 20{,}000\left[1 - e^{-0.05(24)}\right]^2 \approx 9766.59$, or approximately 9767.

b. $N'(t) = 20{,}000(2)\left(1 - e^{-0.05t}\right)\left[-(-0.05)e^{-0.05t}\right] = 2000e^{-0.05t}\left(1 - e^{-0.05t}\right)$. Thus, 24 months after its introduction, the product was selling at the rate of $N'(24) \approx 420.95$, or approximately 421 units per month.

73. a. Here $f(p) = 50e^{-0.02p}$, so $E(p) = -\dfrac{pf'(p)}{f(p)} = -\dfrac{p\left(50e^{-0.02p}\right)(-0.02)}{50e^{-0.02p}} = 0.02p$.

b. Using the result from part (a), we see that $E(p) = 1$ if $0.02p = 1$, or $p = 50$; $E(p) < 1$ if $p < 50$; and $E(p) > 1$ if $p > 50$. Demand is inelastic if $0 < p < 50$, unitary if $p = 50$, and elastic if $p > 50$.

c. Because demand is inelastic when $p = 40$, decreasing the unit price slightly will cause revenue to decrease.

d. Because demand is elastic when $p = 60$, increasing the unit price slightly will cause revenue to decrease.

74. a. Using Equation 7 from Section 3.4 with $f(p) = ae^{-bp}$, we have $E(p) = -\dfrac{pf'(p)}{f(p)} = -\dfrac{pae^{-bp}(-b)}{ae^{-bp}} = bp$.

b. Using the result from part (a), we see that $bp = 1$ if $p = 1/b$, showing that demand is unitary if $p = 1/b$. Because $bp > 1$ if $p \geq 1/b$ and $bp < 1$ if $p \leq 1/b$, we see that demand is elastic if $p > 1/b$ and inelastic if $0 < p < 1/b$.

75. **a.** $W = 2.4e^{1.84h}$, so if $h = 1.6$, $W = 2.4e^{1.84(1.6)} \approx 45.58$, or approximately 45.6 kg.

b. $\Delta W \approx dW = (2.4)(1.84)\,e^{1.84h}\,dh$. With $h = 1.6$ and $dh = \Delta h = 1.65 - 1.6 = 0.05$, we find $\Delta W \approx (2.4)(1.84)\,e^{1.84(1.6)} \cdot (0.05) \approx 4.19$, or approximately 4.2 kg.

76. The number of people is

$$\Delta P \approx f'(10)\,\Delta x = 50{,}000\frac{d}{dx}\left(1 - e^{-0.01x^2}\right)(0.1) = 50{,}000\left(0.02xe^{-0.01x^2}\right)\Big|_{x=10}(0.1)$$

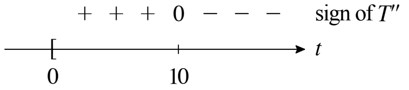
$$= (50{,}000)(0.02)(10)(0.3679)(0.1) \approx 367.9\text{, or 368 people.}$$

77. $P(t) = 80{,}000e^{\sqrt{t}/2-0.09t} = 80{,}000e^{(1/2)t^{1/2}-0.09t}$, so $P'(t) = 80{,}000\left(\frac{1}{4}t^{-1/2} - 0.09\right)e^{(1/2)t^{1/2}-0.09t}$. Setting $P'(t) = 0$, we have $\frac{1}{4}t^{-1/2} = 0.09$, so $t^{-1/2} = 0.36$, $\dfrac{1}{\sqrt{t}} = 0.36$, and $t = \left(\frac{1}{0.36}\right)^2 \approx 7.72$. Evaluating $P(t)$ at each of its endpoints and at the point $t = 7.72$, we find $P(0) = 80{,}000$, $P(7.72) \approx 160{,}207.69$, and $P(8) \approx 160{,}170.71$. We conclude that P is optimized at $t = 7.72$. The optimal price is approximately \$160,208.

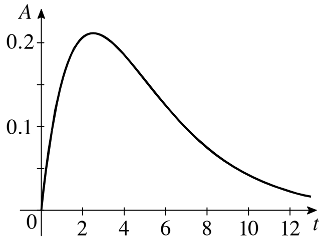
78. We want to find the maximum of dT/dt:

$$T'(t) = -1000\frac{d}{dt}\left[(t+10)\,e^{-0.1t} + 10{,}000\right] = -1000\left[e^{-0.1t} + (t+10)\,e^{-0.1t}(-0.1)\right] = 100te^{-0.1t}\text{, so}$$

$$T''(t) = 100\frac{d}{dt}\left(te^{-0.1t}\right) = 100\left[e^{-0.1t} + te^{-0.1t}(-0.1)\right] = 100e^{-0.1t}(1 - 0.1t).$$ Observe that $T''(t) = 0$ if $t = 10$, a critical number of T'. From the sign diagram of T'', we see that $t = 10$ gives a relative maximum of T'. This is in fact an absolute maximum. Thus, the maximum production will be reached in the tenth year of operation.



79. $A(t) = 0.23te^{-0.4t}$, so $A'(t) = 0.23(1 - 0.4t)\,e^{-0.4t}$. Setting $A'(t) = 0$ gives $t = \frac{1}{0.4} = \frac{5}{2}$. From the graph of A, we see that the proportion of alcohol is highest $2\frac{1}{2}$ hours after drinking. The level is given by $A\left(\frac{5}{2}\right) \approx 0.2115$, or approximately 0.21%.



80. **a.** $p = 8 + 4e^{-2t} + te^{-2t}$, so the price at $t = 0$ is $8 + 4$, or \$12 per unit.

b. $\dfrac{dp}{dt} = -8e^{-2t} + e^{-2t} - 2te^{-2t}$, so $\dfrac{dp}{dt}\Big|_{t=0} = -8e^{-2t} + e^{-2t} - 2te^{-2t}\big|_{t=0} = -8 + 1 = -7$. Thus, the price is decreasing at the rate of \$7/week.

c. The equilibrium price is $\lim\limits_{t\to\infty}\left(8 + 4e^{-2t} + te^{-2t}\right) = 8 + 0 + 0$, or \$8 per unit.

81. **a.** The temperature inside the house is given by $T(0) = 30 + 40e^0 = 70$, or 70°F.

b. The reading is changing at the rate of $T'(1) = 40(-0.98)\,e^{-0.98t}\big|_{t=1} \approx -14.7$. Thus, it is dropping at the rate of approximately 14.7°F/min.

c. The temperature outdoors is given by $\lim\limits_{t\to\infty} T(t) = \lim\limits_{t\to\infty}\left(30 + 40e^{-0.98t}\right) = 30 + 0 = 30$, or 30°F.

82. $\dfrac{dy}{dt} = \dfrac{d}{dt}\left[(y_0 - C)\,e^{-kt/V} + C\right] = (y_0 - C)\left(-\dfrac{k}{V}\right)e^{-kt/V} = \dfrac{k(C - y_0)}{V}e^{-kt/V}$ (g/cc per unit time).

83. $A(t) = \dfrac{150\left(1 - e^{0.022662t}\right)}{1 - 2.5e^{0.022662t}}$.

a. Let $k = 0.022662$. Then $A'(t) = 150\dfrac{d}{dt}\left[\dfrac{1 - e^{kt}}{1 - 2.5e^{kt}}\right] = 150\dfrac{\left(1 - 2.5e^{kt}\right)\left(-ke^{kt}\right) - \left(1 - e^{kt}\right)\left(-2.5ke^{kt}\right)}{\left(1 - 2.5e^{kt}\right)^2}$.

Thus, the rate of formation of chemical C one minute after the interaction begins is

$A'(1) = \dfrac{5.09895e^{0.022662}}{\left(1 - 2.5e^{0.022662}\right)^2} \approx 2.15$, or 2.15 g/min.

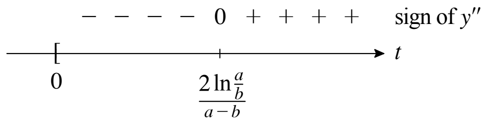
b. The amount of chemical C that is eventually formed is

$\lim\limits_{t\to\infty} A(t) = 150 \lim\limits_{t\to\infty} \dfrac{1 - e^{0.022662t}}{1 - 2.5e^{0.022662t}} = 150 \lim\limits_{t\to\infty} \dfrac{e^{-0.022662t} - 1}{e^{-0.022662t} - 2.5} = \dfrac{150}{2.5} = 60$ g.

84. a. $y = c\left(e^{-bt} - e^{-at}\right)$, so $y' = c\left(-be^{-bt} + ae^{-at}\right) = ca\left(-\frac{b}{a}e^{-bt} + e^{-at}\right) = cae^{-at}\left[-\frac{b}{a}e^{(a-b)t} + 1\right]$. Setting $y' = 0$ gives $-\dfrac{b}{a}e^{(a-b)t} + 1 = 0$, $e^{(a-b)t} = \dfrac{a}{b}$, $\ln e^{(a-b)t} = \ln\dfrac{a}{b}$, and so $t = \dfrac{\ln\frac{a}{b}}{a-b}$. Because $y(0) = 0$ and $\lim\limits_{t\to\infty} y = 0$, $t = \dfrac{\ln\frac{a}{b}}{a-b}$ gives the time at which the concentration is maximal.

b. $y'' = c\left(b^2e^{-bt} - a^2e^{-at}\right) = ca^2e^{-at}\left[\dfrac{b^2}{a^2}e^{(a-b)t} - 1\right]$. Setting $y'' = 0$ gives $e^{(a-b)t} = \dfrac{a^2}{b^2}$, so $t = \dfrac{2\ln\frac{a}{b}}{a-b}$.

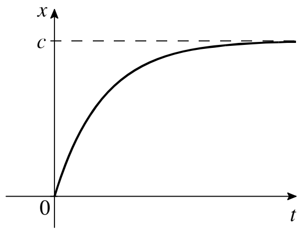
From the sign diagram of y'', we see that the concentration of the drug is decreasing most rapidly when $t = \dfrac{2\ln\frac{a}{b}}{a-b}$ seconds.



85. a. $x(t) = c\left(1 - e^{-at/V}\right)$, so

$x'(t) = \dfrac{d}{dt}\left(c - ce^{-at/V}\right) = \dfrac{ac}{V}e^{-at/V}$. Because $a > 0$, $c > 0$, and $V > 0$, we see that $x'(t)$ is always positive and we conclude that $x(t)$ is always increasing.

b.

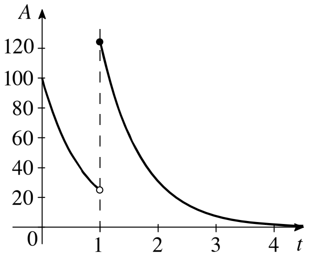


86. We are given that $c\left(1 - e^{-at/V}\right) < m$, so $1 - e^{-at/V} < \dfrac{m}{c}$, $-e^{-at/V} < \dfrac{m}{c} - 1$, and $e^{-at/V} > 1 - \dfrac{m}{c}$. Taking logarithms of both sides of the inequality, we have $-\dfrac{at}{V}\ln e > \ln\dfrac{c-m}{c}$, $-\dfrac{at}{V} > \ln\dfrac{c-m}{c}$, $-t > \dfrac{V}{a}\ln\dfrac{c-m}{c}$, and so $t < \dfrac{V}{a}\left(-\ln\dfrac{c-m}{c}\right) = \dfrac{V}{a}\ln\dfrac{c}{c-m}$. Therefore, the liquid must not be allowed to enter the organ for longer than $t = \dfrac{V}{a}\ln\dfrac{c}{c-m}$ minutes.

87. a. $A(t) = \begin{cases} 100e^{-1.4t} & \text{if } 0 \le t < 1 \\ 100\left(1 + e^{1.4}\right)e^{-1.4t} & \text{if } t \ge 1 \end{cases}$ so $A'(t) = \begin{cases} -140e^{-1.4t} & \text{if } 0 < t < 1 \\ -140\left(1 + e^{1.4}\right)e^{-1.4t} & \text{if } t > 1 \end{cases}$ Thus, after 12 hours the amount of drug is changing at the rate of $A'\left(\frac{1}{2}\right) = -140e^{-0.7} \approx -69.52$, or decreasing at the rate of 70 mg/day. After 2 days, it is changing at the rate of $A'(2) = -140\left(1 + e^{1.4}\right)e^{-2.8} \approx -43.04$, or decreasing at the rate of 43 mg/day.

b. From the graph of A, we see that the maximum occurs at $t = 1$, that is, at the time when she takes the second dose.

c. The maximum amount is $A(1) = 100\left(1 + e^{1.4}\right)e^{-1.4} \approx 124.66$, or 125 mg.



88. False. $f(x) = 3^x = e^{x\ln 3}$, and so $f'(x) = e^{x\ln 3}\frac{d}{dx}(x\ln 3) = (\ln 3)e^{x\ln 3} = (\ln 3)3^x$.

89. False. $f(x) = e^{\pi}$ is a constant function and so $f'(x) = 0$.

90. False. $f'(x) = (\ln\pi)\pi^x$. See Exercise 88.

91. False. $f'(x) = \frac{d}{dx}\left(e^{x^2+x}\right) = (2x+1)e^{x^2+x}$.

92. True. Differentiating both sides of the equation with respect to x, we have $\dfrac{d}{dx}\left(x^2 + e^y\right) = \dfrac{d}{dx}(10)$, so $2x + e^y\dfrac{dy}{dx} = 0$ and thus $\dfrac{dy}{dx} = -\dfrac{2x}{e^y}$.

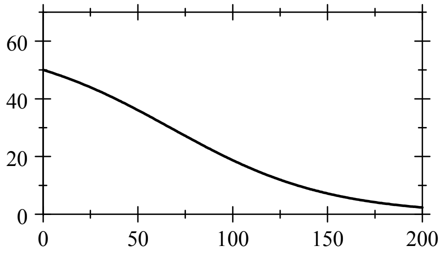
Using Technology page 381

1. 5.4366 **2.** −0.5123 **3.** 12.3929 **4.** 0.0926 **5.** 0.1861 **6.** −1.0311

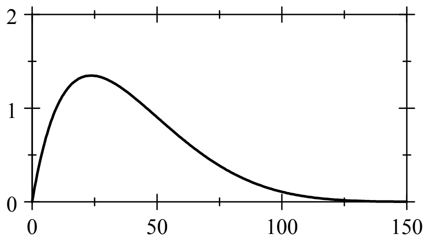
7. a. The initial population of crocodiles is $P(0) = \frac{300}{6} = 50$.

b. $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \dfrac{300e^{-0.024t}}{5e^{-0.024t}+1} = \dfrac{0}{0+1} = 0$.

c.

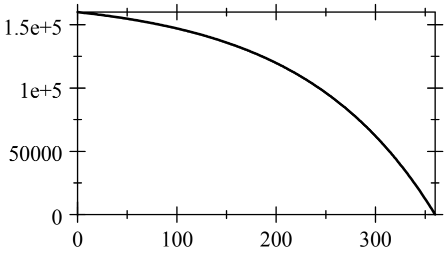


8. a.



b. At $x = 10$, the rate is 57,972/\$1000. If $x = 50$, the rate is −23,418/\$1000. Income distribution is increasing at low income levels and decreasing at higher levels.

9. a.



b. Initially, they owe $B(0) = \$160{,}000$, and their debt is decreasing at the rate of $B'(0) \approx \$87.07$ per month. After 180 payments, they owe $B(180) \approx \$126{,}928.78$ and their debt is decreasing at the rate of $B'(180) \approx \$334.18$ per month.

10. a. At the beginning of June, there are $F(1) \approx 196.20$ or approximately 196 aphids in a typical bean stem. At the beginning of July the number is $F(2) \approx 180.02$, or approximately 180 aphids per bean stem.

b. At the beginning of June, the population of aphids is changing at the rate of $F'(1) \approx 226.02$; that is, it is increasing at the rate of 226 aphids on a typical bean stem per month. At the beginning of July, the population is changing at the rate of $F'(2) \approx -238.3$; that is, it is decreasing at the rate of 238 aphids per month.

5.5 Differentiation of Logarithmic Functions

Concept Questions page 386

1. a. $f'(x) = \frac{1}{x}$.

b. $g'(x) = \frac{f'(x)}{f(x)}$.

2. See the procedure given on page 385 of the text.

Exercises page 387

1. $f(x) = 5\ln x$, so $f'(x) = 5\left(\frac{1}{x}\right) = \frac{5}{x}$.

2. $f(x) = \ln 5x$, so $f'(x) = \frac{5}{5x} = \frac{1}{x}$.

3. $f(x) = \ln(x+1)$, so $f'(x) = \frac{1}{x+1}$.

4. $g(x) = \ln(2x+1)$, so $g'(x) = \frac{2}{2x+1}$.

5. $f(x) = \ln x^8$, so $f'(x) = \frac{8x^7}{x^8} = \frac{8}{x}$.

6. $h(t) = 2\ln t^5$, so $h'(t) = \frac{2}{t^5} \cdot 5t^4 = \frac{10}{t}$.

7. $f(x) = \ln x^{1/2}$, so $f'(x) = \frac{\frac{1}{2}x^{-1/2}}{x^{1/2}} = \frac{1}{2x}$.

8. $f(x) = \ln\left(x^{1/2}+1\right)$, so $f'(x) = \frac{\frac{1}{2}x^{-1/2}}{x^{1/2}+1} = \frac{1}{2\sqrt{x}\left(\sqrt{x}+1\right)}$.

9. $f(x) = \ln\left(\frac{1}{x^2}\right) = \ln x^{-2}$, so $f'(x) = -\frac{2x^{-3}}{x^{-2}} = -\frac{2}{x}$.

10. $f(x) = \ln\frac{1}{2x^3} = \ln 1 - \ln\left(2x^3\right) = -\ln 2 - \ln x^3$, so $f'(x) = -\frac{3x^2}{x^3} = -\frac{3}{x}$.

11. $f(x) = \ln\left(4x^2-5x+3\right)$, so $f'(x) = \frac{8x-5}{4x^2-5x+3} = \frac{8x-5}{4x^2-5x+3}$.

12. $f(x) = \ln\left(3x^2-2x+1\right)$, so $f'(x) = \frac{6x-2}{3x^2-2x+1} = \frac{2(3x-1)}{3x^2-2x+1}$.

13. $f(x) = \ln\left(\frac{2x}{x+1}\right) = \ln 2x - \ln(x+1)$, so

$$f'(x) = \frac{2}{2x} - \frac{1}{x+1} = \frac{1}{x} - \frac{1}{x+1} = \frac{(x+1)-x}{x(x+1)} = \frac{x+1-x}{x(x+1)} = \frac{1}{x(x+1)}.$$

14. $f(x) = \ln(x+1) - \ln(x-1)$, so $f'(x) = \frac{1}{x+1} - \frac{1}{x-1} = \frac{(x-1)-(x+1)}{x^2-1} = -\frac{2}{x^2-1}$.

15. $f(x) = x^2\ln x$, so $f'(x) = x^2\left(\frac{1}{x}\right) + (\ln x)(2x) = x + 2x\ln x = x(1+2\ln x)$.

16. $f(x) = 3x^2 \ln 2x$, so $f'(x) = 6x \ln 2x + 3x^2 \cdot \dfrac{2}{2x} = 6x \ln 2x + 3x = 3x(2\ln 2x + 1)$.

17. $f(x) = \dfrac{2\ln x}{x}$, so $f'(x) = \dfrac{x\left(\frac{2}{x}\right) - 2\ln x}{x^2} = \dfrac{2(1-\ln x)}{x^2}$.

18. $f(x) = \dfrac{3\ln x}{x^2}$, so $f'(x) = \dfrac{x^2\left(\frac{3}{x}\right) - (3\ln x)(2x)}{x^4} = \dfrac{3x(1-2\ln x)}{x^4} = \dfrac{3(1-2\ln x)}{x^3}$.

19. $f(u) = \ln(u-2)^3$, so $f'(u) = \dfrac{3(u-2)^2}{(u-2)^3} = \dfrac{3}{u-2}$.

20. $f(x) = \ln\left(x^3-3\right)^4$, so $f'(x) = \dfrac{4\left(x^3-3\right)^3\left(3x^2\right)}{\left(x^3-3\right)^4} = \dfrac{4\left(3x^2\right)}{x^3-3} = \dfrac{12x^2}{x^3-3}$.

21. $f(x) = (\ln x)^{1/2}$, so $f'(x) = \frac{1}{2}(\ln x)^{-1/2}\left(\frac{1}{x}\right) = \dfrac{1}{2x\sqrt{\ln x}}$.

22. $f(x) = (\ln x + x)^{1/2}$, so $f'(x) = \frac{1}{2}(\ln x + x)^{-1/2}\left(\frac{1}{x}+1\right) = \dfrac{x+1}{2x\sqrt{\ln x + x}}$.

23. $f(x) = (\ln x)^2$, so $f'(x) = 2(\ln x)\left(\dfrac{1}{x}\right) = \dfrac{2\ln x}{x}$.

24. $f(x) = 2(\ln x)^{3/2}$, so $f'(x) = 2\left(\frac{3}{2}\right)(\ln x)^{1/2}\left(\frac{1}{x}\right) = \dfrac{3(\ln x)^{1/2}}{x}$.

25. $f(x) = \ln\left(x^3+1\right)$, so $f'(x) = \dfrac{3x^2}{x^3+1}$.

26. $f(x) = \ln\left(x^2-4\right)^{1/2}$, so $f'(x) = \dfrac{\frac{1}{2}\left(x^2-4\right)^{-1/2}(2x)}{\left(x^2-4\right)^{1/2}} = \dfrac{x}{x^2-4}$.

27. $f(x) = e^x \ln x$, so $f'(x) = e^x \ln x + e^x\left(\frac{1}{x}\right) = \dfrac{e^x(x\ln x + 1)}{x}$.

28. $f(x) = e^x \ln\sqrt{x+3} = \frac{1}{2}e^x \ln(x+3)$, so $f'(x) = \dfrac{1}{2}\left[e^x \ln(x+3) + e^x \cdot \dfrac{1}{x+3}\right] = \dfrac{e^x\left[(x+3)\ln(x+3)+1\right]}{2(x+3)}$.

29. $f(t) = e^{2t}\ln(t+1)$, so $f'(t) = e^{2t}\left(\dfrac{1}{t+1}\right) + \ln(t+1)\cdot\left(2e^{2t}\right) = \dfrac{\left[2(t+1)\ln(t+1)+1\right]e^{2t}}{t+1}$.

30. $g(t) = t^2\ln\left(e^{2t}+1\right)$, so $g'(t) = 2t\ln\left(e^{2t}+1\right) + t^2\left(\dfrac{2e^{2t}}{e^{2t}+1}\right) = \dfrac{2t\left[\left(e^{2t}+1\right)\ln\left(e^{2t}+1\right) + te^{2t}\right]}{e^{2t}+1}$.

31. $f(x)\dfrac{\ln x}{x^2}$, so $f'(x) = \dfrac{x^2\left(\frac{1}{x}\right) - \ln x\,(2x)}{x^4} = \dfrac{1-2\ln x}{x^3}$.

32. $g(t) = \dfrac{t}{\ln t}$, so $g'(t) = \dfrac{(\ln t)(1) - t\left(\frac{1}{t}\right)}{(\ln t)^2} = \dfrac{\ln t - 1}{(\ln t)^2}$.

33. $f'(x)=\frac{d}{dx}\left[\ln(\ln x)\right]=\frac{\frac{d}{dx}(\ln x)}{\ln x}=\frac{\frac{1}{x}}{\ln x}=\frac{1}{x\ln x}.$

34. $g'(x)=\frac{d}{dx}\left[\ln\left(e^x+\ln x\right)\right]=\frac{\frac{d}{dx}\left(e^x+\ln x\right)}{e^x+\ln x}=\frac{e^x+\frac{1}{x}}{e^x+\ln x}=\frac{xe^x+1}{x\left(e^x+\ln x\right)}.$

35. $f(x)=\ln 2+\ln x$, so $f'(x)=\frac{1}{x}$ and $f''(x)=-\frac{1}{x^2}.$

36. $f(x)=\ln(x+5)$, so $f'(x)=\frac{1}{x+5}$ and $f''(x)=\frac{d}{dx}(x+5)^{-1}=-(x+5)^{-2}=-\frac{1}{(x+5)^2}.$

37. $f(x)=\ln\left(x^2+2\right)$, so $f'(x)=\frac{2x}{\left(x^2+2\right)}$ and $f''(x)=\frac{\left(x^2+2\right)(2)-2x(2x)}{\left(x^2+2\right)^2}=\frac{2\left(2-x^2\right)}{\left(x^2+2\right)^2}.$

38. $f(x)=(\ln x)^2$, so $f'(x)=2(\ln x)\left(\frac{1}{x}\right)=\frac{2\ln x}{x}$ and $f''(x)=\frac{x\left(\frac{2}{x}\right)-2\ln x}{x^2}=\frac{2(1-\ln x)}{x^2}.$

39. $f'(x)=\frac{d}{dx}\left(x^2\ln x\right)=\frac{d}{dx}\left(x^2\right)\ln x+\frac{d}{dx}(\ln x)x^2=2x\ln x+\frac{1}{x}\cdot x^2=2x\ln x+x=x(2\ln x+1)$ and

$f''(x)=\frac{d}{dx}\left[x(2\ln x+1)\right]=\frac{d}{dx}(x)(2\ln x+1)+\frac{d}{dx}(2\ln x+1)x=2\ln x+1+\frac{2}{x}\cdot x=2\ln x+3.$

40. $g(x)=e^{2x}\ln x$, so $g'(x)=e^{2x}\frac{d}{dx}\ln x+(\ln x)\frac{d}{dx}e^{2x}=\frac{e^{2x}}{x}+2e^{2x}\ln x$ and

$g''(x)=\frac{2xe^{2x}-e^{2x}}{x^2}+\frac{2e^{2x}}{x}+4e^{2x}\ln x=\frac{2xe^{2x}-e^{2x}+2xe^{2x}+4x^2e^{2x}\ln x}{x^2}=\frac{\left(4x-1+4x^2\ln x\right)e^{2x}}{x^2}.$

41. $y=(x+1)^2(x+2)^3$, so

$\ln y=\ln(x+1)^2(x+2)^3=\ln(x+1)^2+\ln(x+2)^3=2\ln(x+1)+3\ln(x+2).$

Thus, $\frac{y'}{y}=\frac{2}{x+1}+\frac{3}{x+2}=\frac{2(x+2)+3(x+1)}{(x+1)(x+2)}=\frac{5x+7}{(x+1)(x+2)}$ and

$y'=\frac{(5x+7)(x+1)^2(x+2)^3}{(x+1)(x+2)}=(5x+7)(x+1)(x+2)^2.$

42. $y=(3x+2)^4(5x-1)^2$, so $\ln y=4\ln(3x+2)+2\ln(5x-1)$. Thus,

$\frac{dy}{dx}\cdot\frac{1}{y}=\frac{4(3)}{3x+2}+\frac{2(5)}{5x-1}=\frac{12(5x-1)+10(3x+2)}{(3x+2)(5x-1)}=\frac{60x-12+30x+20}{(3x+2)(5x-1)}=\frac{2(45x+4)}{(3x+2)(5x-1)}$, and so

$\frac{dy}{dx}=\frac{2(3x+2)^4(5x-1)^2(45x+4)}{(3x+2)(5x-1)}=2(3x+2)^3(5x-1)(45x+4).$

43. $y=(x-1)^2(x+1)^3(x+3)^4$, so $\ln y=2\ln(x-1)+3\ln(x+1)+4\ln(x+3)$. Thus,

$\frac{y'}{y}=\frac{2}{x-1}+\frac{3}{x+1}+\frac{4}{x+3}=\frac{2(x+1)(x+3)+3(x-1)(x+3)+4(x-1)(x+1)}{(x-1)(x+1)(x+3)}$

$=\frac{2x^2+8x+6+3x^2+6x-9+4x^2-4}{(x-1)(x+1)(x+3)}=\frac{9x^2+14x-7}{(x-1)(x+1)(x+3)}$, and so

$y'=\frac{9x^2+14x-7}{(x-1)(x+1)(x+3)}\cdot y=\frac{\left(9x^2+14x-7\right)(x-1)^2(x+1)^3(x+3)^4}{(x-1)(x+1)(x+3)}$

$=\left(9x^2+14x-7\right)(x-1)(x+1)^2(x+3)^3.$

44. $y = (3x+5)^{1/2}(2x-3)^4$, so $\ln y = \frac{1}{2}\ln(3x+5) + 4\ln(2x-3)$. Thus,

$$\frac{y'}{y} = \frac{1}{2}\cdot\frac{1(3)}{3x+5} + 4\cdot\frac{2}{2x-3} = \frac{3}{2(3x+5)} + \frac{8}{2x-3} = \frac{3(2x-3)+16(3x+5)}{2(3x+5)(2x-3)}$$
$$= \frac{54x+71}{2(3x+5)(2x-3)}, \text{ and so}$$
$$y' = \left(\frac{54x+71}{2}\right)(3x+5)^{-1}(2x-3)^{-1}y = \left(\frac{54x+71}{2}\right)(3x+5)^{-1}(2x-3)^{-1}(3x+5)^{1/2}(2x-3)^4$$
$$= \tfrac{1}{2}(2x-3)^3(54x+71)(3x+5)^{-1/2}.$$

45. $y = \dfrac{(2x^2-1)^5}{\sqrt{x+1}}$, so $\ln y = \ln\dfrac{(2x^2-1)^5}{(x+1)^{1/2}} = 5\ln(2x^2-1) - \frac{1}{2}\ln(x+1)$. Thus,

$$\frac{y'}{y} = \frac{20x}{2x^2-1} - \frac{1}{2(x+1)} = \frac{40x(x+1)-(2x^2-1)}{2(2x^2-1)(x+1)} = \frac{38x^2+40x+1}{2(2x^2-1)(x+1)}, \text{ and so}$$
$$y' = \frac{38x^2+40x+1}{2(2x^2-1)(x+1)}\cdot\frac{(2x^2-1)^5}{\sqrt{x+1}} = \frac{(38x^2+40x+1)(2x^2-1)^4}{2(x+1)^{3/2}}.$$

46. $y = \dfrac{\sqrt{4+3x^2}}{\sqrt[3]{x^2+1}}$, so $\ln y = \frac{1}{2}\ln(4+3x^2) - \frac{1}{3}\ln(x^2+1)$. Thus,

$$\frac{y'}{y} = \frac{6x}{2(4+3x^2)} - \frac{2x}{3(x^2+1)} = \frac{9x(x^2+1)-2x(4+3x^2)}{3(4+3x^2)(x^2+1)}, \text{ and so}$$
$$y' = \frac{3x^3+x}{3(4+3x^2)(x^2+1)}\cdot\frac{\sqrt{4+3x^2}}{(x^2+1)^{1/3}} = \frac{x(3x^2+1)}{3(4+3x^2)^{1/2}(x^2+1)^{4/3}}.$$

47. $y = 3^x$, so $\ln y = x\ln 3$, $\dfrac{1}{y}\cdot\dfrac{dy}{dx} = \ln 3$, and $\dfrac{dy}{dx} = y\ln 3 = 3^x\ln 3$.

48. $y = x^{x+2}$, so $\ln y = \ln x^{x+2} = (x+2)\ln x$, So $\dfrac{y'}{y} = \ln x + (x+2)\left(\dfrac{1}{x}\right) = \dfrac{x\ln x + x + 2}{x}$, and

$$y' = \frac{(x\ln x + x + 2)x^{x+2}}{x}.$$

49. $y = (x^2+1)^x$, so $\ln y = \ln(x^2+1)^x = x\ln(x^2+1)$,

$$\frac{y'}{y} = \ln(x^2+1) + x\left(\frac{2x}{x^2+1}\right) = \frac{(x^2+1)\ln(x^2+1)+2x^2}{x^2+1}, \text{ and}$$
$$y' = \frac{\left[(x^2+1)\ln(x^2+1)+2x^2\right](x^2+1)^x}{x^2+1} = (x^2+1)^{x-1}\left[(x^2+1)\ln(x^2+1)+2x^2\right].$$

50. $y = x^{\ln x}$, so $\ln y = \ln(x^{\ln x}) = (\ln x)^2$. Thus, $\dfrac{y'}{y} = 2(\ln x)\left(\dfrac{1}{x}\right) = \dfrac{2\ln x}{x}$ and so

$$y' = \frac{2\ln x}{x}\cdot x^{\ln x} = 2(\ln x)x^{\ln x - 1}.$$

51. $\dfrac{d}{dx}(\ln y - x\ln x) = \dfrac{d}{dx}(-1)$, so $\dfrac{d}{dx}\ln y - \dfrac{d}{dx}(x\ln x) = 0$, $\dfrac{y'}{y} = \left[\ln x + x\left(\dfrac{1}{x}\right)\right] = \ln x + 1$, and $y' = (\ln x + 1)y$.

52. $\frac{d}{dx}\left(\ln xy - y^2\right) = \frac{d}{dx}(5)$, so $\frac{d}{dx}\left(\ln x + \ln y - y^2\right) = 0$, $\frac{1}{x} + \frac{y'}{y} - 2yy' = 0$, $y + xy' - 2xy^2y' = 0$, $x\left(1 - 2y^2\right)y' = -y$, and so $y' = \frac{y}{x\left(2y^2 - 1\right)}$.

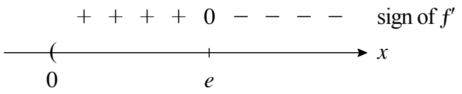
53. $y = x\ln x$. The slope of the tangent line at any point is $y' = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$. In particular, the slope of the tangent line at $(1, 0)$ is $m = \ln 1 + 1 = 1$. Thus, an equation of the tangent line is $y - 0 = 1(x - 1)$, or $y = x - 1$.

54. $y = \ln x^2 = 2\ln x$ and $y' = 2/x$, and this gives the slope of the tangent line at any point (x, y) on the graph of $y = \ln x^2$. In particular, the slope of the tangent line at $(2, \ln 4)$ is $m = \frac{2}{2} = 1$. Therefore, an equation is $y - \ln 4 = 1(x - 2)$, or $y = x + \ln 4 - 2$.

55. $f(x) = \ln x^2 = 2\ln x$ and so $f'(x) = 2/x$. Because $f'(x) < 0$ if $x < 0$ and $f'(x) > 0$ if $x > 0$, we see that f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

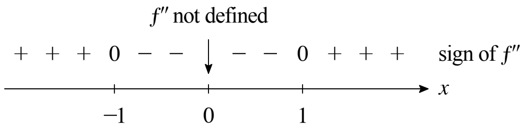
56. $f(x) = \frac{\ln x}{x}$, so $f'(x) = \frac{x\frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$. Observe that $f'(x) = 0$ if $1 - \ln x = 0$, or $x = e$.

The sign diagram of f' on $(0, \infty)$ shows that f is increasing on $(0, e)$ and decreasing on (e, ∞).



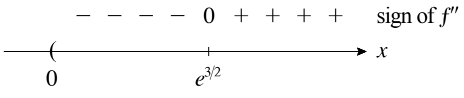
57. $f(x) = x^2 + \ln x^2$, so $f'(x) = 2x + \frac{2x}{x^2} = 2x + \frac{2}{x}$ and $f''(x) = 2 - \frac{2}{x^2}$. To find the intervals of concavity for f, we first set $f''(x) = 0$, giving $2 - \frac{2}{x^2} = 0$, $2 = \frac{2}{x^2}$, $2x^2 = 2$, $x^2 = 1$, and so $x = \pm 1$.

From the sign diagram for f'', we see that f is concave upward on $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 0)$ and $(0, 1)$.

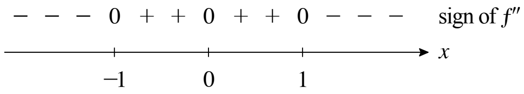


58. $f(x) = \frac{\ln x}{x}$. From Exercise 56, we have $f'(x) = \frac{1 - \ln x}{x^2}$, and so

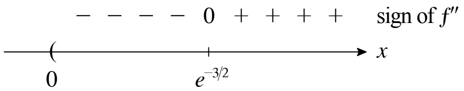
$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)(2x)}{x^4} = \frac{(2\ln x - 3)}{x^3}$. Observe that $f''x) = 0$ implies $2\ln x - 3 = 0$, $\ln x = \frac{3}{2}$, and so $x = e^{3/2}$. From the sign diagram of f'', we see that the graph of f is concave downward on $\left(0, e^{3/2}\right)$ and concave upward on $\left(e^{3/2}, \infty\right)$.



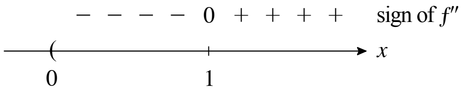
59. $f(x) = \ln\left(x^2 + 1\right)$, so $f'(x) = \frac{2x}{x^2 + 1}$ and $f''(x) = \frac{\left(x^2 + 1\right)(2) - (2x)(2x)}{\left(x^2 + 1\right)^2} = -\frac{2\left(x^2 - 1\right)}{\left(x^2 + 1\right)^2}$. Setting $f''(x) = 0$ gives $x = \pm 1$ as candidates for inflection points of f. From the sign diagram of f'', we see that $(-1, \ln 2)$ and $(1, \ln 2)$ are inflection points of f.



60. $f(x) = x^2 \ln x$, so $f'(x) = 2x \ln x + x^2\left(\frac{1}{x}\right) = 2x \ln x + x$ and $f''(x) = 2\ln x + 2x\left(\frac{1}{x}\right) + 1 = 2\ln x + 3 = 0$ implies that $\ln x = -\frac{3}{2}$, so $x = e^{-3/2}$. From the sign diagram of f'', we see that $\left(e^{-3/2}, -\frac{3}{2}e^{-3}\right)$ is an inflection point of f.



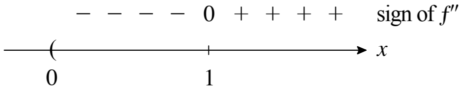
61. $f(x) = x^2 + 2\ln x$, so $f'(x) = 2x + \dfrac{2}{x}$ and $f''(x) = 2 - \dfrac{2}{x^2} = 0$ implies $2 - \dfrac{2}{x^2} = 0$, $x^2 = 1$, and so $x = \pm 1$. We reject the negative root because the domain of f is $(0, \infty)$. The sign diagram of f'' shows that $(1, 1)$ is an inflection point of the graph of f. $f'(1) = 4$. So, an equation of the required tangent line is $y - 1 = 4(x - 1)$ or $y = 4x - 3$.



62. $f(x) = e^{x/2} \ln x$, so $f'(x) = e^{x/2}\left(\dfrac{1}{x}\right) + \frac{1}{2}e^{x/2}\ln x = \left(\dfrac{1}{x} + \dfrac{\ln x}{2}\right)e^{x/2}$ and

$$f''(x) = \left(-\frac{1}{x^2} + \frac{1}{2x}\right)e^{x/2} + \left(\frac{1}{x} + \frac{\ln x}{2}\right)e^{x/2}\left(\frac{1}{2}\right) = \left(-\frac{1}{x^2} + \frac{1}{x} + \frac{1}{4}\ln x\right)e^{x/2}.$$

Thus, $f''(1) = 0$. The sign diagram of f'' shows that $(1, 0)$ is an inflection point. $f(1) = 0$ and $f'(1) = e^{1/2}$, so an equation of the required tangent line is $y - 0 = \sqrt{e}(x - 1)$ or $y = \sqrt{e}x - \sqrt{e}$.



63. $f(x) = x - \ln x$, so $f'(x) = 1 - \dfrac{1}{x} = \dfrac{x-1}{x} = 0$ if $x = 1$, a critical point of f. From the table, we see that f has an absolute minimum at $(1, 1)$ and an absolute maximum at $(3, 3 - \ln 3)$.

x	$\frac{1}{2}$	1	3
$f(x)$	$\frac{1}{2} + \ln 2$	1	$3 - \ln 3$

64. $g(x) = \dfrac{x}{\ln x}$, so $g'(x) = \dfrac{\ln x - x\left(\frac{1}{x}\right)}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2}$. Observe that $g'(x) = 0$ if $x = e$, a critical point of g. From the table, we see that f has an absolute minimum at (e, e) and an absolute maximum at $(5, 3.1067)$.

x	2	e	5
$f(x)$	2.885	e	3.1067

65. $\ln(xy) = x + y$, so $\ln x + \ln y = x + y$. Differentiating with respect to x, we obtain $\dfrac{1}{x} + \dfrac{1}{y}y' = 1 + y'$, $y'\left(\dfrac{1}{y} - 1\right) = 1 - 1/x$, and $y'\left(\dfrac{1-y}{y}\right) = \dfrac{x-1}{x}$. Thus, $y' = \dfrac{y(x-1)}{x(1-y)}$.

66. $\ln x + e^{-y/x} = 10$. Differentiating with respect to x, we obtain $\dfrac{1}{x} + e^{-y/x}\dfrac{d}{dx}\left(-\dfrac{y}{x}\right) = 0$, so $\dfrac{1}{x} + e^{-y/x}\left(\dfrac{-xy' + y \cdot 1}{x^2}\right) = 0$, $\dfrac{xy' - y}{x^2} \cdot e^{-y/x} = \dfrac{1}{x}$, $xy' - y = xe^{y/x}$, $xy' = y + xe^{y/x}$, and finally $\dfrac{dy}{dx} = \dfrac{y + xe^{y/x}}{x}$.

67. $\ln x + xy = 5$. Differentiating with respect to x, we obtain $\dfrac{1}{x} + y + xy' = 0$. Differentiating again, we have $-\dfrac{1}{x^2} + y' + y' + xy'' = 0$, so $xy'' = \dfrac{1}{x^2} - 2y' = \dfrac{1 - 2x^2 y'}{x^2}$. But $y' = \dfrac{-\frac{1}{x} - y}{x} = \dfrac{1 - xy}{x^2}$, so $y'' = \dfrac{1}{x^3}\left[1 + 2x^2\left(\dfrac{1 + xy}{x^2}\right)\right] = \dfrac{3 + 2xy}{x^3}$.

68. $\ln y + y = x$. Differentiating with respect to x, we obtain $\dfrac{y'}{y} + y' = 1$, $y'\left(\dfrac{1}{y} + 1\right) = 1$, $y'\left(\dfrac{1+y}{y}\right) = 1$, and $y' = \dfrac{y}{1+y}$. Differentiating again, we have $y'' = \dfrac{(1+y)\,y' - y\,(y')}{(1+y)^2} = \dfrac{y'}{(1+y)^2} = \dfrac{y}{1+y} \cdot \dfrac{1}{(1+y)^2} = \dfrac{y}{(1+y)^3}$.

69. $\ln y + xy = 1$. Differentiating with respect to x, we obtain $\dfrac{y'}{y} + y + xy' = 0$. Substituting $x = 1$ and $y = 1$ gives $y' + 1 + y' = 0$, so $y' = -\frac{1}{2}$.

70. $\ln x + xe^y = 1$. Differentiating with respect to x, we obtain $\dfrac{1}{x} + e^y + xe^y y' = 0$. Substituting $x = 1$ and $y = 0$ gives $1 + 1 + y' = 0$, so $y' = -2$. Thus, an equation of the tangent line at $(1, 0)$ is $y - 0 = -2\,(x - 1)$ or $y = -2x + 2$.

71. $f(x) = 7.2956 \ln\left(0.0645012x^{0.95} + 1\right)$, so

$$f'(x) = 7.2956 \cdot \frac{\frac{d}{dx}\left(0.0645012x^{0.95} + 1\right)}{0.0645012x^{0.95} + 1} = \frac{7.2956\,(0.0645012)\left(0.95x^{-0.05}\right)}{0.0645012x^{0.95} + 1} = \frac{0.4470462}{x^{0.05}\left(0.0645012x^{0.95} + 1\right)}.$$

Thus, $f'(100) = 0.05799$, or approximately 0.0580%/kg, and $f'(500) = 0.01330$, or approximately 0.0133%/kg.

72. $\ln W = \ln 2.4 + 1.84h$. Differentiating this equation implicitly with respect to h yields $\dfrac{W'}{W} = 1.84$, or $W' = 1.84W$. Therefore, $\Delta W \approx dW = W'\,dh = 1.84W\,dh$. When $h = 1$, $\ln W = \ln 2.4 + 1.84\,(1) \approx 2.71547$, so $W \approx 15.112$. Thus, with $dh = \Delta h = 0.1$, we have $\Delta W = (1.84)\,(15.112)\,(0.1) \approx 2.78061$, and so the weight of the child increases by approximately 2.78 kg.

73. **a.** $W'(t) = \dfrac{d}{dt}\,(49.9 + 17.1 \ln t) = \frac{17.1}{t} > 0$ if $t > 0$, so $W'(t) > 0$ on $[1, 6]$ and W is increasing on $(1, 6)$.

b. $W''(t) = \dfrac{d}{dt}\left(\dfrac{17.1}{t}\right) = -\dfrac{17.1}{t^2} < 0$ on $(1, 6)$, so W is concave downward on $(1, 6)$.

74. **a.** The amount spent in 2012 was $f(0) = 3.7$ (billion dollars). The amount spent in 2014 was $f(2) = 3.7 + 0.84 \ln(2 + 1)$, or approximately \$4.6 billion.

b. $f'(t) = \dfrac{0.84}{t+1}$, so the amount spent annually was growing at the rate of $f'(2) = \dfrac{0.84}{2+1} = 0.28$, or approximately \$0.28 billion/year, in 2014.

75. **a.** $\ln V = \ln\left(C\left(1-\frac{2}{N}\right)^n\right) = \ln C + n\ln\left(1-\frac{2}{N}\right)$, so $\frac{d}{dn}\ln V = \frac{d}{dn}(\ln C) + \frac{d}{dn}\left[n\ln\left(1-\frac{2}{N}\right)\right]$, $\frac{V'}{V} = \ln\left(1-\frac{2}{N}\right)$, and $V' = V\ln\left(1-\frac{2}{N}\right) = C\left(1-\frac{2}{N}\right)^n \ln\left(1-\frac{2}{N}\right)$.

b. The relative rate of change of $V(n)$ is given by $\frac{V'(n)}{V(n)} = \frac{C\left(1-\frac{2}{N}\right)^n \ln\left(1-\frac{2}{N}\right)}{C\left(1-\frac{2}{N}\right)^n} = \ln\left(1-\frac{2}{N}\right)$.

76. **a.** $V(2) = 60{,}000\left(1-\frac{2}{10}\right)^2 = 38{,}400$, or \$38,400.

b. $\frac{V'(2)}{V(2)} = \ln\left(1-\frac{2}{10}\right) \approx -0.223$, or approximately -22.3%.

77. $\ln P(t) = \ln\left(\frac{40+80e^{0.06t}}{20+e^{0.06t}}\right) = \ln\left(40+80e^{0.06t}\right) - \ln\left(20+e^{0.06t}\right)$, so

$$\frac{P'(t)}{P(t)} = \frac{d}{dt}P(t) = \frac{80\left(0.06e^{0.06t}\right)}{40+80e^{0.06t}} - \frac{0.06e^{0.06t}}{20+e^{0.06t}} \text{ and } \left.\frac{P'(t)}{P(t)}\right|_{t=60} = \frac{80\left(0.06e^{3.6}\right)}{40+80e^{3.6}} - \frac{0.06e^{3.6}}{20+e^{3.6}} \approx 0.0204.$$

Therefore, the relative rate of growth of the population five years after the establishment of the biotech research center is approximately 2.04% per month.

78. **a.** The revenue function is $R(x) = px = (200-0.01x\ln x)x = 200x - 0.01x^2\ln x$ and the marginal revenue function is $R'(x) = 200 - 0.01\left[x^2(1/x) + 2x\ln x\right] = 200 - 0.01x - 0.02x\ln x$.

b. The approximate revenue to be realized from the sale of the 500th yacht is $R'(499) = 200 - 0.01(499) - 0.02(499)\ln 499 \approx 133.01$, or approximately \$13,301.

79. $P(x) = 2\ln(2x+1) + 2x - x^2 - 0.3$. We want to maximize the function P with respect to x. Setting $P'(x) = 0$ gives $P'(x) = \frac{2\cdot 2}{2x+1} + 2 - 2x = 0$, or $4 + (2x+1)(2-2x) = 0$, $2x^2 - x - 3 = 0$, and $(2x-3)(x+1) = 0$. Therefore, $x = -1$ or $x = \frac{3}{2}$. We reject the negative root, so $\frac{3}{2}$ is the only critical number of P. Because $P''(x) = 4\frac{d}{dx}(2x+1)^{-1} - 2 = -\frac{4\cdot 2}{(2x+1)^2} - 2 < 0$ for all $x > 0$, we see that the graph of P is concave downward on $(0,\infty)$, implying that $\frac{3}{2}$ gives an absolute maximum for P with value $P\left(\frac{3}{2}\right) \approx 3.22$. Thus, by employing 150 consultants, Seko makes an estimated annual profit of approximately \$3.22 million.

80. **a.** $\ln I = \ln I_0 a^x = \ln I_0 + \ln a^x = \ln I_0 + x(\ln a)$. Therefore, $\frac{I'}{I} = \frac{d}{dx}[\ln I_0 + x(\ln a)] = \frac{d}{dx}[x(\ln a)] = \ln a$ and $I' = (\ln a)I = (\ln a)I_0a^x$.

b. Because $I' = (\ln a)I$, we conclude that I' is proportional to I with $\ln a$ as the constant of proportion.

81. **a.** $100\frac{d}{dx}\left[\ln f(x)\right] = \frac{100f'(x)}{f'(x)}$, and this is precisely the percentage rate of change of f.

b. The percentage rate of growth of the company t years from now is

$$100\frac{d}{dt}[\ln R(t)] = 100\frac{d}{dt}\left[\ln\left(0.1t^{1.5}e^{0.2t}\right)\right] = 100\frac{d}{dt}(\ln 0.1 + 1.5\ln t + 0.2t) = 100\left(\frac{1.5}{t} + 0.2\right).$$

Thus, the percentage rate of growth 3 years from now is $100\left(\frac{1.5}{3} + 0.2\right) = 70$, or 70%/year.

82. **a.** We find $-\dfrac{\frac{d}{dp}\left[\ln f(p)\right]}{\frac{d}{dp}(\ln p)} = -\dfrac{\frac{f'(p)}{f(p)}}{\frac{1}{p}} = -\dfrac{pf'(p)}{f(p)} = E(p)$, proving the assertion.

b. $E(p) = -\dfrac{\frac{d}{dp}\left[\ln f(p)\right]}{\frac{d}{dp}(\ln p)} = -\dfrac{\frac{d}{dp}\left[\ln \frac{e^{-0.1p^{1/2}}}{(2p+1)^{1/2}}\right]}{\frac{1}{p}} = -p\frac{d}{dp}\left[\ln e^{-0.1p^{1/2}} - \ln(2p+1)^{1/2}\right]$

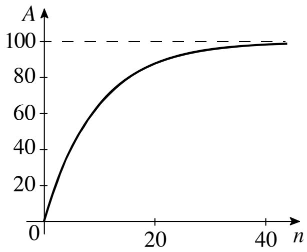
$= -p\frac{d}{dp}\left[-0.1p^{1/2} - \frac{1}{2}\ln(2p+1)\right] = -p\left(-\frac{p^{-1/2}}{20} - \frac{1}{2}\frac{2}{2p+1}\right) = \frac{1}{20}\left(\sqrt{p} + \frac{20p}{2p+1}\right).$

83. The relative rate of change of P is

$$\frac{P'(t)}{P(t)} = \frac{d}{dt}\left[\ln P(t)\right] = \frac{d}{dt}\ln\left(Le^{-\ln(L/P_0)e^{-ct}}\right) = \frac{d}{dt}\left[\ln L - \ln\left(\frac{L}{P_0}\right)e^{-ct}\right] = 0 - \ln\left(\frac{L}{P_0}\right)\left(-ce^{-ct}\right)$$
$$= c\ln\left(\frac{L}{P_0}\right)e^{-ct}.$$

84. **a.** If $0 < r < 100$, then $c = 1 - \dfrac{r}{100}$ satisfies $0 < c < 1$. It suffices to show that $A_1(n) = -\left(1 - \frac{r}{100}\right)^n$ is increasing; that is, it suffices to show that $A_2(n) = -A_1(n) = \left(1 - \frac{r}{100}\right)^n$ is decreasing. Let $y = \left(1 - \frac{r}{100}\right)^n$. Then $\ln y = \ln\left(1 - \frac{r}{100}\right)^n = \ln c^n = n\ln c$. Differentiating both sides with respect to n, we find $\frac{y'}{y} = \ln c$, and so $y' = (\ln c)\left(1 - \frac{r}{100}\right)^n$. This is negative because $\ln c < 0$ and $\left(1 - \frac{r}{m}\right)^n > 0$ for $0 < r < 100$. Therefore, A is an increasing function of n on $(0, \infty)$.

b.



c. $\lim_{n\to\infty} A(n) = \lim_{n\to\infty} 100\left[1 - \left(1 - \frac{r}{100}\right)^n\right] = 100.$

85. **a.** $R = \log\dfrac{10^6 I_0}{I_0} = \log 10^6 = 6.$

b. $I = I_0 10^R$ by definition. Taking the natural logarithm on both sides, we find $\ln I = \ln I_0 10^R = \ln I_0 + \ln 10^R = \ln I_0 + R\ln 10$. Differentiating implicitly with respect to R, we obtain $\dfrac{I'}{I} = \ln 10$. Therefore, $\Delta I \approx dI = \dfrac{dI}{dR}\Delta R = (\ln 10)\, I\, \Delta R$. If $|\Delta R| \leq (0.02)(6) = 0.12$ and $I = 1{,}000{,}000 I_0$, (see part (a)), then $|\Delta I| \leq (\ln 10)(1{,}000{,}000 I_0)(0.12) \approx 276{,}310.21 I_o$. Thus, the error is at most 276,310 times the standard reference intensity.

86. **a.** $R(S_0) = k\ln\frac{S_0}{S_0} = k\ln 1 = 0$ because $\ln 1 = 0$.

b. $\dfrac{dR}{dS} = \dfrac{d}{dS}k\ln\dfrac{S}{S_0} = k\dfrac{d}{dS}(\ln S - \ln S_0) = k\dfrac{d}{dS}(\ln S) = \dfrac{k}{S}$, and so $\dfrac{dr}{dS}$ is inversely proportional to S with k as the constant of proportionality. Our result says that if the stimulus is small, then a small change in S is easily felt. But if the stimulus is larger, then a small change in S is not as discernible.

87. $-C\ln y+Dy=A\ln x-Bx+E$. Differentiating implicitly gives $-C\frac{y'}{y}+Dy'=\frac{Ax'}{x}-Bx'$,

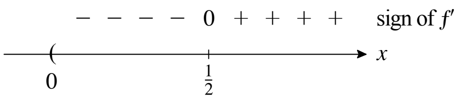
$$\left(D-\frac{C}{y}\right)y'=\left(\frac{A}{x}-B\right)x',\ \left(\frac{Dy-C}{y}\right)y'=\left(\frac{A-Bx}{x}\right)x', \text{ and so } y'=\frac{(A-Bx)\,yx'}{(Dy-C)\,x}.$$

88. We differentiate $Vt=p-k\ln\left(1-\frac{p}{x_0}\right)$ with respect to t, obtaining

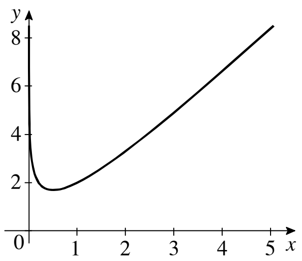
$$V=\frac{dp}{dt}-k\frac{-\frac{1}{x_0}\frac{dp}{dt}}{1-\frac{p}{x_0}}=\frac{dp}{dt}\left(1+\frac{k}{x_0-p}\right)=\frac{dp}{dt}\left(\frac{x_0-p+k}{x_0-p}\right). \text{ Thus, } \frac{dp}{dt}=\frac{V\,(x_0-p)}{x_0-p+k}.$$

89. $f(x)=2x-\ln x$. We first gather the following information on f.

1. The domain of f is $(0,\infty)$.
2. There is no y-intercept.
3. $\lim_{x\to\infty}(2x-\ln x)=\infty$.
4. There is no asymptote.
5. $f'(x)=2-\frac{1}{x}=\frac{2x-1}{x}$. Observe that $f'(x)=0$ at $x=\frac{1}{2}$, a critical point of f. From the sign diagram of f', we conclude that f is decreasing on $\left(0,\frac{1}{2}\right)$ and increasing on $\left(\frac{1}{2},\infty\right)$.

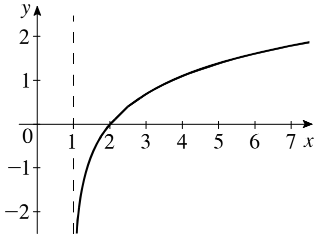


6. The results of part 5 show that $\left(\frac{1}{2},1+\ln 2\right)$ is a relative minimum of f.
7. $f''(x)=\frac{1}{x^2}$ and is positive if $x>0$, so the graph of f is concave upward on $(0,\infty)$.
8. The results of part 7 show that f has no inflection point.



90. $f(x)=\ln(x-1)$. We first gather the following information on f.

1. The domain of f is obtained by requiring that $x-1>0$. We find the domain to be $(1,\infty)$.
2. Because $x\neq 0$, there is no y-intercept. Next, setting $y=0$ gives $x-1=1$, so the x-intercept is 2.
3. $\lim_{x\to 1^+}\ln(x-1)=-\infty$.
4. There is no horizontal asymptote. Because $\lim_{x\to 1^+}\ln(x-1)=-\infty$, $x=1$ is a vertical asymptote.
5. $f'(x)=\frac{1}{x-1}>0$ for $x>1$, so f has no critical number.
6. The results of part 5 show that f is increasing on $(1,\infty)$.
7. $f''(x)=-\frac{1}{(x-1)^2}$. Because $f''(x)<0$ for $x>1$, we see that f is concave downward on $(1,\infty)$.
8. From the results of part 7, we see that f has no inflection point.



91. **a.** $f(x) = b^x$. Taking the logarithm of each side, we have $\ln f(x) = \ln b^x = x \ln b$, So $\frac{d}{dx}\left[\ln f(x)\right] = \frac{d}{dx}(x \ln b)$ and $\frac{f'(x)}{f(x)} = \ln b$. Therefore, $f'(x) = (\ln b) f(x) = (\ln b) b^x$.

b. $f'(x) = \frac{d}{dx}(3^x) = (\ln 3) 3^x$.

92. **a.** $f(x) = \log_b x$, so $x = b^{f(x)}$. Thus, $\frac{d}{dx}(x) = \frac{d}{dx}\left[b^{f(x)}\right]$, $1 = (\ln b) b^{f(x)} f'(x)$, and therefore $f'(x) = \frac{1}{(\ln b) b^{f(x)}} = \frac{1}{(\ln b) x}$.

b. $f'(x) = \frac{d}{dx}\left(\log_{10} x\right) = \frac{1}{(\ln 10) x}$.

93. $f'(x) = \frac{d}{dx}\left(x^3 2^x\right) = x^3 \frac{d}{dx}(2^x) + 2^x \frac{d}{dx}\left(x^3\right) = (\ln 2) x^3 2^x + 3x^2 2^x = x^2 (x \ln 2 + 3) 2^x$.

94. $g'(x) = \frac{d}{dx}\left(\frac{10^x}{x+1}\right) = \frac{(x+1)(\ln 10) 10^x - 10^x}{(x+1)^2} = \frac{[(x+1)(\ln 10) - 1] 10^x}{(x+1)^2}$.

95.
$$h'(x) = \frac{d}{dx}\left(x^2 \log_{10} x\right) = x^2 \frac{d}{dx} \log_{10} x + \left(\log_{10} x\right) \frac{d}{dx}\left(x^2\right) = \frac{x^2}{x \ln 10} + 2x \log_{10} x$$
$$= x\left(\frac{1}{\ln 10} + 2\log_{10} x\right).$$

96. $f(x) = 3^{x^2} + \log_2\left(x^2 + 1\right)$, so
$$f'(x) = \frac{d}{dx}\left[3^{x^2} + \log_2\left(x^2+1\right)\right] = (\ln 3) 3^{x^2} (2x) + \frac{2x}{(\ln 2)\left(x^2+1\right)}$$
$$= 2x\left[(\ln 3) 3^{x^2} + \frac{1}{(\ln 2)\left(x^2+1\right)}\right].$$

97. False. $\ln 5$ is a constant function and $f'(x) = 0$.

98. True. $f(x) = \ln a^x = x \ln a$, so $f'(x) = \frac{d}{dx}(x \cdot \ln a) = \ln a$.

99. If $x \leq 0$, then $|x| = -x$. Therefore, $\ln |x| = \ln(-x)$. Writing $f(x) = \ln |x|$, we have $|x| = -x = e^{f(x)}$. Differentiating both sides with respect to x and using the Chain Rule,we have $-1 = e^{f(x)} \cdot f'(x)$, so $f'(x) = -\frac{1}{e^{f(x)}} = -\frac{1}{-x} = \frac{1}{x}$.

100. Let $f(x) = \ln x$. Then by definition, $f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0} \frac{\ln(h+1)}{h}$. But $f'(x) = \frac{d}{dx} \ln x = \frac{1}{x}$, and so with $h = x$, we have $f'(1) = 1 = \lim_{x\to 0} \frac{\ln(x+1)}{x}$.

5.6 Exponential Functions as Mathematical Models

Concept Questions page 398

1. $Q(t) = Q_0e^{kt}$ where $k > 0$ represents exponential growth and $k < 0$ represents exponential decay. The larger the magnitude of k, the more quickly the former grows and the more quickly the latter decays.

2. The half-life of a radioactive substance is the time it takes for the substance to decay to half its original amount.

3. $Q(t) = \dfrac{A}{1 + Be^{-kt}}$, where A, B, and k are positive constants. Q increases rapidly for small values of t but the rate of increase slows down as Q (always increasing) approaches the number A.

Exercises page 399

1. a. The growth constant is $k = 0.02$.

b. Initially, there are 300 units present.

c.

t	0	10	20	100	1000
Q	300	366	448	2217	1.46×10^{11}

2. a. $k = -0.06$.

b. $Q_0 = 2000$.

c.

t	0	5	10	20	100
Q	2000	1482	1098	602	5

3. a. $Q(t) = Q_0e^{kt}$. Here $Q_0 = 100$ and so $Q(t) = 100e^{kt}$. Because the number of cells doubles in 20 minutes, we have $Q(20) = 100e^{20k} = 200$, $e^{20k} = 2$, $20k = \ln 2$, and so $k = \frac{1}{20}\ln 2 \approx 0.03466$. Thus, $Q(t) = 100e^{0.03466t}$.

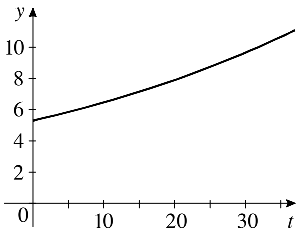
b. We solve the equation $100e^{0.03466t} = 1{,}000{,}000$, obtaining $e^{0.03466t} = 10{,}000$, $0.03466t = \ln 10{,}000$, and so $t = \dfrac{\ln 10{,}000}{0.03466} \approx 266$, or 266 minutes.

c. $Q(t) = 1000e^{0.03466t}$.

4. $Q(t) = 5.3e^{kt}$. Because the population grows at the rate of 2% per year, we have $N(t) = 5.3e^{0.02t}$. Note that $t = 0$ corresponds to the beginning of 1990.

a.

t	$Q(t)$
1990	5.3
1995	5.86
2000	6.47
2005	7.15
2010	7.91
2015	8.74
2020	9.66
2025	10.67



b. $Q'(t) = 5.3(0.02)e^{0.02t} = 0.106e^{0.02t}$, so the rate of growth in the year 2010 is $Q'(20) = 0.106e^{0.02(20)} \approx 0.1581$, or approximately 0.16 billion people per year.

5. a. We solve the equation $5.3e^{0.02t} = 3(5.3)$, obtaining $e^{0.02t} = 3$, $0.02t = \ln 3$, and so $t = \dfrac{\ln 3}{0.02} \approx 54.93$. Thus, the world population will triple in approximately 54.93 years.

b. If the growth rate is 1.8%, then proceeding as before, we find $N(t) = 5.3e^{0.018t}$. If $t = 54.93$, the population would be $N(54.93) = 5.3e^{0.018(54.93)} \approx 14.25$, or approximately 14.25 billion.

6. The resale value of the machinery at any time t is given by $V(t) = 500{,}000e^{-kt}$, where $t = 0$ represents three years ago. We have $V(3) = 320{,}000 = 500{,}000e^{-3k}$, which gives $e^{-3k} = \dfrac{320{,}000}{500{,}000} = 0.64$. Therefore, $-3k \ln e = \ln 0.64$ and $k = \dfrac{\ln 0.64}{-3} \approx 0.149$. Four years from now, the resale value of the machinery will be $V(7) = 500{,}000e^{-(0.149)(7)} \approx 176{,}198$, or approximately \$176,198.

7. $P(h) = p_0e^{-kh}$, so $P(0) = p_0 = 15$. Thus, $P(4000) = 15e^{-4000k} = 12.5$, $e^{-4000k} = \frac{12.5}{15}$, $-4000k = \ln\left(\frac{12.5}{15}\right)$, and so $k = 0.00004558$. Therefore, $P(12{,}000) = 15e^{-0.00004558(12{,}000)} = 8.68$, or 8.7 lb/in^2. The rate of change of atmospheric pressure with respect to altitude is given by $P'(h) = \dfrac{d}{dh}\left(15e^{-0.00004558h}\right) = -0.0006837e^{-0.00004558h}$. Thus, the rate of change of atmospheric pressure with respect to altitude when the altitude is 12,000 feet is $P'(12{,}000) = -0.0006837e^{-0.00004558(12{,}000)} \approx -0.00039566$. That is, it is declining at the rate of approximately 0.0004 lb/in^2/ft.

8. We are given that $Q(280) = 20$. Using this condition, we have $Q(280) = Q_0 \cdot 2^{-280/140} = 20$. Thus, $Q_0 \cdot 2^{-2} = 20$, so $\frac{1}{4}Q_0 = 20$ and $Q_0 = 80$. Thus, the initial amount was 80 mg.

9. Suppose the amount of P-32 at time t is given by $Q(t) = Q_0e^{-kt}$, where Q_0 is the amount present initially and k is the decay constant. Because this element has a half-life of 14.2 days, we have $\frac{1}{2}Q_0 = Q_0e^{-14.2k}$, so $e^{-14.2k} = \frac{1}{2}$, $-14.2k = \ln\frac{1}{2}$, and $k = -\frac{\ln(1/2)}{14.2} \approx 0.0488$. Therefore, the amount of P-32 present at any time t is given by $Q(t) = 100e^{-0.0488t}$. In particular, the amount left after 7.1 days is given by $Q(7.1) = 100e^{-0.0488(7.1)} = 100e^{-0.34648} \approx 70.717$, or 70.717 grams. The rate at which the element decays is $Q'(t) = \dfrac{d}{dt}\left(100e^{-0.0488t}\right) = 100(-0.0488)e^{-0.0488t} = -4.88e^{-0.0488t}$. Therefore, $Q'(7.1) = -4.88e^{-0.0488(7.1)} \approx -3.451$; that is, it is decreasing at the rate of 3.451 g/day.

10. Suppose the amount of Sr-90 present at time t is given by $Q(t) = Q_0e^{-kt}$, where Q_0 is the amount present initially and k is the decay constant. Because this element has a half-life of 27 years, we find $\frac{1}{2}Q_0 = Q_0e^{-27k}$, $e^{-27k} = \frac{1}{2}$, $-27k = \ln\frac{1}{2}$, and so $k = -\frac{1}{27}\ln\frac{1}{2}$. Therefore, the amount of Sr-90 present at time t is given by $Q(t) = Q_0e^{(1/27)\ln(1/2)\cdot t} = Q_0e^{(t/27)\ln(1/2)} = Q_0\left(\frac{1}{2}\right)^{t/27}$. To find t when $Q(t) = \frac{1}{4}Q_0$, we calculate $\frac{1}{4}Q_0 = Q_0\left(\frac{1}{2}\right)^{t/27}$, $\left(\frac{1}{2}\right)^{t/27} = \frac{1}{4}$, $\frac{1}{27}t\ln\frac{1}{2} = \ln\frac{1}{4}$, $t = 27\dfrac{\ln\frac{1}{4}}{\ln\frac{1}{2}} = 27\left(\dfrac{-\ln 4}{-\ln 2}\right) \approx 54$, or approximately 54 years.

11. We solve the equation $0.2Q_0 = Q_0e^{-0.00012t}$, obtaining $\ln 0.2 = -0.00012t$ and $t = \dfrac{\ln 0.2}{-0.00012} \approx 13{,}412$, or approximately 13,412 years.

12. We solve the equation $0.18Q_0 = Q_0e^{-0.00012t}$, obtaining $\ln 0.18 = -0.00012t$ and $t = \dfrac{\ln 0.18}{-0.00012} \approx 14{,}290$, or approximately 14,290 years.

13. **a.** $f(t) = 157e^{-0.55t}$, so $f'(t) = 157(-0.55)e^{-0.55t} = -86.35e^{-0.55t}$. So the number of annual bank failures was changing at the rate of $f'(1) \approx -49.82$; that is, it was dropping at the rate of approximately 50/year in 2011.

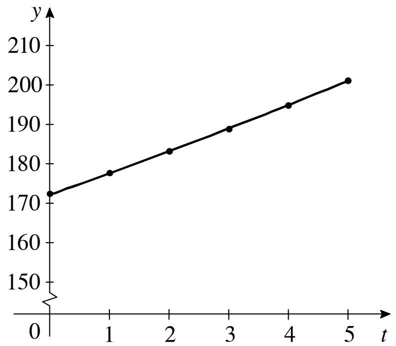
b. The projected number of failures in 2013 is $f(3) = 157e^{-0.55(3)} \approx 30.15$, or approximately 30.

14. $f(t) = 172.2e^{0.031t}$.

a. The projected number of online shoppers (in millions) from 2010 through 2015 is shown in the table.

t	0	1	2	3	4	5
$f(t)$	172.2	177.6	183.2	189.0	194.9	201.1

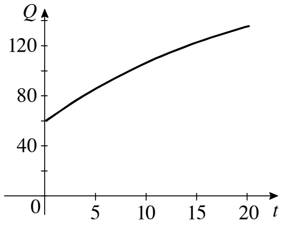
b.



15. a. $S = S_0e^{-kt}$, so $S(0) = S_0 = 100$. Thus, $S(t) = 100e^{kt}$. Next, $S(5) = 150$ gives $100e^{5k} = 150$, so $e^{5k} = \frac{150}{100} = 1.5$, $5k = \ln 1.5$, and $k \approx 0.0811$. Thus, the model is $S(t) = 100e^{0.0811t}$.

b. The sales of Garland Corporation in 2013 were $S(3) = 100e^{0.0811(3)} \approx 127.5$, or approximately \$127.5 million.

16.

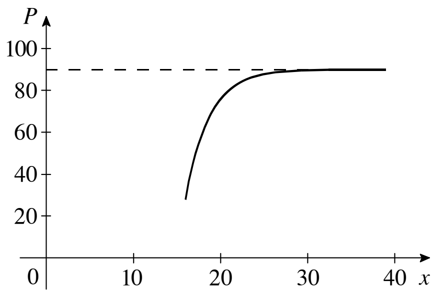


a. $Q(0) = 120\left(1 - e^{0}\right) + 60 = 60$, or 60 wpm.

b. $Q(10) = 120\left(1 - e^{-0.5}\right) + 60 = 107.22$, or approximately 107 wpm.

c. $Q(20) = 120\left(1 - e^{-1}\right) + 60 = 135.65$, or approximately 136 wpm.

17.



a. The percentage of 16-year-olds with a driver's license was $P(16) = 90\left[1 - e^{-0.37(16-15)}\right] \approx 27.8$, or approximately 27.8.

b. The percentage of 20-year-olds with a driver's license was $P(2) = 90\left[1 - e^{-0.37(20-15)}\right] \approx 75.8$, or approximately 75.8.

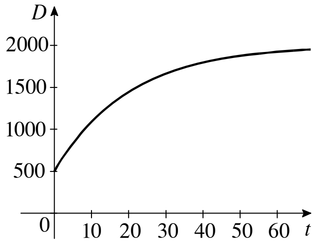
c. The percentage of 39-year-olds with a driver's license was $P(39) = 90\left[1 - e^{-0.37(39-15)}\right] \approx 90.0$, or approximately 90.0.

18. a. $S(t) = 50{,}000 + Ae^{-kt}$. Using the condition $S(1) = 83{,}515$ and $S(3) = 65{,}055$, we have $S(1) = 50{,}000 + Ae^{-k} = 83{,}515$ and $S(3) = 50{,}000 + Ae^{-3k} = 65{,}055$. The first equation gives $Ae^{-k} = 33{,}515$ and the second gives $Ae^{-3k} = 15055$, so $\dfrac{Ae^{-k}}{Ae^{-3k}} = \dfrac{33{,}515}{15{,}055}$, $e^{2k} = \dfrac{33{,}515}{15{,}055}$, and $k = \frac{1}{2}\ln\frac{33{,}515}{15{,}055} \approx 0.40014$.

b. $A = 33{,}515e^{k} = 33{,}515e^{0.40014} = 50{,}006$, so $S(t) = 50{,}000 + 50{,}006e^{-0.40014t}$. In particular, $S(4) = 50{,}000 + 50006e^{-0.40014(4)} \approx 60{,}090$, or approximately \$60,090.

c. $S'(t) = \dfrac{d}{dt}\left(50{,}000 + 50{,}006e^{-0.40014t}\right) = 50{,}006\,(-0.40014)\,e^{-0.40014t} = -20{,}009.4e^{-0.40014t}$, and so $S'(4) = -20{,}009.4e^{-0.40014(4)} \approx -4037.6$. That is, the sales volume is falling by approximately \$4038/week.

19.



a. After 1 month, the demand is $D(1) = 2000 - 1500e^{-0.05} \approx 573$, after 12 months it is $D(12) = 2000 - 1500e^{-0.6} \approx 1177$, after 24 months it is $D(24) = 2000 - 1500e^{-1.2} \approx 1548$, and after 60 months, it is $D(60) = 2000 - 1500e^{-3} \approx 1925$.

b. $\lim_{t\to\infty} D(t) = \lim_{t\to\infty} \left(2000 - 1500e^{-0.05t}\right) = 2000$, and we conclude that the demand is expected to stabilize at 2000 computers per month.

c. $D'(t) = -1500e^{-0.05t}(-0.05) = 75e^{-0.05t}$. Therefore, the rate of growth after 10 months is $D'(10) = 75e^{-0.5} \approx 45.49$, or approximately 46 computers per month.

20. a. The proportion that will fail after 3 years is $P(3) = 100\left(1 - e^{-0.3}\right) \approx 25.92\%$. Therefore, 74% will be usable.

b. $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} 100\left(1 - e^{-0.1t}\right) = 100$, so all will eventually fail, as one might expect.

21. a. The length is given by $f(5) = 200\left(1 - 0.956e^{-0.18\cdot 5}\right) \approx 122.26$, or approximately 122.3 cm.

b. $f'(t) = 200(-0.956)e^{-0.18t}(-0.18) = 34.416e^{-0.18t}$, so a 5-year-old is growing at the rate of $f'(5) = 34.416e^{-0.18(5)} \approx 13.9925$, or approximately 14 cm/yr.

c. The maximum length is given by $\lim_{t\to\infty} 200\left(1 - 0.956e^{-0.18t}\right) = 200$, or 200 cm.

22. a. $Q(1) = \dfrac{1000}{1 + 199e^{-0.8}} \approx 11.06$, or 11 children.

b. $Q(10) = \dfrac{1000}{1 + 199e^{-8}} \approx 937.4$, or 937 children.

c. $\lim_{t\to\infty} \dfrac{1000}{1 + 199e^{-0.8t}} = 1000$, or 1000 children.

23. a. $N(0) = \dfrac{400}{1 + 39} = 10$ flies.

b. $\lim_{t\to\infty} \dfrac{400}{1 + 39e^{-0.16t}} = 400$ flies.

c. $N(20) = \dfrac{400}{1 + 39e^{-0.16(20)}} \approx 154.5$, or 154 flies.

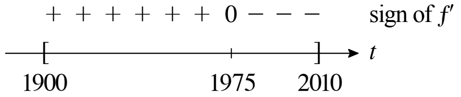
d. $N'(t) = \dfrac{d}{dt}\left[400\left(1 + 39e^{-0.16t}\right)^{-1}\right] = -400\left(1 + 39e^{-0.16t}\right)^{-2} \dfrac{d}{dt}\left(39e^{-0.16t}\right) = \dfrac{2496e^{-0.16t}}{\left(1 + 39e^{-0.16t}\right)^2}$, so $N'(20) = \dfrac{2496e^{-0.16\cdot 20}}{\left(1 + 39e^{-0.16\cdot 20}\right)^2} \approx 15.17$, or approximately 15 fruit flies per day.

24. The projected population of citizens aged 45–64 in 2010 is $P(20) = \dfrac{197.9}{1 + 3.274e^{-0.0361(20)}} \approx 76.3962$, or 76.4 million.

25. $f(t) = \dfrac{40e^{-(t-1975)/20}}{\left[1+e^{-(t-1975)/20}\right]^2}$. Let $u = (t-1975)/20$. Then $\dfrac{du}{dt} = \dfrac{1}{20}$ and

$$f'(t) = f'(u)\frac{du}{dt} = 40 \cdot \frac{\left(1+e^{-u}\right)^2(-1) - e^{-u}(2)\left(1+e^{-u}\right)e^{-u}(-1)}{\left(1+e^{-u}\right)^4}\left(\frac{1}{20}\right)$$

$$= 2 \cdot \frac{\left(1+e^{-u}\right)e^{-u}\left[-\left(1+e^{-u}\right)+2e^{-u}\right]}{\left(1+e^{-u}\right)^4} = \frac{2e^{-u}\left(e^{-u}-1\right)}{\left(1+e^{-u}\right)^3} = \frac{2e^{-(t-1975)/20}\left[e^{-(t-1975)/20}-1\right]}{\left[1+e^{-(t-1975)/20}\right]^3}.$$



Setting $f'(t) = 0$ gives $e^{-(t-1975)/20} = 1$, so $-(t-1975)/20 = \ln 1 = 0$ and $t = 1975$ is a critical number of f. From the sign diagram, we see that f has a relative maximum value at $t = 1975$. Since $t = 1975$ is the only critical number in the interval $(1900, 2010)$, we see that it gives an absolute maximum value of $f(1975) \approx 10$. We conclude that the maximum rate of production of crude oil in the U.S. occurred around 1975 and was approximately 10 million barrels per day.

26. The expected population of the U.S. in 2020 is $P(3) = \dfrac{616.5}{1+4.02e^{-0.5(3)}} \approx 324.99$, or approximately 325 million people.

27. The first of the given conditions implies that $f(0) = 300$, that is, $300 = \dfrac{3000}{1+Be^0} = \dfrac{3000}{1+B}$. Thus, $1+B = 10$, and $B = 9$. Therefore, $f(t) = \dfrac{3000}{1+9e^{-kt}}$. Next, the condition $f(2) = 600$ gives the equation $600 = \dfrac{3000}{1+9e^{-2k}}$, so $1+9e^{-2k} = 5$, $e^{-2k} = \frac{4}{9}$, and $k = -\frac{1}{2}\ln\frac{4}{9}$. Therefore, $f(t) = \dfrac{3000}{1+9e^{(1/2)t\cdot\ln(4/9)}} = \dfrac{3000}{1+9\left(\frac{4}{9}\right)^{t/2}}$. The number of students who had heard about the policy four hours later is given by $f(4) = \dfrac{3000}{1+9\left(\frac{4}{9}\right)^2} = 1080$, or 1080 students. To find the rate at which the rumor was spreading at any time time, we compute

$$f'(t) = \tfrac{d}{dt}\left[3000\left(1+9e^{-0.405465t}\right)^{-1}\right] = (3000)(-1)\left(1+9e^{-0.405465}\right)^{-2}\tfrac{d}{dt}\left(9e^{-0.405465t}\right)$$

$$= -3000(9)(-0.405465)e^{-0.405465t}\left(1+9e^{-0.405465t}\right)^{-2} = \frac{10947.555e^{-0.405465t}}{\left(1+9e^{-0.405465t}\right)^2}.$$

In particular, the rate at which the rumor was spreading 4 hours after the ceremony is given by

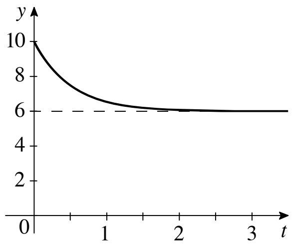
$f'(4) = \dfrac{10947.555e^{-0.405465\cdot 4}}{\left(1+9e^{-0.405465\cdot 4}\right)^2} \approx 280.26$. Thus, the rumor is spreading at the rate of 280 students per hour.

28. **a.** $f(t) = 6+4e^{-2t}$, so $f'(t) = -8e^{-2t} < 0$ for all t in $(0,\infty)$. Thus, f is decreasing on $(0,\infty)$.

b. $f''(t) = 16e^{-2t} > 0$ for all t in $(0,\infty)$, so f is concave upward on $(0,\infty)$.

c. $\lim\limits_{t\to\infty} f(t) = \lim\limits_{t\to\infty}\left(6+4e^{-2t}\right) = 6$.

d.



29. $x(t) = \dfrac{15\left(1-\left(\frac{2}{3}\right)^{3t}\right)}{1-\frac{1}{4}\left(\frac{2}{3}\right)^{3t}}$, so $\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} \dfrac{15\left[1-\left(\frac{2}{3}\right)^{3t}\right]}{1-\frac{1}{4}\left(\frac{2}{3}\right)^{3t}} = \dfrac{15(1-0)}{1-0} = 15$, or 15 lb.

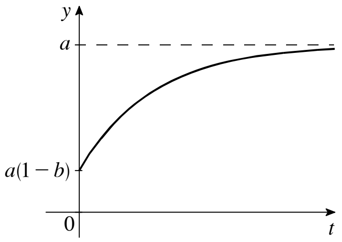
30. a. $f'(t) = \frac{d}{dt}\left[a\left(1-be^{-kt}\right)\right] = \frac{d}{dt}(a) - \frac{d}{dt}abe^{-kt} = 0 - be^{-kt}(-k) = bke^{-kt}$. Because $f'(t) > 0$ for all $t \geq 0$, f is increasing on $(0, \infty)$.

b. $f''(t) = \frac{d}{dt}\left(bke^{-kt}\right) = -bk^2e^{-kt} < 0$ on $(0, \infty)$, and the conclusion follows.

c. $\lim\limits_{t\to\infty} f(t) = \lim\limits_{t\to\infty}\left[a\left(1-be^{-kt}\right)\right] = \lim\limits_{t\to\infty} a - \lim\limits_{t\to\infty} abe^{-kt}$

$= a - 0 = a$.

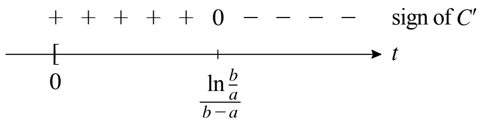
d.



31. a. $C(t) = \dfrac{k}{b-a}\left(e^{-at} - e^{-bt}\right)$, so

$C'(t) = \dfrac{k}{b-a}\left(-ae^{-at} + be^{-bt}\right) = \dfrac{kb}{b-a}\left[e^{-bt} - \left(\dfrac{a}{b}\right)e^{-at}\right] = \dfrac{kb}{b-a}e^{-bt}\left[1 - \dfrac{a}{b}e^{(b-a)t}\right]$.

$C'(t) = 0$ implies that $1 = \dfrac{a}{b}e^{(b-a)t}$, or $t = \dfrac{\ln\left(\frac{b}{a}\right)}{b-a}$. The sign diagram of C' shows that this value of t gives a maximum.



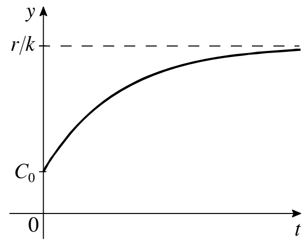
b. $\lim\limits_{t\to\infty} C(t) = 0$.

32. a. $\lim\limits_{t\to\infty}\left[\frac{r}{k} - \left(\frac{r}{k} - C_0\right)e^{-kt}\right] = \frac{r}{k}$, and this shows that in the long run the concentration of the glucose solution approaches $\frac{r}{k}$.

b. $C'(t) = -\left(\frac{r}{k} - C_0\right)e^{-kt}(-k) = k\left(\frac{r}{k} - C_0\right)e^{-kt} > 0$ for all $t > 0$ because $\frac{r}{k} > C_0$ for all $t > 0$. Thus, C is increasing on $(0, \infty)$.

c. $C''(t) = -k^2\left(\frac{r}{k} - C_0\right)e^{-kt} < 0$ because $\frac{r}{k} > C_0$ for all $t > 0$. Thus, the graph of C is concave downward.

d.



33. a. We solve $Q_0e^{-kt} = \frac{1}{2}Q_0$ for t, obtaining $e^{-kt} = \frac{1}{2}$, $\ln e^{-kt} = \ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$, $-kt = -\ln 2$, and so $\bar{t} = \frac{\ln 2}{k}$.

b. $\bar{t} = \frac{\ln 2}{0.0001238} \approx 5598.927$, or approximately 5599 years.

34. **a.** $Q'(t)=\frac{d}{dt}\left(\frac{A}{1+Be^{-kt}}\right)=A\frac{d}{dt}\left(1+Be^{-kt}\right)^{-1}=-A\left(1+Be^{-kt}\right)^{-2}\frac{d}{dt}\left(1+Be^{-kt}\right)$

$=-A\left(1+Be^{-kt}\right)^{-2}\left(-kBe^{-kt}\right)=\frac{kABe^{-kt}}{\left(1+Be^{-kt}\right)^2}$ (1)

Next,

$kQ\left(1-\frac{Q}{A}\right)=k\left(\frac{A}{1+Be^{-kt}}\right)\left(1-\frac{1}{1+Be^{-kt}}\right)=k\left(\frac{A}{1+Be^{-kt}}\right)\left(\frac{1+Be^{-kt}-1}{1+Be^{-kt}}\right)=\frac{kABe^{-kt}}{\left(1+Be^{-kt}\right)^2}$ (2).

The desired result follows by comparing equations (1) and (2).

b. Because $Q(t)<A$, $Q'=kQ\left(1-\frac{Q}{A}\right)>0$, and we see that Q is increasing on $(0,\infty)$.

35. **a.** From the results of Exercise 34, we have $Q'=kQ\left(1-\frac{Q}{A}\right)$, so

$Q''=\frac{d}{dt}\left(kQ-\frac{k}{A}Q^2\right)=kQ'-\frac{2k}{A}QQ'=\frac{k}{A}Q'(A-2Q)$. Setting $Q''=0$ gives $Q=\frac{A}{2}$ since $Q'>0$ for all t. Furthermore, $Q''>0$ if $Q<\frac{A}{2}$ and $Q''<0$ if $Q>\frac{A}{2}$, So the graph of Q has an inflection point when $Q=\frac{A}{2}$. To find the value of t, we solve the equation $\frac{A}{2}=\frac{A}{1+Be^{-kt}}$, obtaining $1+Be^{-kt}=2$, $Be^{-kt}=1$, $e^{-kt}=\frac{1}{B}$, $-kt=\ln\frac{1}{B}=-\ln B$, and so $t=\frac{\ln B}{k}$.

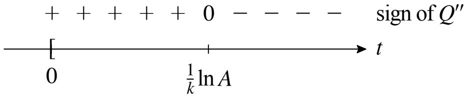
b. The quantity Q increases most rapidly at the instant of time when it reaches one-half of the maximum quantity. This occurs at $t=\frac{\ln B}{k}$.

36. $Q(t)=\frac{A}{1+Be^{-kt}}$, so $Q(t_1)=\frac{A}{1+Be^{-kt_1}}=Q_1$ implies that $A=Q_1+Q_1e^{-kt_1}B$, so $e^{-kt_1}=\frac{A-Q_1}{BQ_1}$ (1). Next, we have $Q(t_2)=\frac{A}{1+Be^{-kt_2}}=Q_2$, and this leads to $e^{-kt_2}=\frac{A-Q_2}{BQ_2}$ (2). Dividing equation (1) by equation (2) gives $\frac{e^{-kt_1}}{e^{-kt_2}}=\frac{A-Q_1}{BQ_1}\cdot\frac{BQ_2}{A-Q_2}$, so $e^{k(t_2-t_1)}=\frac{Q_2(A-Q_1)}{Q_1(A-Q_2)}$, $k(t_2-t_1)=\ln\frac{Q_2(A-Q_1)}{Q_1(A-Q_2)}$, and $k=\frac{1}{t_2-t_1}\ln\frac{Q_2(A-Q_1)}{Q_1(A-Q_2)}$.

37. We use the result of Exercise 36 with $t_1=14$, $t_2=21$, $A=600$, $Q_1=76$, and $Q_2=167$ to obtain $k=\frac{1}{21-14}\ln\left[\frac{167(600-76)}{76(600-167)}\right]\approx 0.14$.

38. **a.** $Q'(t)=Ce^{-Ae^{-kt}}\frac{d}{dt}\left(-Ae^{-kt}\right)=-ACe^{-Ae^{-kt}}\cdot e^{-kt}(-k)=ACke^{\left(-Ae^{-kt}-kt\right)}$.

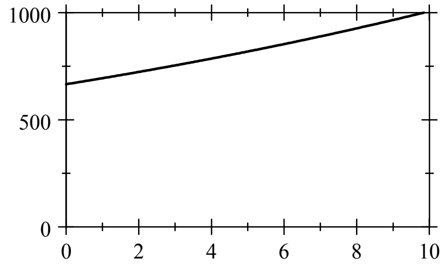
b. $Q''(t)=ACke^{\left(-Ae^{-kt}-kt\right)}\left[-k-Ae^{-kt}(-k)\right]=0$, if $Ae^{-kt}=1$, so $e^{-kt}=\frac{1}{A}$, $-kt=\ln\frac{1}{A}$, and $t=-\frac{1}{k}\ln\frac{1}{A}=\frac{1}{k}\ln A$. The sign diagram shows that $t=\frac{1}{k}\ln A$ is an inflection point, and so the growth is most rapid at this time.



c. $\lim_{t\to\infty}Q(t)=C$.

Using Technology page 403

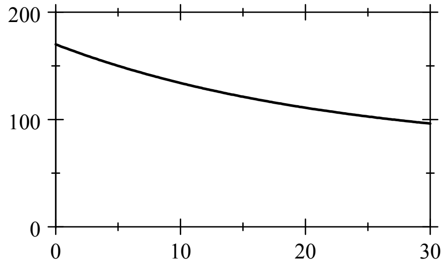
1. a.



b. $T(0) = 666$ million; $T(8) \approx 926.8$ million.

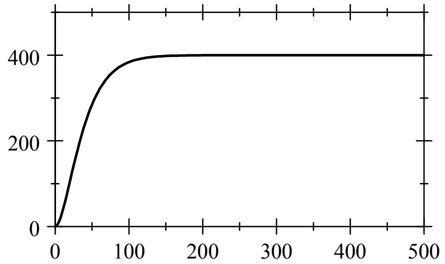
c. $T'(8) \approx 38.3$ million/yr/yr.

2. a.



b. $T(t) = 120$ when $t \approx 15.54$ min.

3. a.

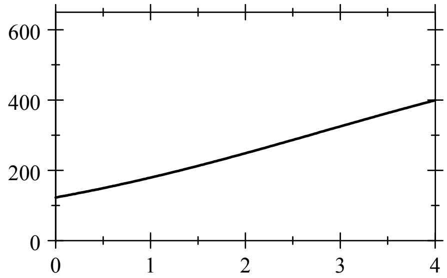


b. $\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \left[-20(t+20)e^{-0.05t} + 400\right]$

$= 400,$

so Starr will eventually sell 400,000 copies of Laser Beams.

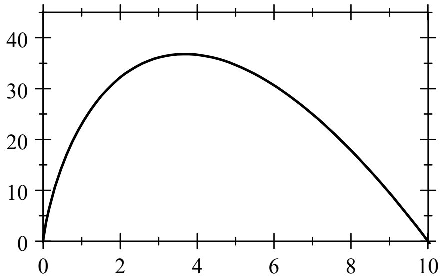
4. a.



b. $P(3) \approx 325$ million.

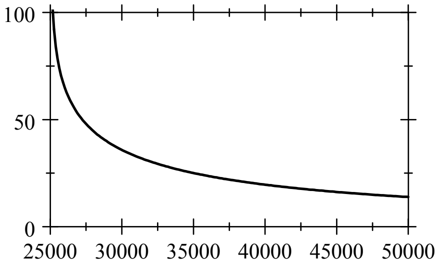
c. $P'(3) \approx 76.84$ million per 30 years.

5. a.



b. $R'(x) = 0$ when $x \approx 3.68$.

6. a.

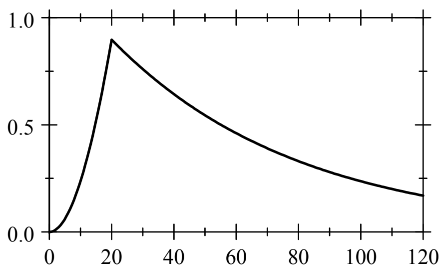


b. $f(x) = 25$ when $x \approx \$35{,}038.78$/yr.

c. $\lim_{x\to 25{,}000^+} f(x) = \infty$. If Christine withdraws \$25,000/yr she will be withdrawing only the interest, and so the account will never be depleted.

d. $\lim_{x\to\infty} f(x) = 0$. If Christine withdraws everything in her account, it is depleted immediately.

7. a.



b. The initial concentration is 0.

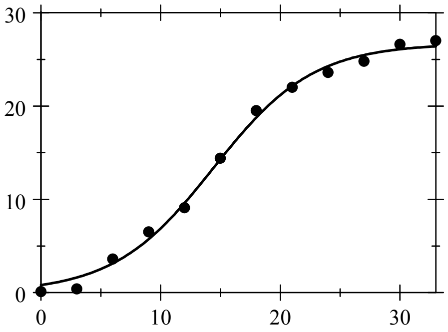
c. $C(10) \approx 0.237 \text{ g/cm}^3$.

d. $C(30) \approx 0.760 \text{ g/cm}^3$.

e. $\lim_{t\to\infty} C(t) = 0$.

8. a. $f(t) = \dfrac{26.71}{1 + 31.74e^{-0.24t}}$.

b.



c. At midnight, the snowfall was accumulating at the rate of $f'(12) \approx 1.476$, or approximately 1.5 in/hr. At noon on February 7, it was accumulating at the rate of $f'(24) \approx 0.530$, or approximately 0.5 in/hr.

d. The inflection point is approximately (14.4, 13.4), so snow was accumulating at the greatest rate at about 2:24 A.M. on February 7. The rate of accumulation was $f'(14.4) \approx 1.60$, or approximately 1.6 in/hr.

CHAPTER 5 Concept Review Questions page 405

1. power, 0, 1, exponential

2. a. $(-\infty, \infty)$, $(0, \infty)$ **b.** $(0, 1)$, $(-\infty, \infty)$

3. a. $(0, \infty)$, $(-\infty, \infty)$, $(1, 0)$ **b.** $< 1, > 1$

4. a. x **b.** x

5. accumulated amount, principal, nominal interest rate, number of conversion periods, term

6. $\left(1 + \dfrac{r}{m}\right)^m - 1$ **7.** Pe^{rt}

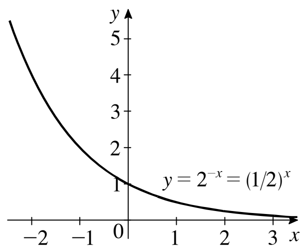
8. a. $e^{f(x)} \cdot f'(x)$ **b.** $\dfrac{f'(x)}{f(x)}$

9. a. initially, growth **b.** decay **c.** time, one-half

10. a. horizontal asymptote, C **b.** horizontal asymptote, A, carrying capacity

CHAPTER 5 Review Exercises page 406

1.



$\left(\frac{1}{2}\right)^x = \frac{1}{2^x} = 2^{-x}$, so the two graphs are the same.

2. If $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$, then $\log_{2/3}\left(\frac{27}{8}\right) = -3$.

3. $16^{-3/4} = 0.125$ is equivalent to $-\frac{3}{4} = \log_{16} 0.125$.

4. $\log_4(2x+1) = 2$, $(2x+1) = 4^2 = 16$, $2x = 15$, and so $x = \frac{15}{2}$.

5. $\ln(x-1) + \ln 4 = \ln(2x+4) - \ln 2$, so $\ln(x-1) - \ln(2x+4) = -\ln 2 - \ln 4 = -(\ln 2 + \ln 4)$, $\ln\left(\frac{x-1}{2x+4}\right) = -\ln 8 = \ln\frac{1}{8}$, $\left(\frac{x-1}{2x+4}\right) = \frac{1}{8}$, $8x - 8 = 2x + 4$, $6x = 12$, and so $x = 2$. Check: LHS $= \ln(2-1) + \ln 4 = \ln 4$; RHS $= \ln(4+4) - \ln 2 = \ln 8 - \ln 2 = \ln\frac{8}{2} = \ln 4$.

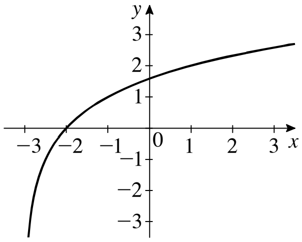
6. $\ln 30 = \ln(2 \cdot 3 \cdot 5) = \ln 2 + \ln 3 + \ln 5 = x + y + z$.

7. $\ln 3.6 = \ln\frac{36}{10} = \ln 36 - \ln 10 = \ln 6^2 - \ln(2 \cdot 5) = 2\ln 6 - \ln 2 - \ln 5 = 2(\ln 2 + \ln 3) - \ln 2 - \ln 5$

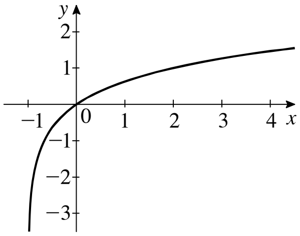
$= 2(x+y) - x - z = x + 2y - z$.

8. $\ln 75 = \ln\left(3 \cdot 5^2\right) = \ln 3 + 2\ln 5 = y + 2z$.

9. We first sketch the graph of $y = 2^{x+3}$, then reflect this graph with respect to the line $y = x + 3$.



10. We first sketch the graph of $y = 3^{x+1}$, then reflect this graph with respect to the line $y = x + 1$.



11. a. Using Formula (6) with $P = 10{,}000$, $r = 0.06$, $m = 365$ and $t = 2$, we have

$$A = 10{,}000\left(1 + \frac{0.06}{365}\right)^{365(2)} = 11{,}274.86, \text{ or } \$11{,}274.86.$$

b. Using Formula (10) with $P = 10{,}000$, $r = 0.06$, and $t = 2$, we have $A = 10{,}000e^{0.06(2)} = 11{,}274.97$, or \$11,274.97.

12. Using Formula (6), with $A = 10{,}000$, $P = 12{,}000$, $m = 4$, and $t = 3$, we have $A = 10{,}000\left(1 + \frac{r}{4}\right)^{4(3)} = 12{,}000$, or $\left(1 + \frac{r}{4}\right)^{12} = 1.2$. Solving for r, we have $\frac{r}{4} = (1.2)^{1/12} - 1$, so $r = 4\left[(1.2)^{1/12} - 1\right] \approx 0.0612$, or 6.12% per year.

13. Using Formula (6) with $A = 10{,}000$, $P = 15{,}000$, $r = 0.06$, and $m = 4$, we have $A = 10{,}000\left(1 + \frac{0.06}{4}\right)^{4t} = 15{,}000$, or $\left(1 + \frac{0.06}{4}\right)^{4t} = 1.5$. Solving for t, we have $4t \ln(1.015) = \ln 1.5$, so $t = \dfrac{\ln 1.5}{4 \ln 1.015} \approx 6.808$, or approximately 6.8 years.

14. Using Formula (7) to compute the effective rate of interest with $r = 0.08$ and $m = 4$, we have $r_{\text{eff}} = \left(1 + \dfrac{r}{m}\right)^m - 1$, or $0.08 = \left(1 + \dfrac{r}{4}\right)^4 - 1$. Solving for r, we have $\left(1 + \dfrac{r}{4}\right)^4 = 1.08$, $\dfrac{r}{4} = 1.08^{1/4} - 1$, and so $r = 4\left[(1.08)^{1/4} - 1\right] \approx 0.0777$, or approximately 7.77%/yr.

15. $f(x) = xe^{2x}$, so $f'(x) = e^{2x} + xe^{2x}(2) = (1 + 2x)e^{2x}$.

16. $f(t) = \sqrt{t}e^t + t$, so $f'(t) = \frac{1}{2}t^{-1/2}e^t + t^{1/2}e^t + 1 = \dfrac{e^t}{2\sqrt{t}} + \sqrt{t}e^t + 1$.

17. $g(t) = \sqrt{t}e^{-2t}$, so $g'(t) = \frac{1}{2}t^{-1/2}e^{-2t} + \sqrt{t}e^{-2t}(-2) = \dfrac{1 - 4t}{2\sqrt{t}e^{2t}}$.

18. $g(x) = e^x\left(1 + x^2\right)^{1/2}$, so

$$\begin{aligned} g'(x) &= e^x \tfrac{d}{dx}\left(1 + x^2\right)^{1/2} + \left(1 + x^2\right)^{1/2} \tfrac{d}{dx}e^x = e^x \tfrac{1}{2}\left(1 + x^2\right)^{-1/2}(2x) + \left(1 + x^2\right)^{1/2} e^x \\ &= e^x\left(1 + x^2\right)^{-1/2}\left(x + 1 + x^2\right) = \frac{e^x\left(x^2 + x + 1\right)}{\sqrt{1 + x^2}}. \end{aligned}$$

19. $y = \dfrac{e^{2x}}{1 + e^{-2x}}$, so $y' = \dfrac{\left(1 + e^{-2x}\right)e^{2x}(2) - e^{2x} \cdot e^{-2x}(-2)}{\left(1 + e^{-2x}\right)^2} = \dfrac{2\left(e^{2x} + 2\right)}{\left(1 + e^{-2x}\right)^2}$.

20. $f(x) = e^{2x^2 - 1}$, so $f'(x) = e^{2x^2 - 1}(4x) = 4xe^{2x^2 - 1}$.

21. $f(x) = xe^{-x^2}$, so $f'(x) = e^{-x^2} + xe^{-x^2}(-2x) = \left(1 - 2x^2\right)e^{-x^2}$.

22. $g(x) = \left(1 + e^{2x}\right)^{3/2}$, so $g'(x) = \frac{3}{2}\left(1 + e^{2x}\right)^{1/2} \cdot e^{2x}(2) = 3e^{2x}\left(1 + e^{2x}\right)^{1/2}$.

23. $f(x) = x^2e^x + e^x$, so $f'(x) = 2xe^x + x^2e^x + e^x = \left(x^2 + 2x + 1\right)e^x = (x + 1)^2 e^x$.

24. $g(t) = t \ln t$, so $g'(t) = \ln t + t\left(\frac{1}{t}\right) = \ln t + 1$.

25. $f(x) = \ln\left(e^{x^2} + 1\right)$, so $f'(x) = \dfrac{e^{x^2}(2x)}{e^{x^2} + 1} = \dfrac{2xe^{x^2}}{e^{x^2} + 1}$.

26. $f(x) = \dfrac{x}{\ln x}$, so $f'(x) = \dfrac{\ln x \dfrac{d}{dx}x - x\dfrac{d}{dx}\ln x}{(\ln x)^2} = \dfrac{\ln x - x \cdot \dfrac{1}{x}}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2}$.

27. $f(x) = \dfrac{\ln x}{x + 1}$, so $f'(x) = \dfrac{(x + 1)\left(\frac{1}{x}\right) - \ln x}{(x + 1)^2} = \dfrac{1 + \frac{1}{x} - \ln x}{(x + 1)^2} = \dfrac{x - x\ln x + 1}{x(x + 1)^2}$.

28. $y = (x + 1)e^x$, so $y' = e^x + (x + 1)e^x = (x + 2)e^x$.

29. $y = \ln\left(e^{4x} + 3\right)$, so $y' = \dfrac{e^{4x}(4)}{e^{4x} + 3} = \dfrac{4e^{4x}}{e^{4x} + 3}$.

30. $f(r) = \dfrac{re^r}{1 + r^2}$, so $f'(r) = \dfrac{\left(1 + r^2\right)\left(e^r + re^r\right) - re^r(2r)}{\left(1 + r^2\right)^2} = \dfrac{\left(r^3 - r^2 + r + 1\right)e^r}{\left(1 + r^2\right)^2}$.

31. $f(x) = \dfrac{\ln x}{1 + e^x}$, so

$$f'(x) = \frac{(1 + e^x)\dfrac{d}{dx}\ln x - \ln x\dfrac{d}{dx}(1 + e^x)}{(1 + e^x)^2} = \frac{(1 + e^x)\left(\dfrac{1}{x}\right) - (\ln x)\, e^x}{(1 + e^x)^2} = \frac{1 + e^x - xe^x \ln x}{x(1 + e^x)^2}$$
$$= \frac{1 + e^x(1 - x\ln x)}{x(1 + e^x)^2}.$$

32. $g(x) = \dfrac{e^{x^2}}{1 + \ln x}$, so $g'(x) = \dfrac{(1 + \ln x)\, e^{x^2}(2x) - e^{x^2}\left(\frac{1}{x}\right)}{(1 + \ln x)^2} = \dfrac{\left(2x^2 + 2x^2 \ln x - 1\right)e^{x^2}}{x(1 + \ln x)^2}$.

33. $y = \ln(3x + 1)$, so $y' = \dfrac{3}{3x + 1}$ and $y'' = 3\dfrac{d}{dx}(3x + 1)^{-1} = -3(3x + 1)^{-2}(3) = -\dfrac{9}{(3x + 1)^2}$.

34. $y = x\ln x$, so $y' = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$ and $y'' = \frac{1}{x}$.

35. $h'(x) = g'(f(x))\, f'(x)$. But $g'(x) = 1 - \dfrac{1}{x^2}$ and $f'(x) = e^x$, so $f(0) = e^0 = 1$ and $f'(0) = e^0 = 1$. Therefore, $h'(0) = g'(f(0))\, f'(0) = g'(1)\, f'(0) = 0 \cdot 1 = 0$.

36. $h'(1) = g'(f(1))\, f'(1)$ by the Chain Rule. Now $g'(x) = \dfrac{(x - 1) - (x + 1)}{(x - 1)^2} = -\dfrac{2}{(x - 1)^2}$, $f'(x) = \dfrac{1}{x}$, and $f(1) = 0$, so $h'(1) = -\dfrac{2}{(-1)^2} \cdot 1 = -2$.

37. $y = \left(2x^3 + 1\right)\left(x^2 + 2\right)^3$, so $\ln y = \ln\left(2x^3 + 1\right) + 3\ln\left(x^2 + 2\right)$,

$$\frac{y'}{y} = \frac{6x^2}{2x^3 + 1} + \frac{3(2x)}{x^2 + 2} = \frac{6x^2\left(x^2 + 2\right) + 6x\left(2x^3 + 1\right)}{\left(2x^3 + 1\right)\left(x^2 + 2\right)} = \frac{6x^4 + 12x^2 + 12x^4 + 6x}{\left(2x^3 + 1\right)\left(x^2 + 2\right)}$$
$$= \frac{18x^4 + 12x^2 + 6x}{\left(2x^3 + 1\right)\left(x^2 + 2\right)},$$

and so $y' = 6x\left(3x^3 + 2x + 1\right)\left(x^2 + 2\right)^2$.

38. $f(x) = \dfrac{x\left(x^2 - 2\right)^2}{x - 1}$, so $\ln f(x) = \ln x + 2\ln\left(x^2 - 2\right) - \ln(x - 1)$. Thus,

$$\frac{f'(x)}{f(x)} = \frac{1}{x} + \frac{2(2x)}{x^2 - 2} - \frac{1}{x - 1} = \frac{\left(x^2 - 2\right)(x - 1) + 4x^2(x - 1) - x\left(x^2 - 2\right)}{x(x - 1)\left(x^2 - 2\right)} = \frac{4x^3 - 5x^2 + 2}{x(x - 1)\left(x^2 - 2\right)}, \text{ and so}$$

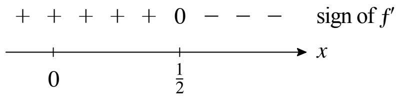
$$f'(x) = \frac{4x^3 - 5x^2 + 2}{x(x - 1)\left(x^2 - 2\right)} \frac{x\left(x^2 - 2\right)^2}{x - 1} = \frac{\left(4x^3 - 5x^2 + 2\right)\left(x^2 - 2\right)}{(x - 1)^2}.$$

39. $y = e^{-2x}$, so $y' = -2e^{-2x}$. This gives the slope of the tangent line to the graph of $y = e^{-2x}$ at any point (x, y). In particular, the slope of the tangent line at $\left(1, e^{-2}\right)$ is $y'(1) = -2e^{-2}$. The required equation is $y - e^{-2} = -2e^{-2}(x - 1)$, or $y = \frac{1}{e^2}(-2x + 3)$.

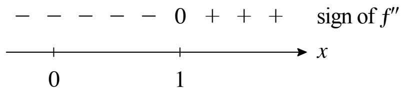
40. $y = xe^{-x}$, so $y' = e^{-x} + xe^{-x}(-1) = (1 - x)e^{-x}$. The slope of the tangent line at $\left(1, e^{-1}\right)$ is 0. Therefore, an equation of the tangent line is $y = 1/e$.

41. $f(x) = xe^{-2x}$. We first gather the following information on f.

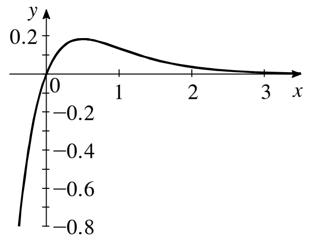
1. The domain of f is $(-\infty, \infty)$.
2. Setting $x = 0$ gives 0 as the y-intercept.
3. $\lim_{x\to-\infty} xe^{-2x} = -\infty$ and $\lim_{x\to\infty} xe^{-2x} = 0$.
4. The results of part 3 show that $y = 0$ is a horizontal asymptote.
5. $f'(x) = e^{-2x} + xe^{-2x}(-2) = (1 - 2x)e^{-2x}$. Observe that $f'(x) = 0$ at $x = \frac{1}{2}$, a critical point of f. The sign diagram of f' shows that f is increasing on $\left(-\infty, \frac{1}{2}\right)$ and decreasing on $\left(\frac{1}{2}, \infty\right)$.



6. The results of part 5 show that $\left(\frac{1}{2}, \frac{1}{2}e^{-1}\right)$ is a relative maximum.
7. $f''(x) = -2e^{-2x} + (1 - 2x)e^{-2x}(-2) = 4(x - 1)e^{-2x} = 0$ if $x = 1$. The sign diagram of f'' shows that the graph of f is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.

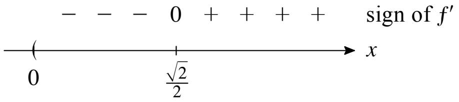


8. f has an inflection point at $\left(1, 1/e^2\right)$.



42. $f(x) = x^2 - \ln x$. We first gather the following information on f.

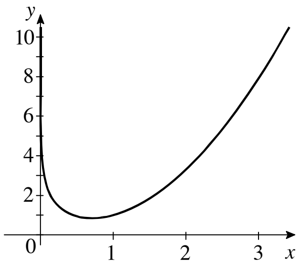
1. The domain of f is $(0, \infty)$.
2. There is no y-intercept.
3. $\lim_{x\to\infty}\left(x^2 - \ln x\right) = \infty$.
4. There is no asymptote.
5. $f'(x) = 2x - \frac{1}{x} = \frac{2x^2 - 1}{x}$. Setting $f'(x) = 0$ gives $x = \pm\frac{\sqrt{2}}{2}$. We reject the negative root, so $x = \frac{\sqrt{2}}{2}$is a critical point of f. The sign diagram of f' shows that f is decreasing on $\left(0, \frac{\sqrt{2}}{2}\right)$ and increasing on $\left(\frac{\sqrt{2}}{2}, \infty\right)$.



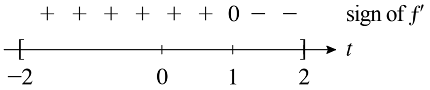
6. The results of part 5 show that $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}(1+\ln 2)\right)$ is a relative minimum of f.

7. $f''(x) = 2 + \dfrac{1}{x^2}$. Observe that $f''(x) > 0$ for all x in $(0, \infty)$, so the graph of f is concave upward on $(0, \infty)$.

8. The results of part 7 show that f has no inflection point.



43. $f(t) = te^{-t}$, so $f'(t) = e^{-t} + t\left(-e^{-t}\right) = e^{-t}(1-t)$. Setting $f'(t) = 0$ gives $t = 1$ as the only critical point of f. From the sign diagram of f' we see that $f(1) = e^{-1} = 1/e$ is the absolute maximum value of f.



44. $g(t) = \dfrac{\ln t}{t}$, so $g'(t) = \dfrac{t\left(\frac{1}{t}\right) - \ln t}{t^2} = \dfrac{1-\ln t}{t^2}$. Observe that $g'(t) = 0$ if $t = e$, but this value lies outside the interval $[1, 2]$. We calculate $g(1) = 0$ and $g(2) = \frac{1}{2}\ln 2$, and conclude that g has an absolute minimum at $(1, 0)$ and an absolute maximum at $\left(2, \frac{1}{2}\ln 2\right)$.

45. We want to solve the equation $8.2 = 4.5e^{r(5)}$. We have $e^{5r} = \frac{8.2}{4.5}$, so $r = \frac{1}{5}\ln\frac{8.2}{4.5} \approx 0.120$, and so the annual rate of return is 12%.

46. Using the present value formula for compound interest with $A = 30{,}000$, $m = 12$, $t = 5$ and $r = 0.05$, we have $A = 30{,}000\left(1 + \frac{0.05}{12}\right)^{-12(5)} \approx 23{,}376.16$, or \$23,376.16.

47. $P = 119{,}346e^{-0.06(4)} \approx 93{,}880.89$, or \$93,880.89.

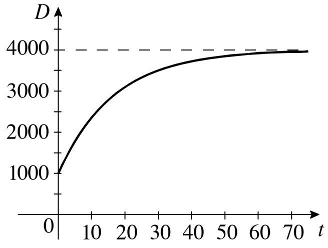
48. We solve the equation $2 = 1(1+0.075)^t$ for t. Taking logarithms of both sides, we have $\ln 2 = \ln(1.075)^t \approx t\ln 1.075$. Thus, $t = \dfrac{\ln 2}{\ln 1.075} \approx 9.58$, or 9.6 years.

49. **a.** $Q(t) = 2000e^{kt}$. Now $Q(120) = 18{,}000$ gives $2000e^{120k} = 18{,}000$, $e^{120k} = 9$, and so $120k = \ln 9$. Thus, $k = \frac{1}{120}\ln 9 \approx 0.01831$ and $Q(t) = 2000e^{0.01831t}$.

b. $Q(240) = 2000e^{0.01831(240)} \approx 161{,}992$, or approximately 162,000.

50. We have $Q(t) = Q_0e^{-kt}$, where Q_0 is the amount of radium present initially. Because the half-life of radium is 1600 years, we have $\frac{1}{2}Q_0 = Q_0e^{-1600k}$, $e^{-1600k} = \frac{1}{2}$, $-1600k = \ln\frac{1}{2} = -\ln 2$, and $k = \dfrac{\ln 2}{1600} \approx 0.0004332$.

51.



a. $D(1) = 4000 - 3000e^{-0.06} = 1175$,
$D(12) = 4000 - 3000e^{-0.72} = 2540$, and
$D(24) = 4000 - 3000e^{-1.44} = 3289$.

b. $\lim\limits_{t\to\infty} D(t) = \lim\limits_{t\to\infty}\left(4000 - 3000e^{-0.06t}\right) = 4000$.

52. $M(0) = 200(1) = 200$, or 200 g, and $M'(t) = -0.14(200)e^{-0.14t} = -28e^{-0.14t}$. At $t = 2$, we have $M'(2) = -28e^{-0.14(2)} \approx -21.16$, or approximately -21.2 g/yr.

53. $C(t) = 1486e^{-0.073t} + 500$.

- **a.** The average energy consumption of the York refrigerator/freezer at the beginning of 1972 is given by $C(0) = 1486e^{-0.073(0)} + 500 = 1486 + 500 = 1986$, or 1986 kWh/yr.
- **b.** To show that the average energy consumption of the York refrigerator is decreasing over the years in question, we compute $C'(t) = 1486e^{-0.073t}(-0.073) = -108.48e^{-0.073t}$ and note that $C'(t) < 0$ for all t. Therefore, $C(t)$ is decreasing over the interval $(0, 20)$.
- **c.** To see if the York refrigerator/freezer satisfied the 1990 requirement, we compute $C(18) = 1486e^{-0.073(18)} + 500 = 399.35 + 500 = 899.35$, or 899.35 kWh/yr. Because this is less than 950 kWh/yr, we conclude that York satisfied the requirement.

54. $f(t) = 1.5 + 1.8te^{-1.2t}$, so $f'(t) = 1.8\frac{d}{dt}\left(te^{-1.2t}\right) = 1.8\left[e^{-1.2t} + te^{-1.2t}(-1.2)\right] = 1.8e^{-1.2t}(1 - 1.2t)$. $f'(0) = 1.8$, $f'(1) = -0.11$, $f'(2) = -0.23$, and $f'(3) = -0.13$. Thus, measured in barrels per \$1000 of output per decade, the amount of oil used is increasing by 1.8 in 1965, decreasing by 0.11 in 1966, and so on.

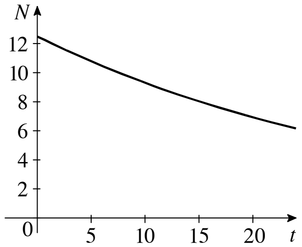
55.
- **a.** The price at $t = 0$ is $18 - 3e^0 - 6e^0 = 9$, or \$9/unit.
- **b.** $\frac{dp}{dt} = 6e^{-2t} + 2e^{-t/3}$, so $\left.\frac{dp}{dt}\right|_{t=0} = 6e^0 + 2e^0 = 8$. Thus, the price is increasing at the rate of \$8/unit/week.
- **c.** The equilibrium price is given by $\lim_{t\to\infty} p = \lim_{t\to\infty}\left(18 - 3e^{-2t} - 6e^{-t/3}\right) = 18$, or \$18/unit.

56. We have $Q(10) = 90$, and so $\dfrac{3000}{1 + 499e^{-10k}} = 90$, $1 + 499e^{-10k} = \dfrac{3000}{90}$, $499e^{-10k} = \dfrac{2910}{90}$, $e^{-10k} = \dfrac{2910}{90(499)}$, $-10k = \ln\dfrac{2910}{90(499)}$, and $k = -\dfrac{1}{10}\ln\dfrac{2910}{90(499)} \approx 0.2737$. Thus, $N(t) = \dfrac{3000}{1 + 499e^{-0.2737t}}$. The number of students who have contracted the flu by the 20th day is $N(20) = \dfrac{3000}{1 + 499e^{-0.2737(20)}} \approx 969.93$, or approximately 970 students.

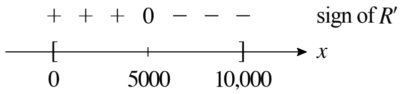
57. $P(t) = \dfrac{12}{1 + 3e^{-0.2747t}}$.

- **a.** $P(0) = \dfrac{12}{1 + 3e^0} = \dfrac{12}{4} = 3$, or 3 billion.
- **b.** $P(8) = \dfrac{12}{1 + 3e^{-0.2747(8)}} \approx 9.00$, or approximately 9 billion.
- **c.** $P'(t) = \dfrac{12(-0.2747)3e^{-0.2747t}}{\left(1 + 3e^{-0.2747t}\right)^2} = \dfrac{9.8892e^{-0.2747t}}{\left(1 + 3e^{-0.2747t}\right)^2}$, so $P'(6) = \dfrac{9.8892e^{-0.2747(6)}}{\left[1 + 3e^{-0.2747(6)}\right]^2} \approx 0.698567$. The population of the world is expected to be increasing by approximately 698.567 million per decade in 2030.

58.
- **a.** The infant mortality rate in 1980 is given by $N(0) = 12.5e^{-0.0294(0)} = 12.5$, or 12.5 per 1000 live births. In 1990, it is $N(10) = 12.5e^{-0.0294(10)} \approx 9.3$, or approximately 9.3 per 1000 live births. In 2000, it is $N(20) = 12.5e^{-0.0294(20)} \approx 6.9$, or approximately 6.9 per 1000 live births.
- **b.**



59. The revenue is given by $R = px = 20e^{-0.0002x}x = 20xe^{-0.0002x}$. To find the maximum of this function, we compute $R'(x) = 20e^{-0.0002x} - 20x\left(0.0002e^{-0.0002x}\right) = 20e^{-0.0002x}(1 - 0.0002x) = 0$ if $1 - 0.0002x = 0$, or $x = 5000$. From the sign diagram of R', we see that $(5000, 36787.9)$ is a maximum, so 5000 pairs of socks should be produced to yield a maximum revenue of approximately \$36,788.

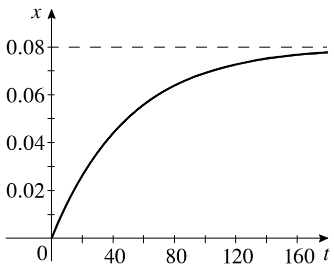


60. a. The initial concentration is $x(0) = 0.08\left(1 - e^{-0.02\cdot 0}\right) = 0$, or 0 g/cm^3.

b. The concentration after 30 seconds is $x(30) = 0.08\left(1 - e^{-0.02\cdot 30}\right) = 0.03609$, or 0.0361 g/cm^3.

c. The concentration in the long run is given by $\lim_{t\to\infty} 0.08\left(1 - e^{-0.02t}\right) = 0.08$, or 0.08 g/cm^3.

d.



CHAPTER 5 Before Moving On... page 408

1. $\dfrac{100}{1+2e^{0.3t}} = 40$, so $1 + 2e^{0.3t} = \dfrac{100}{40} = 2.5$, $2e^{0.3t} = 1.5$, $e^{0.3t} = \dfrac{1.5}{2} = 0.75$, $0.3t = \ln 0.75$, and so $t = \dfrac{\ln 0.75}{0.3} \approx -0.959$.

2. $A = 3000\left(1 + \frac{0.08}{52}\right)^{4(52)} = 4130.37$, or \$4130.37.

3. $f(x) = e^{\sqrt{x}}$, so $f'(x) = \dfrac{d}{dx}e^{x^{1/2}} = e^{x^{1/2}}\dfrac{d}{dx}\left(x^{1/2}\right) = e^{x^{1/2}}\left(\frac{1}{2}x^{-1/2}\right) = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$.

4. $y = x\ln\left(x^2+1\right)$, so $\dfrac{dy}{dx} = x\dfrac{d}{dx}\ln\left(x^2+1\right) + \ln\left(x^2+1\right)\dfrac{d}{dx}(x) = x\cdot\dfrac{2x}{x^2+1} + \ln\left(x^2+1\right) = \dfrac{2x^2}{x^2+1} + \ln\left(x^2+1\right)$.

Thus, $\left.\dfrac{dy}{dx}\right|_{x=1} = \dfrac{2}{1+1} + \ln 2 = 1 + \ln 2$.

5. $y = e^{2x}\ln 3x$, so $y' = e^{2x}\dfrac{d}{dx}\ln 3x + \ln 3x\cdot\dfrac{d}{dx}e^{2x} = \dfrac{e^{2x}}{x} + 2e^{2x}\ln 3x$ and

$$y'' = \frac{d}{dx}\left(x^{-1}e^{2x}\right) + 2e^{2x}\frac{d}{dx}\ln 3x + (\ln 3x)\frac{d}{dx}\left(2e^{2x}\right) = -x^{-2}e^{2x} + 2x^{-1}e^{2x} + 2e^{2x}\left(\frac{1}{x}\right) + 4e^{2x}\cdot\ln 3x$$

$$= -\frac{1}{x^2}e^{2x} + \frac{4e^{2x}}{x} + 4(\ln 3x)e^{2x} = e^{2x}\left(\frac{4x^2\ln 3x + 4x - 1}{x^2}\right).$$

6. $T(0) = 200$ gives $70 + ce^0 = 70 + C = 200$, so $C = 130$. Thus, $T(t) = 70 + 130e^{-kt}$. $T(3) = 180$ implies $70 + 130e^{-3k} = 180$, so $130e^{-3k} = 110$, $e^{-3k} = \dfrac{110}{130}$, $-3k = \ln \dfrac{11}{13}$, and $k = -\frac{1}{3}\ln\frac{11}{13} \approx 0.0557$. Therefore, $T(t) = 70 + 130e^{-0.0557t}$. So when $T(t) = 150$, we have $70 + 130e^{-0.0557t} = 150$, $130e^{-0.0557t} = 80$, $e^{-0.0557t} = \frac{80}{130} = \frac{8}{13}$, $-0.0557t = \ln\frac{8}{13}$, and finally $t = -\dfrac{\ln\frac{8}{13}}{0.0557} \approx 8.716$, or approximately 8.7 minutes.

CHAPTER 5 Explore & Discuss

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Equating the two expressions for y, we obtain $y_0 b^{kx} = y_0 e^{px}$, so $b^x = e^{px}$. Taking natural logarithms of both sides, we obtain $\ln b^{kx} = \ln e^{px}$, $kx \ln b = px \ln e = px$, and so $p = k \ln b$. Therefore, $y = y_0 e^{(k \ln b)x}$, as was to be shown.

Page 382

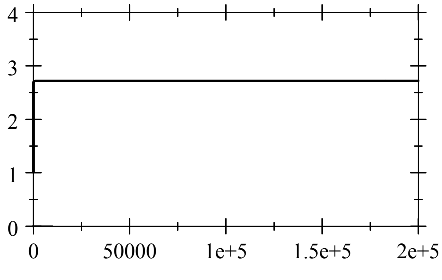
1, 2. $f'(x) = \displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{h\to 0} \left[\frac{1}{h}\ln\left(\frac{x+h}{x}\right)\right] = \lim_{h\to 0} \ln\left(1 + \frac{h}{x}\right)^{1/h}$.

3. From part 2, we have

$$f'(x) = \lim_{m\to\infty} \ln\left(1 + \frac{1}{m}\right)^{m/x} = \frac{1}{x}\lim_{m\to\infty} \ln\left(1 + \frac{1}{m}\right)^m = \frac{1}{x}\ln\left(\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m\right) = \frac{1}{x}\ln e = \frac{1}{x}.$$

CHAPTER 5 Exploring with Technology

Page 341



Using ZOOM and TRACE, we see that $f(x)$ approaches 2.71828... as x gets larger and larger.

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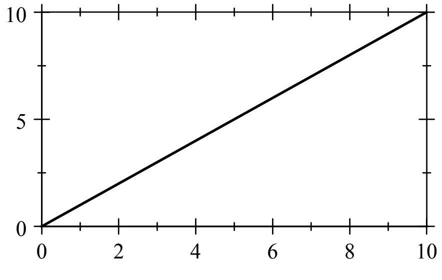
1, 2.



The graphs are the same.

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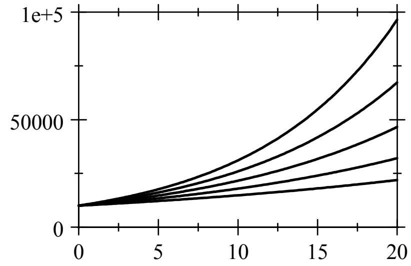
1. $A_1(t) = 10{,}000\,(1.04)^t$, $A_2(t) = 10{,}000\,(1.06)^t$, $A_3(t) = 10{,}000\,(1.08)^t$, $A_4(t) = 10{,}000\,(1.1)^t$, and $A_5(t) = 10{,}000\,(1.12)^t$.

2.



3. $A_1(20) \approx 21{,}911.23$, $A_2(20) \approx 32{,}071.35$, $A_3(20) \approx 46{,}609.57$, $A_4(20) \approx 67{,}275.00$, and $A_5(20) \approx 96{,}462.93$. These give the accumulated amounts after 20 years when \$10,000 is invested at 4%, 6%, 8%, 10%, and 12% per year, compounded annually.

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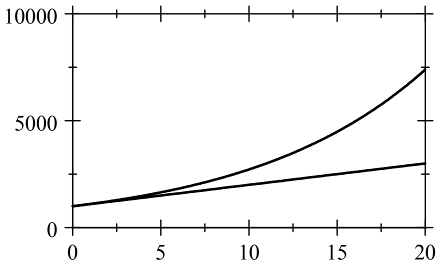
1, 2.



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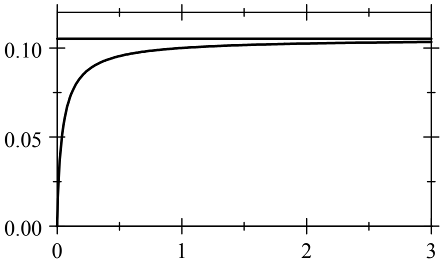
1.



2. From the graph in part 1, we see that the graph of y_1 approaches the line $y_2 = e^{0.1} - 1$ as x gets larger and larger. This shows that

$$\lim_{m\to\infty}\left[\left(1+\frac{0.1}{m}\right)^m - 1\right] = e^{0.1} - 1.$$

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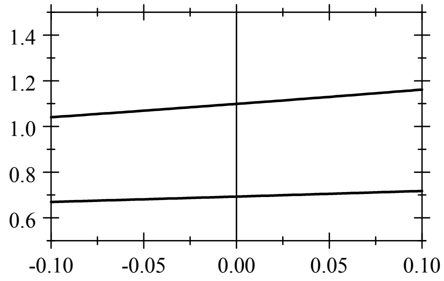
1. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{b^{x+h} - b^x}{h} = \lim_{h\to 0} \dfrac{b^x\left(b^h - 1\right)}{h} = b^x \lim_{h\to 0} \dfrac{b^h - 1}{h}$ because b^x is a constant with respect to the limiting process.

2. These results are immediately obvious if you replace b by 2 and 3, respectively.

3. We graph $f(x) = \dfrac{2^x - 1}{x}$ and $g(x) = \dfrac{3^x - 1}{x}$ using the viewing rectangle $[-0.1, 0.1] \times [0.5, 1.5]$.
Using ZOOM and TRACE, we see that $\lim_{h\to 0} f(h) \approx 0.69$ and $\lim_{h\to 0} g(h) \approx 1.10$.



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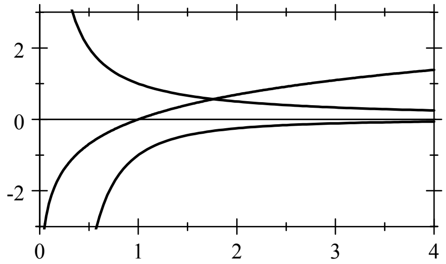

1. The graph of f' is always positive, but decreases to zero as $x \to \infty$. This tells us that the graph of f is increasing more slowly as $x \to \infty$.

2. Because $f'' < 0$ for all values of $x > 0$, we see that f is concave downward for $x > 0$. But $f''(x) \to 0$, showing that the bend (concavity) of the graph of f is less pronounced as x gets larger.

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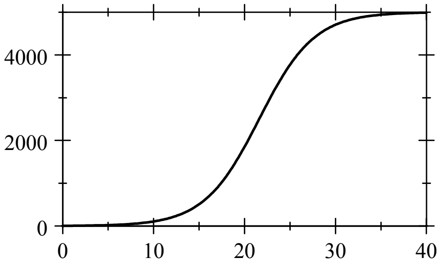
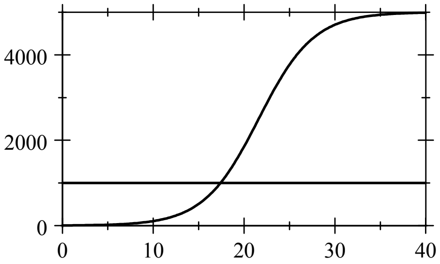
1. The graph of $Q(t)$ is shown below.



2.



The point of intersection of the graphs is approximately (17.405, 1000). Therefore, it takes approximately 17.4 days for the first 1000 soldiers to contract the flu.

6 INTEGRATION

6.1 Antiderivatives and the Rules of Integration

Concept Questions page 418

1. An antiderivative of a continuous function f on an interval I is a function F such that $F'(x) = f(x)$ for every x in I. For example, an antiderivative of $f(x) = x^2$ on $(-\infty, \infty)$ is the function $F(x) = \frac{1}{3}x^3$ on $(-\infty, \infty)$.

2. If $f'(x) = g'(x)$ for all x in I, then $f(x) = g(x) + C$ for all x in I, where C is an arbitrary constant.

3. The indefinite integral of f is the family of functions $F(x) + C$, where F is an antiderivative of f and C is an arbitrary constant.

4. No, the power rule holds only for $n \neq -1$. Rather, $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$, $x \neq 0$.

Exercises page 418

1. $F(x) = \frac{1}{3}x^3 + 2x^2 - x + 2$, so $F'(x) = x^2 + 4x - 1 = f(x)$.

2. $F(x) = xe^x + \pi$, so $F'(x) = xe^x + e^x = e^x(x+1) = f(x)$.

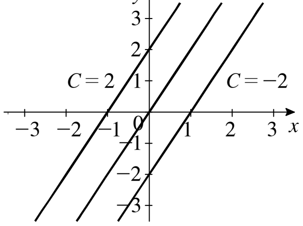
3. $F(x) = \left(2x^2-1\right)^{1/2}$, so $F'(x) = \frac{1}{2}\left(2x^2-1\right)^{-1/2}(4x) = 2x\left(2x^2-1\right)^{-1/2} = f(x)$.

4. $F(x) = x\ln x - x$, so $F'(x) = x\left(\frac{1}{x}\right) + \ln x - 1 = \ln x = f(x)$.

5. a. $G'(x) = \frac{d}{dx}(2x) = 2 = f(x)$

b. $F(x) = G(x) + C = 2x + C$

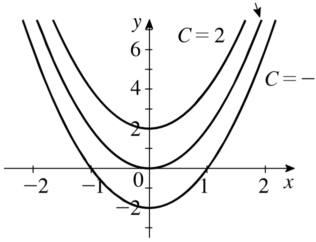
c.



6. a. $G'(x) = 4x = f(x)$, and so G is an antiderivative of f.

b. $H(x) = G(x) + C = 2x^2 + C$, where C is an arbitrary constant.

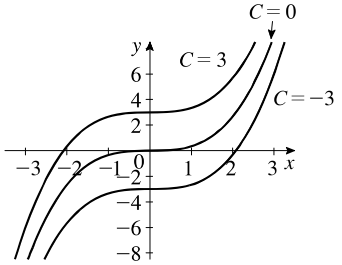
c.



7. **a.** $G'(x) = \dfrac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2 = f(x)$

b. $F(x) = G(x) + C = \frac{1}{3}x^3 + C$

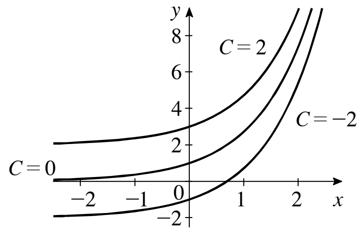
c.



8. **a.** $G(x) = e^x$, so $G'(x) = e^x = f(x)$.

b. $F(x) = e^x + C$, where C is an arbitrary constant.

c.



9. $\int 6\,dx = 6x + C.$

10. $\int \sqrt{2}\,dx = \sqrt{2}x + C.$

11. $\int x^3\,dx = \frac{1}{4}x^4 + C.$

12. $\int 2x^5\,dx = 2\left(\frac{1}{6}x^6\right) + C = \frac{1}{3}x^6 + C.$

13. $\int x^{-4}\,dx = -\frac{1}{3}x^{-3} + C.$

14. $\int 3t^{-7}\,dt = 3\left(-\frac{1}{6}t^{-6}\right) + C = -\frac{1}{2}t^{-6} + C.$

15. $\int x^{2/3}\,dx = \frac{3}{5}x^{5/3} + C.$

16. $\int 2u^{3/4}\,du = 2\left(\frac{4}{7}u^{7/4}\right) + C = \frac{8}{7}u^{7/4} + C.$

17. $\int x^{-5/4}\,dx = -4x^{-1/4} + C.$

18. $\int 3x^{-2/3}\,dx = 3\left(\dfrac{x^{1/3}}{\frac{1}{3}}\right) + C = 9x^{1/3} + C.$

19. $\displaystyle\int \frac{2}{x^3}\,dx = 2\int x^{-3}\,dx = -x^{-2} + C = -\frac{1}{x^2} + C.$

20. $\displaystyle\int \frac{1}{3x^5}\,dx = \frac{1}{3}\int x^{-5}\,dx = \frac{1}{3}\left(-\frac{1}{4}x^{-4}\right) + C$

$\displaystyle = -\frac{1}{12x^4} + C.$

21. $\int \pi\sqrt{t}\,dt = \pi\int t^{1/2}\,dt = \pi\left(\frac{2}{3}t^{3/2}\right) + C$

$= \frac{2\pi}{3}t^{3/2} + C.$

22. $\displaystyle\int \frac{3}{\sqrt{t}}\,dt = 3\int t^{-1/2}\,dt = 6t^{1/2} + C = 6\sqrt{t} + C.$

23. $\int (3 - 4x)\,dx = \int 3\,dx - 4\int x\,dx = 3x - 2x^2 + C.$

24. $\int \left(1 + u + u^2\right) du = u + \frac{1}{2}u^2 + \frac{1}{3}u^3 + C.$

25. $\int \left(x^2 + x + x^{-3}\right) dx = \int x^2\,dx + \int x\,dx + \int x^{-3}\,dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{2}x^{-2} + C.$

26. $\int \left(0.3t^2 + 0.02t + 2\right) dt = 0.3\left(\frac{1}{3}t^3\right) + 0.02\left(\frac{1}{2}t^2\right) + 2t + C = 0.1t^3 + 0.01t^2 + 2t + C.$

27. $\int 5e^x\,dx = 5e^x + C.$

28. $\int (1 + e^x)\,dx = x + e^x + C.$

29. $\int (1 + x + e^x)\,dx = x + \frac{1}{2}x^2 + e^x + C.$

30. $\int \left(2 + x + 2x^2 + e^x\right) dx = 2x + \frac{1}{2}x^2 + \frac{2}{3}x^3 + e^x + C.$

31. $\int \left(4x^3 - \dfrac{2}{x^2} - 1\right) dx = \int \left(4x^3 - 2x^{-2} - 1\right) dx = x^4 + 2x^{-1} - x + C = x^4 + \dfrac{2}{x} - x + C.$

32. $\int \left(6x^3 + 3x^{-2} - x\right) dx = \tfrac{3}{2}x^4 - 3x^{-1} - \tfrac{1}{2}x^2 + C.$

33. $\int \left(x^{5/2} + 2x^{3/2} - x\right) dx = \tfrac{2}{7}x^{7/2} + \tfrac{4}{5}x^{5/2} - \tfrac{1}{2}x^2 + C.$

34. $\int \left(t^{3/2} + 2t^{1/2} - 4t^{-1/2}\right) dt = \tfrac{2}{5}t^{5/2} + \tfrac{4}{3}t^{3/2} - 8t^{1/2} + C.$

35. $\int \left(x^{1/2} + 2x^{-1/2}\right) dx = \tfrac{2}{3}x^{3/2} + 4x^{1/2} + C.$

36. $\int \left(x^{2/3} - x^{-2}\right) dx = \tfrac{3}{5}x^{5/3} + \tfrac{1}{x} + C.$

37. $\int \left(\dfrac{u^3 + 2u^2 - u}{3u}\right) du = \tfrac{1}{3}\int \left(u^2 + 2u - 1\right) du = \tfrac{1}{9}u^3 + \tfrac{1}{3}u^2 - \tfrac{1}{3}u + C.$

38. $\int \left(x^2 - x^{-2}\right) dx = \tfrac{1}{3}x^3 + x^{-1} = \tfrac{1}{3}x^3 + \tfrac{1}{x} + C.$

39. $\int (2t + 1)(t - 2)\, dt = \int \left(2t^2 - 3t - 2\right) dt = \tfrac{2}{3}t^3 - \tfrac{3}{2}t^2 - 2t + C.$

40. $\int u^{-2}\left(1 - u^2 + u^4\right) du = \int \left(u^{-2} - 1 + u^2\right) du = -u^{-1} - u + \tfrac{1}{3}u^3 + C.$

41. $\int \dfrac{1}{x^2}\left(x^4 - 2x^2 + 1\right) dx = \int \left(x^2 - 2 + x^{-2}\right) dx = \tfrac{1}{3}x^3 - 2x - x^{-1} + C = \tfrac{1}{3}x^3 - 2x - \tfrac{1}{x} + C.$

42. $\int t^{1/2}\left(t^2 + t - 1\right) dt = \int \left(t^{5/2} + t^{3/2} - t^{1/2}\right) dt = \tfrac{2}{7}t^{7/2} + \tfrac{2}{5}t^{5/2} - \tfrac{2}{3}t^{3/2} + C.$

43. $\int \dfrac{ds}{(s+1)^{-2}} = \int (s+1)^2\, ds = \int \left(s^2 + 2s + 1\right) ds = \tfrac{1}{3}s^3 + s^2 + s + C.$

44. $\int \left(x^{1/2} + 3x^{-1} - 2e^x\right) dx = \tfrac{2}{3}x^{3/2} + 3\ln|x| - 2e^x + C.$

45. $\int \left(e^t + t^e\right) dt = e^t + \dfrac{1}{e+1}t^{e+1} + C.$

46.
$$\int \left(\frac{1}{x^2} - \frac{1}{\sqrt[3]{x^2}} + \frac{1}{\sqrt{x}}\right) dx = \int \left(x^{-2} - x^{-2/3} + x^{-1/2}\right) dx = -x^{-1} - 3x^{1/3} + 2x^{1/2} + C$$
$$= -\tfrac{1}{x} - 3x^{1/3} + 2\sqrt{x} + C.$$

47. $\int \dfrac{x^3 + x^2 - x + 1}{x^2}\, dx = \int \left(x + 1 - \dfrac{1}{x} + \dfrac{1}{x^2}\right) dx = \tfrac{1}{2}x^2 + x - \ln|x| - x^{-1} + C.$

48. $\int \dfrac{t^3 + \sqrt[3]{t}}{t^2}\, dt = \int \left(t + t^{-5/3}\right) dt = \tfrac{1}{2}t^2 - \tfrac{3}{2}t^{-2/3} + C.$

49.
$$\int \frac{\left(x^{1/2} - 1\right)^2}{x^2}\, dx = \int \frac{x - 2x^{1/2} + 1}{x^2}\, dx = \int \left(x^{-1} - 2x^{-3/2} + x^{-2}\right) dx$$
$$= \ln|x| + 4x^{-1/2} - x^{-1} + C = \ln|x| + \frac{4}{\sqrt{x}} - \frac{1}{x} + C.$$

50. $\int (x+1)^2 \left(1-\frac{1}{x}\right) dx = \int \left(x^2+2x+1\right)\left(1-\frac{1}{x}\right) dx = \int \left(x^2+x-1-\frac{1}{x}\right) dx$
$= \frac{1}{3}x^3 + \frac{1}{2}x^2 - x - \ln|x| + C.$

51. $\int f'(x)\,dx = \int (3x+1)\,dx = \frac{3}{2}x^2 + x + C$. The condition $f(1) = 3$ gives $f(1) = \frac{3}{2} + 1 + C = 3$, so $C = \frac{1}{2}$. Therefore, $f(x) = \frac{3}{2}x^2 + x + \frac{1}{2}$.

52. $f(x) = \int f'(x)\,dx = \int \left(3x^2 - 6x\right) dx = x^3 - 3x^2 + C$. Using the given initial condition, we have $f(2) = 8 - 12 + C = 4$, so $C = 8$. Therefore, $f(x) = x^3 - 3x^2 + 8$.

53. $f'(x) = 3x^2 + 4x - 1$, so $f(x) = x^3 + 2x^2 - x + C$. Using the given initial condition, we have $f(2) = 8 + 2(4) - 2 + C = 9$, so $16 - 2 + C = 9$, or $C = -5$. Therefore, $f(x) = x^3 + 2x^2 - x - 5$.

54. $f(x) = \int f'(x)\,dx = \int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = 2x^{1/2} + C$. Using the given condition, we obtain $f(4) = 2\sqrt{4} + C = 4 + C = 2$, so $C = -2$. Therefore, $f(x) = 2\sqrt{x} - 2$.

55. $f(x) = \int f'(x)\,dx = \int \left(1 + \frac{1}{x^2}\right) dx = \int \left(1 + x^{-2}\right) dx = x - \frac{1}{x} + C$. Using the given initial condition, we have $f(1) = 1 - 1 + C = 3$, so $C = 3$. Therefore, $f(x) = x - \frac{1}{x} + 3$.

56. $f(x) = \int \left(e^x - 2x\right) dx = e^x - x^2 + C$. Using the initial condition, we have $f(0) = e^0 - 0 + C = 1 + C = 2$, so $C = 1$. Thus, $f(x) = e^x - x^2 + 1$.

57. $f(x) = \int \frac{x+1}{x}\,dx = \int \left(1 + \frac{1}{x}\right) dx = x + \ln|x| + C$. Using the initial condition, we have $f(1) = 1 + \ln 1 + C = 1 + C = 1$, so $C = 0$. Thus, $f(x) = x + \ln|x|$.

58. $f'(x) = 1 + e^x + \frac{1}{x}$, so $f(x) = xe^x + \ln|x| + C$. Using the initial condition, we have $f(1) = 1 + e + \ln 1 + C$, and so $3 + e = 1 + e + C$ and $C = 2$. Therefore, $f(x) = xe^x + \ln|x| + 2$.

59. $f(x) = \int f'(x)\,dx = \int \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2}\left(2x^{1/2}\right) + C = x^{1/2} + C$, and $f(2) = \sqrt{2} + C = \sqrt{2}$ implies $C = 0$. Thus, $f(x) = \sqrt{x}$.

60. $f(t) = \int f'(t)\,dt = \int \left(t^2 - 2t + 3\right) dt = \frac{1}{3}t^3 - t^2 + 3t + C$. $f(1) = \frac{1}{3} - 1 + 3 + C = 2$ implies $C = -\frac{1}{3}$, so $f(t) = \frac{1}{3}t^3 - t^2 + 3t - \frac{1}{3}$.

61. $f'(x) = e^x + x$, so $f(x) = e^x + \frac{1}{2}x^2 + C$ and $f(0) = e^0 + \frac{1}{2}(0) + C = 1 + C$. Thus, $3 = 1 + C$, and so $2 = C$. Therefore, $f(x) = e^x + \frac{1}{2}x^2 + 2$.

62. $f(x) = \int \left(\frac{2}{x} + 1\right) dx = 2\ln|x| + x + C$. $f(1) = 2\ln 1 + 1 + C = 2$, so $f(x) = 2\ln|x| + x + 1$.

63. The net amount on deposit in branch A is given by the area under the graph of f from $t = 0$ to $t = 180$. On the other hand, the net amount on deposit in branch B is given by the area under the graph of g over the same interval. Branch A has a larger amount on deposit because the rate at which money was deposited into branch A was always greater than the rate at which money was deposited into branch B over the period in question.

64. Because $f(t) \geq g(t)$ for all t in $[0, T]$, we see that the velocity of car A is always greater than or equal to that of car B. We conclude accordingly that after T seconds, car A will be ahead of car B.

65. The number in year t is $N(t) = \int R(t)\,dt = \int 14.3\,dt = 14.3t + C$. To determine C, we use the condition $N(0) = 90.1$, giving $C = 90.1$. Therefore, $N(t) = 14.3t + 90.1$. The estimated number of users in 2015 is $N(4) = 14.3(4) + 90.1 = 147.3$, or 147.3 million.

66. The percentage in year t is $P(t) = \int R(t)\,dt = \int 0.7dt = 0.7t + C$. To determine C, we use the condition $P(0) = 75$, giving $C = 75$. Therefore, $P(t) = 0.7t + 75$. The percentage of households that owned multiple sets in 2010 was $P(10) = 0.7(10) + 75 = 82$.

67. Let f be the position function of the maglev. Then $f'(t) = v(t)$. Therefore, $f(t) = \int f'(t)\,dt = \int v(t)\,dt = \int (0.2t + 3)\,dt = 0.1t^2 + 3t + C$. If we measure the position of the maglev from the station, then the required function is $f(t) = 0.1t^2 + 3t$.

68. **a.** $R(x) = \int R'(x)\,dx = \int (-0.009x + 12)\,dx = -0.0045x^2 + 12x + C$. But $R(0) = C = 0$, and so $R(x) = -0.0045x^2 + 12x$.

b. $R(x) = px$, and so $-0.0045x^2 + 12x = px$. Thus, $p = -0.0045x + 12$.

69. $P'(x) = -0.004x + 20$, so $P(x) = -0.002x^2 + 20x + C$. Because $C = -16{,}000$, we find that $P(x) = -0.002x^2 + 20x - 16{,}000$. The company realizes a maximum profit when $P'(x) = 0$, that is, when $x = 5000$ units. Next, $P(5000) = -0.002(5000)^2 + 20(5000) - 16{,}000 = 34{,}000$. Thus, the maximum profit of \$34,000 is realized at a production level of 5000 units.

70. $C(x) = \int C'(x)\,dx = \int (0.002x + 100)\,dx = 0.001x^2 + 100x + k$, but $C(0) = k = 4000$, and so $C(x) = 0.001x^2 + 100x + 4000$.

71. **a.** The amount of wind energy generated in year t is $A(t) = \int r(t)\,dt = \int (5.018t - 3.204)\,dt = 2.509t^2 - 3.204t + C$. To determine C, we use the condition $A(0) = 1.8$, giving $C = 1.8$. Therefore, $A(t) = 2.509t^2 - 3.204t + 1.8$.

b. The amount of wind energy generated in 2012 was $A(7) = 2.509(7)^2 - 3.204(7) + 1.8 = 102.313$, or approximately 102.3 terawatt-hours.

c. The amount generated in 2013 was $A(8) = 2.509(8)^2 - 3.204(8) + 1.8 = 136.744$, or approximately 136.7 terawatt-hours.

72. **a.** $f(t) = \int r(t)\,dt = \int (0.0058t + 0.159)\,dt = 0.0029t^2 + 0.159t + C$. $f(0) = 1.6$, and so $0 + 0 + C = 1.6$, or $C = 1.6$. Therefore, $f(t) = 0.0029t^2 + 0.159t + 1.6$.

b. The national health expenditure in 2015 will be $f(13) = 0.0029\left(13^2\right) + 0.159(13) + 1.6 = 4.1571$, or approximately \$4.16 trillion.

73. The total number of acres grown in year t is $N(t) = \int R(t)\,dt = \int (150t + 14.82)\,dt = 75t^2 + 14.82t + C$. Using the condition $N(0) = 27.2$, we find $N(0) = C = 27.2$. Therefore, $N(t) = 75t^2 + 14.82t + 27.2$. The number of acres grown in 2012 is given by $N(6) = 75(6)^2 + 14.82(6) + 27.2 \approx 2816.12$, or approximately 2816.1 acres.

74. The position of the car is $s(t) = \int f(t)\,dt = \int 2\sqrt{t}\,dt = \int 2t^{1/2}\,dt = 2\left(\frac{2}{3}t^{3/2}\right) + C = \frac{4}{3}t^{3/2} + C$. $s(0) = 0$ implies that $s(0) = C = 0$, so $s(t) = \frac{4}{3}t^{3/2}$.

75. **a.** $h(t) = \int h'(t)\,dt = \int (-32t+4)\,dt = -16t^2 + 4t + C$. But $h(0) = C = 400$, so $h(t) = -16t^2 + 4t + 400$.

b. It strikes the ground when $h(t) = 0$; that is, when $-16t^2 + 4t + 400 = 0$. Using the quadratic formula, we find that $t = \dfrac{-4 \pm \sqrt{16 - 4(-16)(400)}}{2(-16)} \approx 5.13$ or -4.88. We disregard the negative root since t must be nonnegative, and conclude that $t \approx 5.13$.

c. Its velocity is $-32(5.13) + 4 = 160.16$, or approximately 160.16 ft/sec downward.

76. The rate of change of the population at any time t is $P'(t) = 4500t^{1/2} + 1000$. Therefore, $P(t) = 3000t^{3/2} + 1000t + C$. But $P(0) = 30{,}000$, and this implies that $P(t) = 3000t^{3/2} + 1000t + 30{,}000$. Finally, the projected population 9 years after the construction has begun is $P(9) = 3000(9)^{3/2} + 1000(9) + 30{,}000 = 120{,}000$.

77. The number of new subscribers at any time is $N(t) = \int \left(100 + 210t^{3/4}\right) dt = 100t + 120t^{7/4} + C$. The given condition implies that $N(0) = 5000$. Using this condition, we find $C = 5000$. Therefore, $N(t) = 100t + 120t^{7/4} + 5000$. The number of subscribers 16 months from now is $N(16) = 100(16) + 120(16)^{7/4} + 5000$, or 21,960.

78. $v(r) = \int v'(r)\,dr = \int -kr\,dr = -\frac{1}{2}kr^2 + C$. But $v(R) = -\frac{1}{2}kR^2 + C = 0$, so $C = \frac{1}{2}kR^2$. Therefore, $v(R) = -\frac{1}{2}kr^2 + \frac{1}{2}kR^2 = \frac{1}{2}k\left(R^2 - r^2\right)$.

79. $h(t) = \int h'(t)\,dt = \int \left(-3t^2 + 192t\right) dt = -t^3 + 96t^2 + C = -t^3 + 96t^2 + C$. $h(0) = C = 0$ implies $h(t) = -t^3 + 96t^2$. The altitude 30 seconds after liftoff is $h(30) = -30^3 + 96(30)^2 = 59{,}400$ ft.

80. **a.** $N(t) = \int N'(t)\,dt = \int \left(-3t^2 + 12t + 45\right) dt = -t^3 + 6t^2 + 45t + C$. But $N(0) = C = 0$, and so $N(t) = -t^3 + 6t^2 + 45t$.

b. The number is $N(4) = -4^3 + 6(4)^2 + 45(4) = 212$.

81. $C(x) = \int C'(x)\,dx = \int \left(0.000009x^2 - 0.009x + 8\right) dx = 0.000003x^3 - 0.0045x^2 + 8x + k$. $C(0) = k = 120$, and so $C(x) = 0.000003x^3 - 0.0045x^2 + 8x + 120$. Thus, $C(500) = 0.000003(500)^3 - 0.0045(500)^2 + 8(500) + 120 = \$3370$.

82. **a.** $f(x) = \int r(x)\,dx = \int \left(0.004641x^2 - 0.3012x + 4.9\right) dx = 0.001547x^3 - 0.1506x^2 + 4.9x + C$. $f(30) = 0.14$ yields $0.001547\left(30^3\right) - 0.1506\left(30^2\right) + 4.9(30) + C = 0.14$, so $C = -53.09$. Thus, $f(x) = 0.001547x^3 - 0.1506x^2 + 4.9x - 53.09$.

b. The risk of Down syndrome when the maternal age is 40 at delivery is given by $f(40) = 0.001547\left(40^3\right) - 0.1506\left(40^2\right) + 4.9(40) - 53.09 \approx 0.959$, or approximately 0.96%. When the maternal age is 45, the risk is $f(45) \approx 3.416$, or approximately 3.42%.

83. **a.** We have the initial-value problem $R'(t) = 8\sqrt{2}t^{1/2} - 32t^3$ with $R(0) = 0$. Integrating, we find $R(t) = \int \left(8\sqrt{2}t^{1/2} - 32t^3\right) dt = \frac{16\sqrt{2}}{3}t^{3/2} - 8t^4 + C$. $R(0) = 0$ implies that $C = 0$, so $R(t) = \frac{16\sqrt{2}}{3}t^{3/2} - 8t^4$.

b. $R\left(\frac{1}{2}\right) = \frac{16\sqrt{2}}{3}\left(\frac{1}{2}\right)^{3/2} - 8\left(\frac{1}{2}\right)^4 \approx 2.166$, so after $\frac{1}{2}$ hr, approximately 2.2 inches of rain had fallen.

84. **a.** The approximate percentage in year t is
$P(t) = \int r(t)\,dt = \int \left(-0.00025142t^2 + 0.02116t + 0.0328\right) dt = -0.000083807t^3 + 0.01058t^2 + 0.0328t + C$.
To find C, we use the condition $P(0) = 6.2$, obtaining $C = 6.2$. Therefore,
$P(t) = -0.000083807t^3 + 0.01058t^2 + 0.0328t + 6.2$.

b. The percentage is $P(53) \approx 25.1807$, or approximately 25%.

85. **a.** The percentage of people 12 and older using social networking sites or services in year t is
$P(t) = \int R(t)\,dt = \int 5.92t^{-0.158}\,dt \approx 7.031t^{0.842} + C$. To find C, we use the condition $P(1) = 7$, obtaining $7.031 + C = 7$, so $C = -0.031$. Therefore, $P(t) = 7.031t^{0.842} - 0.031$.

b. The percentage in 2013 was $P(5) = 7.031(5)^{0.842} - 0.031 \approx 27.23$, or approximately 27.2%.

86. **a.** $E(t) = \int 31.863t^{-0.61}\,dt = 81.7t^{0.39} + C$. Using the condition $E(0) = 81.7$ gives $C = 81.7$, so
$E(t) = 81.7\left(t^{0.39} + 1\right)$.

b. U.S. coal exports in 2013 were $E(3) = 81.7\left(3^{0.39} + 1\right) \approx 207.1$, or approximately 207.1 million short tons.

87. $S'(W) = 0.131773W^{-0.575}$, so $S = \int 0.131773W^{-0.575}\,dW = 0.310054W^{0.425} + C$. Now
$S(70) = 0.310054(70)^{0.425} + C = 1.886277 + C = 1.886277$, so $C = -0.000007 \approx 0$. Thus,
$S(75) = 0.310054(75)^{0.425} \approx 1.9424$.

88. $A(t) = \int A'(t)\,dt = \int \left(3.2922t^2 - 0.366t^3\right) dt = 1.0974t^3 - 0.0915t^4 + C$. Now $A(0) = C = 0$, so
$A(t) = 1.0974t^3 - 0.0915t^4$.

89. **a.** Let y denote the height of a typical preschool child. Then $R(t) = 25.8931e^{-0.993t} + 6.39$
and $y = \int R(t)\,dt = -\frac{25.8931}{0.993}e^{-0.993t} + 6.39t + C = -26.0756e^{-0.993t} + 6.39t + C$.
$y\left(\frac{1}{4}\right) = -26.0756e^{-(0.993)(1/4)} + 6.39\left(\frac{1}{4}\right) + C = 60.30$. Therefore, $C = 79.045$, and so
$y(t) = -26.0756e^{-0.993t} + 6.39t + 79.045$.

b. $y(1) = -26.0756e^{-0.993} + 6.39 + 79.045 \approx 75.7749$, or approximately 75.77 cm.

90. Denote the constant acceleration by k. Then if $s = f(t)$ is the position function of the car, we have $f''(t) = k$, so $f'(t) = v(t) = \int k\,dt = kt + C_1$. We have $v(0) = 66$, and this gives $C_1 = 66$. Therefore, $f'(t) = v(t) = kt + 66$, and so $s = f(t) = \int f'(t)\,dt = \int v(t)\,dt = (kt + 66)\,dt = \frac{1}{2}kt^2 + 66 + C_2$. Next we use the condition that $s = 0$ when $t = 0$ to obtain $s = f(t) = \frac{1}{2}kt^2 + 66t$. To find the time it takes for the car to go from 66 ft/sec to 88 ft/sec, we use the expression for $v(t)$ to write $88 = kt + 66$, giving $t = \frac{22}{k}$. Finally using the expression for s and the condition that the car covered 440 ft during this period, we have $440 = \frac{1}{2}k\left(\frac{22}{k}\right)^2 + 66\left(\frac{22}{k}\right) = \frac{242}{k} + \frac{1452}{k} = \frac{1694}{k}$, so $k = 3.85$. Therefore, the car was accelerating at the rate of 3.85 ft/sec^2.

91. Denote the constant deceleration by k. Then $f''(t) = -k$, so $f'(t) = v(t) = -kt + C_1$. Next, the given condition implies that $v(0) = 88$. This gives $C_1 = 88$, so $f'(t) = -kt + 88$. Now $s = f(t) = \int f'(t)\,dt = \int(-kt + 88)\,dt = -\frac{1}{2}kt^2 + 88t + C_2$, and $f(0) = 0$ gives $s = f(t) = -\frac{1}{2}kt^2 + 88t$. Because the car is brought to rest in 9 seconds, we have $v(9) = -9k + 88 = 0$, or $k = \frac{88}{9} \approx 9.78$, so the deceleration is 9.78 ft/sec^2. The distance covered is $s = f(9) = -\frac{1}{2}\left(\frac{88}{9}\right)(81) + 88(9) = 396$, so the stopping distance is 396 ft.

92. The acceleration is $\frac{dv}{dt}$, where r is the velocity of the aircraft. So suppose that $\dfrac{dv}{dt} = c$, a constant. Then $v = \int \dfrac{dv}{dt}\,dt = c\,dt = dt + k$, where k is the constant of integration. We have $v\,(0) = 160 \text{ mph} = \frac{160}{60} \times 88 = \frac{704}{3}$ ft/sec. This gives $v\,(0) = k = \frac{704}{3}$, so $v\,(t) = ct + \frac{704}{3}$. Because the aircraft was brought to rest in 1 second, we have $v\,(1) = 0$. Using this condition, we find $v\,(1) = c + \frac{704}{3} = 0$, and $c = -\frac{704}{3}$ ft/sec^2, and the deceleration is equivalent to $\frac{704}{3} \times \frac{1}{32} = \frac{22}{3}$, that is, $7\frac{1}{3}$ g's.

93. The time taken by runner A to cross the finish line is $t = \frac{200}{22} = \frac{100}{11}$ sec. Let a be the constant acceleration of runner B as he begins to spurt. Then $\dfrac{dv}{dt} = a$, so the velocity of runner B as he runs towards the finish line is $v = \int a\,dt = at + c$. At $t = 0$, $v = 20$ and so $v = at = 20$. Now $\frac{ds}{dt} = v = at + 20$, so $s = \int (at + 20)\,dt = \frac{1}{2}at^2 + 20t + k$, where k is the constant of integration. Next, $s\,(0) = 0$ gives $s = \frac{1}{2}at^2 + 20t = \left(\frac{1}{2}at + 20\right)t$. In order for runner B to cover 220 ft in $\frac{100}{11}$ sec, we must have $\left[\frac{1}{2}a\left(\frac{100}{11}\right) + 20\right]\frac{100}{11} = 220$, so $\frac{50}{11}a + 20 = \frac{220 \cdot 11}{100} = \frac{121}{5}$, $\frac{50}{11}a = \frac{121}{5} - 20 = \frac{21}{5}$, and $a = \frac{21}{5} \cdot \frac{11}{50} = 0.924$ ft/sec^2. Therefore, runner B must have an acceleration of at least 0.924 ft/sec^2.

94. $h\,(t) = \int h'\,(t) = \int \left(-\frac{1}{25}\right)\left(\sqrt{20} - \frac{1}{50}t\right)dt = -\frac{1}{25}\left(\sqrt{20}t - \frac{1}{100}t^2\right) + C$. Next, we use the initial condition $h\,(0) = 20$ to obtain $h\,(0) = C = 20$. Therefore, the required expression is $h\,(t) = -\frac{1}{25}\left(\sqrt{20}t - \frac{1}{100}t^2\right) + 20$.

95. Suppose the acceleration is k. The distance covered is $s = f\,(t)$ and satisfies $f''\,(t) = k$. Thus, $f'\,(t) = v\,(t) = \int k\,dt = kt + C_1$. Next, $v\,(0) = 0$ gives $v\,(t) = kt$, and so $s = f\,(t) = \int kt\,dt = \frac{1}{2}kt^2 + C_2$. Now $f\,(0) = 0$ gives $s = \frac{1}{2}kt^2$. If it traveled 800 ft, we have $800 = \frac{1}{2}kt^2$, so $t = \frac{40}{\sqrt{k}}$. Its speed at this time is $v\,(t) = kt = k\left(\frac{40}{\sqrt{k}}\right) = 40\sqrt{k}$. We want the speed to be at least 240 ft/sec, so we require $40\sqrt{k} > 240$, implying that $k > 36$. In other words, the acceleration must be at least 36 ft/sec^2.

96. True. See the proof in Section 6.1 of the text.

97. False. $\int f\,(x)\,dx = F\,(x) + C$, where C is an arbitrary constant.

98. True. Use the Sum Rule followed by the Constant Multiple Rule.

99. False. $\int \frac{d}{dx}\left[f\,(x)\right]dx = \int f'\,(x)\,dx = f\,(x) + C$, where C is a constant of integration.

100. False. Take $f\,(x) = 1$ and $g\,(x) = 1$. Then $\int f\,(x)\,g\,(x)\,dx = \int 1\,dx = x + C$, whereas $\left[\int f\,(x)\,dx\right]\left[\int g\,(x)\,dx\right] = \left(\int 1\,dx\right)\left(\int 1\,dx\right) = (x + C)\,(x + D) = x^2 + (C + D)\,x + CD$.

6.2 Integration by Substitution

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1. To find $I = \int f\,(g\,(x))g'\,(x)\,dx$ by the Method of Substitution, let $u = g\,(x)$, so that $du = g'\,(x)\,dx$. Making the substitution, we obtain $I = \int f\,(u)\,du$, which can be integrated with respect to u. Finally, replace u by $u = g\,(x)$ to evaluate the integral.

2. For $I = xe^{-x^2}dx$, we let $u = -x^2$, so that $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}du$. Then $I = -\frac{1}{2}\int e^u\,du$, which is easily integrated. But the substitution does not work for $J = \int e^{-x^2}\,dx$ because it does not reduce J to the form $f(u)\,du$, where f is easily integrable.

Exercises

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1. Put $u = 4x + 3$, so $du = 4\,dx$ and $dx = \frac{1}{4}\,du$. Then $\int 4(4x+3)^4\,dx = \int u^4\,du = \frac{1}{5}u^5 + C = \frac{1}{5}(4x+3)^5 + C$.

2. Let $u = 2x^2 + 1$, so $du = 4x\,dx$. Then $\int 4x\left(2x^2+1\right)^7 dx = \int u^7\,du = \frac{1}{8}u^8 + C = \frac{1}{8}\left(2x^2+1\right)^8 + C$.

3. Let $u = x^3 - 2x$, so $du = \left(3x^2 - 2\right)dx$. Then $\int\left(x^3-2x\right)^2\left(3x^2-2\right)dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\left(x^3-2x\right)^3 + C$.

4. Put $u = x^3 - x^2 + x$, so $du = \left(3x^2 - 2x + 1\right)dx$. Then $\int\left(3x^2-2x+1\right)\left(x^3-x^2+x\right)^4 dx = \int u^4\,du = \frac{1}{5}u^5 + C = \frac{1}{5}\left(x^3-x^2+x\right)^5 + C$.

5. Let $u = 2x^2 + 3$, so $du = 4x\,dx$. Then

$$\int \frac{4x}{\left(2x^2+3\right)^3}\,dx = \int \frac{1}{u^3}\,du = \int u^{-3}\,du = -\tfrac{1}{2}u^{-2} + C = -\frac{1}{2\left(2x^2+3\right)^2} + C.$$

6. Let $u = x^3 + 2x$, so $du = \left(3x^2+2\right)dx$. Then

$$\int \frac{3x^2+2}{\left(x^3+2x\right)^2}\,dx = \int \frac{du}{u^2} = \int u^{-2}\,du = -u^{-1} + C = -\frac{1}{x^3+2x} + C.$$

7. Put $u = t^3 + 2$, so $du = 3t^2\,dt$ and $t^2\,dt = \frac{1}{3}\,du$. Then $\int 3t^2\sqrt{t^3+2}\,dt = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\left(t^3+2\right)^{3/2} + C$.

8. Let $u = t^3 + 2$, so $du = 3t^2\,dt$. Then $\int 3t^2\left(t^3+2\right)^{3/2} dt = \int u^{3/2}\,du = \frac{2}{5}u^{5/2} + C = \frac{2}{5}\left(t^3+2\right)^{5/2} + C$.

9. Let $u = x^2 - 1$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. Then $\int 2\left(x^2-1\right)^9 x\,dx = 2\int \frac{1}{2}u^9\,du = \frac{1}{10}u^{10} + C = \frac{1}{10}\left(x^2-1\right)^{10} + C$.

10. Let $u = 2x^3 + 3$, so $du = 6x^2\,dx$ and $x^2\,dx = \frac{1}{6}\,du$. Then $\int x^2\left(2x^3+3\right)^4 dx = \frac{1}{6}\int u^4\,du = \frac{1}{30}u^5 + C = \frac{1}{30}\left(2x^3+3\right)^5 + C$.

11. Let $u = 1 - x^5$, so $du = -5x^4\,dx$ and $x^4\,dx = -\frac{1}{5}\,du$. Then

$$\int \frac{x^4}{1-x^5}\,dx = -\frac{1}{5}\int \frac{du}{u} = -\tfrac{1}{5}\ln|u| + C = -\tfrac{1}{5}\ln\left|1-x^5\right| + C.$$

12. Let $u = x^3 - 1$, so $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$. Then

$$\int \frac{x^2}{\sqrt{x^3-1}}\,dx = \frac{1}{3}\int \frac{du}{\sqrt{u}} = \frac{1}{3}\int u^{-1/2}\,du = \tfrac{2}{3}u^{1/2} + C = \tfrac{2}{3}\sqrt{x^3-1} + C.$$

13. Let $u = x - 2$, so $du = dx$. Then $\displaystyle\int \frac{2}{x-2}\,dx = 2\int \frac{du}{u} = 2\ln|u| + C = \ln u^2 + C = \ln(x-2)^2 + C$.

14. Let $u = x^3 - 3$, so $du = 3x^2\,dx$ and $\frac{1}{3}du = x^2\,dx$. Then $\displaystyle\int \frac{x^2}{x^3-3}\,dx = \int \frac{du}{3u} = \tfrac{1}{3}\ln|u| + C = \tfrac{1}{3}\ln\left|x^3 - 3\right| + C.$

15. Let $u = 0.3x^2 - 0.4x + 2$. Then $du = (0.6x - 0.4)\,dx = 2\,(0.3x - 0.2)\,dx$. Thus, $\displaystyle\int \frac{0.3x - 0.2}{0.3x^2 - 0.4x + 2}\,dx = \int \frac{1}{2u}\,du = \tfrac{1}{2}\ln|u| + C = \tfrac{1}{2}\ln\left(0.3x^2 - 0.4x + 2\right) + C.$

16. Let $u = 0.2x^3 + 0.3x$. Then $du = \left(0.6x^2 + 0.3\right)dx = 0.3\left(2x^2 + 1\right)dx$. Thus, $\displaystyle\int \frac{2x^2+1}{0.2x^3 + 0.3x}\,dx = \int \frac{1}{0.3u}\,du = \tfrac{1}{0.3}\ln|u| + C = \tfrac{10}{3}\ln\left|0.2x^3 + 0.3x\right| + C.$

17. Let $u = 3x^2 - 1$, so $du = 6x\,dx$ and $x\,dx = \frac{1}{6}\,du$. Then $\displaystyle\int \frac{2x}{3x^2 - 1}\,dx = 2\int \frac{x}{3x^2-1}\,dx = \frac{1}{3}\int \frac{du}{u} = \tfrac{1}{3}\ln|u| + C = \tfrac{1}{3}\ln\left|3x^2 - 1\right| + C.$

18. $I = \displaystyle\int \frac{x^2 - 1}{x^3 - 3x + 1}\,dx$. Let $u = x^3 - 3x + 1$. Then $du = \left(3x^2 - 3\right)dx = 3\left(x^2 - 1\right)dx$ and $\left(x^2 - 1\right)dx = \frac{1}{3}\,du$. Therefore, $I = \int \frac{1}{3}u^{-1}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln\left|x^3 - 3x + 1\right| + C$.

19. Let $u = -2x$, so $du = -2\,dx$ and $dx = -\frac{1}{2}\,du$. Then $\int e^{-2x}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-2x} + C$.

20. Let $u = -0.02x$, so $du = -0.02\,dx$ and $dx = -\frac{1}{0.02}\,du = -50\,du$. Then $\int e^{-0.02x}\,dx = -50\int e^u\,du = -50e^{-0.02x} + C$.

21. Let $u = 2 - x$, so $du = -dx$ and $dx = -du$. Then $\int e^{2-x}\,dx = -\int e^u\,du = -e^u + C = -e^{2-x} + C$.

22. Let $u = 2t + 3$, so $du = 2\,dt$ and $dt = \frac{1}{2}\,du$. Then $\int e^{2t+3}\,dt = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{2t+3} + C$.

23. Let $u = -x^2$, so $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}\,du$. Then $\int xe^{-x^2}\,dx = \int -\frac{1}{2}e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$.

24. Let $u = x^3 - 1$, so $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$. Then $\int x^2e^{x^3-1}\,dx = \int \frac{1}{3}e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3-1} + C$.

25. $\int \left(e^x - e^{-x}\right)dx = \int e^x\,dx - \int e^{-x}\,dx = e^x - \int e^{-x}\,dx$. To evaluate the second integral on the right, let $u = -x$ so $du = -dx$ and $dx = -du$. Then $\int \left(e^x - e^{-x}\right)dx = e^x + \int e^u\,du = e^x + e^u + C = e^x + e^{-x} + C$.

26. $\int \left(e^{2x} + e^{-3x}\right)dx = \int e^{2x}\,dx + \int e^{-3x}\,dx$. To evaluate the first integral, let $u = 2x$, and to evaluate the second, let $u = -3x$. We find $\int \left(e^{2x} + e^{-3x}\right)dx = \frac{1}{2}e^{2x} - \frac{1}{3}e^{-3x} + C$.

27. Let $u = 1 + e^x$, so $du = e^x\,dx$. Then $\displaystyle\int \frac{2e^x}{1+e^x}\,dx = 2\int \frac{e^x}{1+e^x}\,dx = 2\int \frac{du}{u} = 2\ln|u| + C = 2\ln\left(1 + e^x\right) + C.$

28. Let $u = 1 + e^{2x}$, so $du = 2e^{2x}\,dx$. Then $e^{2x}\,dx = \frac{1}{2}\,du$, so $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}\,dx = \frac{1}{2}\int \frac{du}{u} = \tfrac{1}{2}\ln|u| + C = \tfrac{1}{2}\ln\left(1 + e^{2x}\right) + C.$

29. Let $u = \sqrt{x} = x^{1/2}$. Then $du = \frac{1}{2}x^{-1/2}\,dx$ and $2\,du = x^{-1/2}\,dx$, so $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int 2e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C.$

30. Let $u = e^{-1/x}$. Then $du = -\frac{1}{x^2}e^{-1/2}\,dx$, so $\int \frac{e^{-1/x}}{x^2}\,dx = \int -u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}e^{-2/x} + C$.

31. Let $u = e^{3x} + x^3$, so $du = \left(3e^{3x} + 3x^2\right)dx = 3\left(e^{3x} + x^2\right)dx$ and $\left(e^{3x} + x^2\right)dx = \frac{1}{3}du$. Then $\int \frac{e^{3x} + x^2}{\left(e^{3x} + x^3\right)^3}\,dx = \frac{1}{3}\int \frac{du}{u^3} = \frac{1}{3}\int u^{-3}\,du = -\frac{u^{-2}}{6} + C = -\frac{1}{6\left(e^{3x} + x^3\right)^2} + C$.

32. Let $u = e^x + e^{-x}$, so $du = e^x - e^{-x}\,dx$. Then $\int \frac{e^x - e^{-x}}{\left(e^x + e^{-x}\right)^{3/2}}\,dx = \int \frac{du}{u^{3/2}} = \int u^{-3/2}\,du = -2u^{-1/2} + C = -2\left(e^x + e^{-x}\right)^{-1/2} + C$.

33. Let $u = e^{2x} + 1$, so $du = 2e^{2x}\,dx$ and $\frac{1}{2}du = e^{2x}\,dx$. Then $\int e^{2x}\left(e^{2x} + 1\right)^3 dx = \int \frac{1}{2}u^3\,du = \frac{1}{8}u^4 + C = \frac{1}{8}\left(e^{2x} + 1\right)^4 + C$.

34. Let $u = 1 + e^{-x}\,dx$, so $du = -e^{-x}\,dx$. Then $\int e^{-x}\left(1 + e^{-x}\right)dx = \int -u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\left(1 + e^{-x}\right)^2 + C$.

35. Let $u = \ln 5x$, so $du = \frac{1}{x}\,dx$. Then $\int \frac{\ln 5x}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln 5x)^2 + C$.

36. Let $v = \ln u$, so $dv = \frac{1}{u}\,du$. Then $\int \frac{(\ln u)^3}{u}\,du = \int v^3\,dv = \frac{1}{4}v^4 + C = \frac{1}{4}(\ln u)^4 + C$.

37. Let $u = 3\ln x$, so $du = \frac{3}{x}\,dx$. Then $3\int \frac{1}{x\ln x}\,dx = 3\int \frac{du}{u} = 3\ln|u| + C = 3\ln|\ln x| + C$.

38. Let $u = \ln x$, so $du = \frac{1}{x}\,dx$. Then $\int \frac{1}{x(\ln x)^2}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = -u^{-1} + C = -\frac{1}{\ln x} + C$.

39. Let $u = \ln x$, so $du = \frac{1}{x}\,dx$. Then $\int \frac{\sqrt{\ln x}}{x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\ln x)^{3/2} + C$.

40. Let $u = \ln x$, so $du = \frac{1}{x}\,dx$. Then $\int \frac{(\ln x)^{7/2}}{x}\,dx = \int u^{7/2}\,du = \frac{2}{9}u^{9/2} + C = \frac{2}{9}(\ln x)^{9/2} + C$.

41. $\int \left(xe^{x^2} - \frac{x}{x^2 + 2}\right)dx = \int xe^{x^2}\,dx - \int \frac{x}{x^2 + 2}\,dx$. To evaluate the first integral, let $u = x^2$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. Then $\int xe^{x^2}\,dx = \frac{1}{2}\int e^u\,du + C_1 = \frac{1}{2}e^u + C_1 = \frac{1}{2}e^{x^2} + C_1$. To evaluate the second integral, let $u = x^2 + 2$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. Then $\int \frac{x}{x^2 + 2}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C_2 = \frac{1}{2}\ln\left(x^2 + 2\right) + C_2$. Therefore, $\int \left(xe^{x^2} - \frac{x}{x^2 + 2}\right)dx = \frac{1}{2}e^{x^2} - \frac{1}{2}\ln\left(x^2 + 2\right) + C$.

42. $\int \left(xe^{-x^2} + \frac{e^x}{e^x + 3}\right)dx = \int xe^{-x^2}\,dx + \int \frac{e^x}{e^x + 3}\,dx$. To evaluate the first integral, let $u = -x^2$, so $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}\,du$. Then $\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C_1 = -\frac{1}{2}e^{-x^2} + C_1$. To evaluate the second integral, let $u = e^x + 3$, so $du = e^x\,dx$. Then $\int \frac{e^x}{e^x + 3}\,dx = \int \frac{du}{u} = \ln|u| + C_2 = \ln\left(e^x + 3\right) + C_2$. Therefore, $\int \left(xe^{-x^2} - \frac{e^x}{e^x + 3}\right)dx = -\frac{1}{2}e^{-x^2} + \ln\left(e^x + 3\right) + C$.

43. Let $u=\sqrt{x}-1$, so $du=\frac{1}{2}x^{-1/2}\,dx=\dfrac{1}{2\sqrt{x}}\,dx$ and $dx=2\sqrt{x}\,du$. Also, we have $\sqrt{x}=u+1$, so $x=(u+1)^2=u^2+2u+1$ and $dx=2\,(u+1)\,du$. Thus,

$$\begin{aligned}\int\frac{x+1}{\sqrt{x}-1}\,dx&=\int\frac{u^2+2u+2}{u}\cdot 2\,(u+1)\,du=2\int\frac{\left(u^3+3u^2+4u+2\right)}{u}\,du\\&=2\int\left(u^2+3u+4+\tfrac{2}{u}\right)du=2\left(\tfrac{1}{3}u^3+\tfrac{3}{2}u^2+4u+2\ln|u|\right)+C\\&=2\left[\tfrac{1}{3}\left(\sqrt{x}-1\right)^3+\tfrac{3}{2}\left(\sqrt{x}-1\right)^2+4\left(\sqrt{x}-1\right)+2\ln\left|\sqrt{x}-1\right|\right]+C.\end{aligned}$$

44. Let $v=e^{-u}+u$. Then $dv=\left(-e^{-u}+1\right)du$ and $-dv=\left(e^{-u}-1\right)du$. Therefore,

$$\int\frac{e^{-u}-1}{e^{-u}+u}\,du=\int\left(-\frac{dv}{v}\right)=-\ln|v|=-\ln\left|e^{-u}+u\right|+C.$$

45. Let $u=x-1$, so $du=dx$. Also, $x=u+1$, and so

$$\begin{aligned}\textstyle\int x\,(x-1)^5\,dx&=\textstyle\int(u+1)\,u^5\,du=\int\left(u^6+u^5\right)du=\frac{1}{7}u^7+\frac{1}{6}u^6+C=\frac{1}{7}\,(x-1)^7+\frac{1}{6}\,(x-1)^6+C\\&=\frac{(6x+1)\,(x-1)^6}{42}+C.\end{aligned}$$

46. $\displaystyle\int\frac{t}{t+1}\,dt=\int\left(1-\frac{1}{t+1}\right)dt=\int dt-\int\frac{1}{t+1}\,dt=t-\ln|t+1|+C.$

47. Let $u=1+\sqrt{x}$, so $du=\frac{1}{2}x^{-1/2}\,dx$ and $dx=2\sqrt{x}=2\,(u-1)\,du$. Then

$$\begin{aligned}\int\frac{1-\sqrt{x}}{1+\sqrt{x}}dx&=\int\left(\frac{1-(u-1)}{u}\right)\cdot 2\,(u-1)\,du=2\int\frac{(2-u)\,(u-1)}{u}\,du=2\int\frac{-u^2+3u-2}{u}\,du\\&=2\int\left(-u+3-\frac{2}{u}\right)du=-u^2+6u-4\ln|u|+C\\&=-\left(1+\sqrt{x}\right)^2+6\left(1+\sqrt{x}\right)-4\ln\left(1+\sqrt{x}\right)+C\\&=-1-2\sqrt{x}-x+6+6\sqrt{x}-4\ln\left(1+\sqrt{x}\right)+C=-x+4\sqrt{x}+5-4\ln\left(1+\sqrt{x}\right)+C.\end{aligned}$$

48. Let $u=1-\sqrt{x}$, so $du=-\dfrac{1}{2\sqrt{x}}\,dx$ and $dx=-2\sqrt{x}\,du$. Then $\sqrt{x}=1-u$ and $dx=-2\,(1-u)\,du$, so

$$\begin{aligned}\int\frac{1+\sqrt{x}}{1-\sqrt{x}}\,dx&=\int\frac{2-u}{u}\,(-2)\,(1-u)\,du=-2\int\frac{(u-2)\,(u-1)}{u}\,du=-2\int\frac{u^2-3u+2}{u}\,du\\&=-2\int\left(u-3+\frac{2}{u}\right)du=-2\left(\tfrac{1}{2}u^2-3u+2\ln|u|\right)+C=6u-u^2-4\ln|u|+C\\&=6\left(1-\sqrt{x}\right)-\left(1-\sqrt{x}\right)^2-4\ln\left(1-\sqrt{x}\right)+C.\end{aligned}$$

49. $I=\int v^2\,(1-v)^6\,dv$. Let $u=1-v$, so $du=-dv$. Also, $1-u=v$, and so $(1-u)^2=v^2$. Therefore,

$$\begin{aligned}I&=\textstyle\int-\left(1-2u+u^2\right)u^6du=\int-\left(u^6-2u^7+u^8\right)du=-\left(\frac{1}{7}u^7-\frac{1}{4}u^8+\frac{1}{9}u^9\right)+C\\&=-u^7\left(\tfrac{1}{7}-\tfrac{1}{4}u+\tfrac{1}{9}u^2\right)+C=-\tfrac{1}{252}\,(1-v)^7\left[36-63\,(1-v)+28\left(1-2v+v^2\right)\right]\\&=-\tfrac{1}{252}\,(1-v)^7\left[36-63+63v+28-56v+28v^2\right]=-\tfrac{1}{252}\,(1-v)^7\left(28v^2+7v+1\right)+C.\end{aligned}$$

50. Let $u = x^2 + 1$ so $x^2 = u - 1$, $du = 2x\,dx$, and $x\,dx = \frac{1}{2}du$. Then
$$\int x^3 \left(x^2+1\right)^{3/2} dx = \int x^2 \left(x^2+1\right)^{3/2} x\,dx = \int (u-1)\,u^{3/2}\tfrac{1}{2}\,du = \tfrac{1}{2}\int \left(u^{5/2} - u^{3/2}\right)du$$
$$= \tfrac{1}{2}\left(\tfrac{2}{7}u^{7/2} - \tfrac{2}{5}u^{5/2}\right) + C = \tfrac{1}{35}u^{5/2}\,(5u-7) + C = \tfrac{1}{35}\left(x^2+1\right)^{5/2}\left(5x^2-2\right) + C.$$

51. $f(x) = \int f'(x)\,dx = 5\int (2x-1)^4\,dx$. Let $u = 2x - 1$, so $du = 2\,dx$ and $dx = \frac{1}{2}\,du$. Then $f(x) = \frac{5}{2}\int u^4\,du = \frac{1}{2}u^5 + C = \frac{1}{2}(2x-1)^5 + C$. Next, $f(1) = 3$ implies $\frac{1}{2} + C = 3$, so $C = \frac{5}{2}$. Therefore, $f(x) = \frac{1}{2}(2x-1)^5 + \frac{5}{2}$.

52. $f(x) = \displaystyle\int f'(x)\,dx = \int \frac{3x^2}{2\sqrt{x^3-1}}\,dx$. Let $u = x^3 - 1$, so $du = 3x^2\,dx$. Then $f(x) = \displaystyle\int \frac{du}{2\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\left(2u^{1/2}\right) + C = u^{1/2} + C = \left(x^3-1\right)^{1/2} + C$. Next, $f(1) = (0) + C = 1$, so $C = 1$. Therefore, $f(x) = \sqrt{x^3+1} + 1$.

53. $f(x) = \int -2xe^{-x^2+1}\,dx$. Let $u = -x^2 + 1$, so $du = -2x\,dx$. Then $f(x) = \int e^u du = e^u + C = e^{-x^2+1} + C$. The condition $f(1) = 0$ implies $f(1) = 1 + C = 0$, so $C = -1$. Therefore, $f(x) = e^{-x^2+1} - 1$.

54. $f(x) = \displaystyle\int f'(x)\,dx = \int\left(1 - \frac{2x}{x^2+1}\right)dx = \int dx - \int \frac{2x}{x^2+1}\,dx$. Let us make the substitution $u = x^2 + 1$ for the second integral on the right. With $du = 2x\,dx$, we find $f(x) = \int dx - \int \dfrac{du}{u} = x - \ln|u| + C = x - \ln\left(x^2+1\right) + C$. The condition that the graph of f passes through $(0, 2)$ translates into the condition $f(0) = 2$. Using this condition, we find $f(0) = C = 2$. Therefore, the required function is $f(x) = x - \ln\left(x^2+1\right) + 2$.

55. $N'(t) = 2000\,(1 + 0.2t)^{-3/2}$. Let $u = 1 + 0.2t$, so $du = 0.2\,dt$ and $5\,du = dt$. Then $N(t) = (5)(2000)\int u^{-3/2}\,du = -20{,}000u^{-1/2} + C = -20{,}000\,(1+0.2t)^{-1/2} + C$. Next, $N(0) = -20{,}000\,(1)^{-1/2} + C = 1000$. Therefore, $C = 21{,}000$ and $N(t) = -\dfrac{20{,}000}{\sqrt{1+0.2t}} + 21{,}000$. In particular, $N(5) = -\frac{20{,}000}{\sqrt{2}} + 21{,}000 \approx 6858$.

56. The number of viewers in the tth year is given by $N(t) = \int 3\left(2 + \frac{1}{2}t\right)^{-1/3} dt$. To evaluate this integral, let $u = 2 + \frac{1}{2}t$, so $du = \frac{1}{2}dt$ and $dt = 2\,du$. Then $N(t) = 6\int u^{-1/3}\,du = 9u^{2/3} + C = 9\left(2 + \frac{1}{2}t\right)^{2/3} + C$. The given condition implies that $N(1) = 9\left(\frac{5}{2}\right)^{2/3} + C$. Using this condition, we see that $N(1) = 9\left(\frac{5}{2}\right)^{2/3} + C = 9\left(\frac{5}{2}\right)^{2/3}$, so $C = 0$. Therefore, $N(t) = 9\left(2 + \frac{1}{2}t\right)^{2/3}$. The number of viewers in the 2013 season is given by $N(6) = 9\,(5)^{2/3} \approx 26.32$, or approximately 26.3 million.

57. The amount of CNP fraud in year t is $A(t) = \displaystyle\int [-R(t)]\,dt = -\int \frac{92.07}{t+1}\,dt = -92.07\ln(t+1) + C$. To determine C, we use the condition $A(0) = 328.9$ to obtain $C = 328.9$. Thus, $A(t) = 328.9 - 92.07\ln(t+1)$. The amount of CNP fraud in 2012 was $A(4) = 328.9 - 92.07\ln 5 \approx 180.72$, or approximately 180.7 million GBP.

58. $p(x) = -\int \frac{250x}{\left(16+x^2\right)^{3/2}}\,dx = -250\int \frac{x}{\left(16+x^2\right)^{3/2}}\,dx$. Let $u = 16 + x^2$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. Then $p(x) = -\frac{250}{2}\int u^{-3/2}\,du = (-125)(-2)\,u^{-1/2} + C = \frac{250}{\sqrt{16+x^2}} + C$. $p(3) = \frac{250}{\sqrt{16+9}} + C = 50$ implies $C = 0$, and so $p(x) = \frac{250}{\sqrt{16+x^2}}$.

59. The population t years from now will be $P(t) = \int r(t)\,dt = \int 400\left(1 + \frac{2t}{24+t^2}\right)dt = \int 400\,dt + 800\int \frac{t}{24+t^2}\,dt$. In order to evaluate the second integral on the right, let $u = 24 + t^2$, so $du = 2t\,dt$. We obtain $P(t) = 400t + 800\int \frac{\frac{1}{2}\,du}{u} = 400t + 400\,|\ln u| + C = 400\left[t + \ln\left(24+t^2\right)\right] + C$. To find C, use the condition $P(0) = 60{,}000$, giving $400(0 + \ln 24) + C = 60{,}000$ and $C = 60{,}000 - 400\ln 24 = 58{,}728.78$. Thus, $P(t) = 400\left[t + \ln\left(24+t^2\right)\right] + 58{,}728.78$, and so the population 5 years from now will be $400\left[5 + \ln\left(24+5^2\right)\right] + 58{,}728.78 \approx 62{,}285.51$, or approximately 62,286.

60. Let $u = 2t + 4$, so $du = 2\,dt$. Then $r(t) = \int \frac{30}{\sqrt{2t+4}}\,dt = 30\int \frac{1}{2}u^{-1/2}\,du = 30u^{1/2} + C = 30\sqrt{2t+4} + C$. $r(0) = 60 + C = 0$, so $C = -60$. Therefore, $r(t) = 30\left(\sqrt{2t+4} - 2\right)$. Then $r(16) = 30\left(\sqrt{36} - 2\right) = 120$ ft, so the polluted area is $\pi r^2 = \pi(120)^2 = 14{,}400\pi$, or $14{,}400\pi$ ft^2.

61. Let $u = 1 + 1.09t$. Then $du = 1.09\,dt$, so

$$\int \frac{5.45218}{(1+1.09t)^{0.9}}\,dt = 5.45218\int (1+1.09t)^{-0.9}\,dt = \frac{5.45218}{1.09}\int u^{-0.9}\,du = 50.02u^{0.1} + C$$
$$= 50.02\,(1+1.09t)^{0.1} + C.$$

Then $g(0) = 50.02 + C = 50.02$ implies that $C = 0$, so $g(t) = 50.02\,(1+1.09t)^{0.1}$ and $g(100) = 50.02\,(110)^{0.1} \approx 80.04$.

62. $N(t) = \int N'(t)\,dt = 6\int e^{-0.05t}\,dt = \frac{6}{-0.05}e^{-0.05t}$. Letting $u = -0.05t$, we have $N(t) = -120e^{-0.05t} + C$. $N(0) = 60$ implies that $-120 + C = 60$, so $C = 180$. Therefore, $N(t) = -120e^{-0.05t} + 180$.

63. a. The number of online viewers in year t is $N(t) = \int r(t)\,dt = \int 9.045e^{0.067t}\,dt = 135e^{0.067t} + C$. To determine C, we use the condition $N(0) = 135$, giving $C = 0$. Therefore, $N(t) = 135e^{0.067t}$.

b. The number of viewers in 2012 was $N(4) = 135e^{0.067(4)} \approx 176.492$, or approximately 176.5 million. The number in 2013 was $N(5) = 135e^{0.067(5)} \approx 188.722$, or approximately 188.7 million.

64. a. $S(t) = \int R(t)\,dt = \int 0.538434e^{0.234t}\,dt = 2.301e^{0.234t} + C$. To find C, we use the condition $S(0) = 2.9$, obtaining $2.301 + C = 2.9$, so $C = 0.599$. Therefore, $S(t) = 2.301e^{0.234t} + 0.599$.

b. Media ad spending in 2016 is projected to be $S(4) = 2.301e^{0.234(4)} + 0.599 \approx 6.4660$, or approximately \$6.5 billion.

65. a. $R(t) = \int r(t)\,dt = \int 3.1182e^{0.163(t+1)}\,dt \approx 19.1301e^{0.163(t+1)} + C$. To find C, we use the condition $R(0) = 22.7$, obtaining $19.1301 + C = 22.7$, so $C = 3.5699$. Thus, $R(t) = 19.1301e^{0.163(t+1)} + 3.5699$.

b. Online ad revenue in 2012 was $R(3) \approx 40.2877$, or approximately \$40.3 billion.

66. Let $u = 1 + 2.449e^{-0.3277t}$, so $du = -0.802537e^{-0.3277t}\,dt$ and $e^{-0.3277t}\,dt = -1.24605\,du$. Then
$$h(t) = \int \frac{52.8706e^{-0.3277t}}{\left(1 + 2.449e^{-0.3277t}\right)^2}\,dt = 52.8706\,(-1.24605)\int \frac{du}{u^2} = 65.8794u^{-1} + C = \frac{65.8794}{1 + 2.449e^{-0.3277t}} + C.$$
$h(0) = \dfrac{65.8794}{1 + 2.449} + C = 19.4$, so $h(t) = \dfrac{65.8794}{1 + 2.449e^{-0.3277t}} + 0.3$ and hence
$h(8) = \dfrac{65.8794}{1 + 2.449e^{-0.3277(8)}} + 0.3 \approx 56.22$, or 56.22 inches.

67. $A(t) = \int A'(t)\,dt = r\int e^{-at}\,dt$. Let $u = -at$, so $du = -a\,dt$ and $dt = -\frac{1}{a}\,du$. Then
$A(t) = r\left(-\frac{1}{a}\right)\int e^u\,du = -\frac{r}{a}e^u + C = -\frac{r}{a}e^{-at} + C$. $A(0) = 0$ implies $-\frac{r}{a} + C = 0$, so $C = \frac{r}{a}$. Therefore,
$A(t) = -\frac{r}{a}e^{-at} + \frac{r}{a} = \frac{r}{a}\left(1 - e^{-at}\right)$.

68. $x(t) = \int x'(t)\,dt = \dfrac{1}{V}(ac - bx_0)\int e^{-bt/V}\,dt$. Let $u = -\dfrac{bt}{V}$, so $du = -\dfrac{b}{V}\,dt$ and $dt = -\dfrac{V}{b}\,du$.
Then $x(t) = \dfrac{1}{V}(ac - bx_0)\displaystyle\int\left(-\frac{V}{b}e^u\right)du = \left(-\frac{ac}{b} + x_0\right)e^u = \left(-\frac{ac}{b} + x_0\right)e^{-bt/V} + C$. Because
$x(0) = \left(-\dfrac{ac}{b} + x_0\right) + C = x_0$, we have $C = \dfrac{ac}{b}$, and hence $x(t) = \dfrac{ac}{b} + \left(x_0 - \dfrac{ac}{b}\right)e^{-bt/V}$

69. True. Let $I = \int xf\left(x^2\right)dx$ and put $u = x^2$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so
$I = \frac{1}{2}\int f(u)\,du = \frac{1}{2}\int f(x)\,dx$.

70. False. Let $f(x) = x$, $a = 2$, and $b = 3$. Then $\int f(ax + b)\,dx = \int (2x + 3)\,dx = x^2 + 3x + C_1$. On the other hand $a\int f(x)\,dx = 2\int x\,dx = x^2 + C_2 \neq \int f(ax + b)\,dx$.

71. True. Put $u = kx$. Then $du = k\,dx$ and $dx = (1/k)\,du$. Thus,
$I = \int e^{kx} f\left(e^{kx}\right)dx = \int e^u f\left(e^u\right)(1/k)\,du = \frac{1}{k}\int e^u f\left(e^u\right)du$. Next, we put $w = e^u$, so $dw = e^u du$. Then
$I = \frac{1}{k}\int f(w)\,dw = \frac{1}{k}\int f(x)\,dx$.

72. True. Put $u = \ln x$, so $du = \dfrac{dx}{x}$. Then $\displaystyle\int \frac{f(\ln x)}{x}\,dx = \int f(u)\,du = \int f(x)\,dx$.

6.3 Area and the Definite Integral

Concept Questions page 442

1. See page 438 in the text.

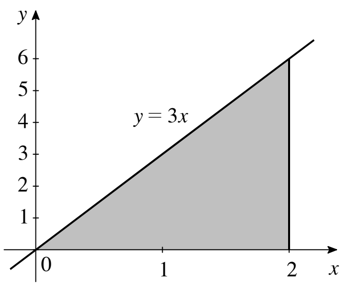
2. See pages 440–441 in the text.

Exercises page 442

1. $\frac{1}{3}(1.9 + 1.5 + 1.8 + 2.4 + 2.7 + 2.5) = \frac{12.8}{3} \approx 4.27$.

2. $\frac{1}{4}(4.5 + 8.0 + 8.5 + 6.0 + 4.0 + 3.0 + 2.5 + 2.0) = \frac{38.5}{4} = 9.625$.

3. a.



$A = \frac{1}{2}(2)(6) = 6.$

b. $\Delta x = \frac{2}{4} = \frac{1}{2}$, so $x_1 = 0, x_2 = \frac{1}{2}, x_3 = 1, x_4 = \frac{3}{2}$. Thus,

$$A \approx \frac{1}{2}\left[3(0) + 3\left(\frac{1}{2}\right) + 3(1) + 3\left(\frac{3}{2}\right)\right] = \frac{9}{2} = 4.5.$$

c. $\Delta x = \frac{2}{8} = \frac{1}{4}$, so $x_1 = 0, \ldots, x_8 = \frac{7}{4}$. Thus,

$$\begin{aligned} A &\approx \frac{1}{4}\left[3(0) + 3\left(\frac{1}{4}\right) + 3\left(\frac{1}{2}\right) + 3\left(\frac{3}{4}\right)\right. \\ &\qquad \left. + 3(1) + 3\left(\frac{5}{4}\right) + 3\left(\frac{3}{2}\right) + 3\left(\frac{7}{4}\right)\right] \\ &= \frac{21}{4} = 5.25. \end{aligned}$$

d. Yes.

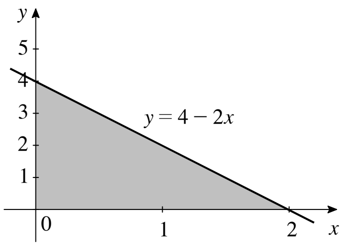
4. a. $A = 6$. See the graph in the solution to Exercise 3.

b. $x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2$. Thus, $A \approx \frac{1}{2}\left[3\left(\frac{1}{2}\right) + 3(1) + 3\left(\frac{3}{2}\right) + 3(2)\right] = 7.5.$

c. $x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, \ldots, x_8 = 2$. Thus, $A \approx \frac{1}{4}\left[3\left(\frac{1}{4}\right) + 3\left(\frac{1}{2}\right) + 3\left(\frac{3}{4}\right) + \cdots + 3\left(\frac{7}{4}\right) + 3(2)\right] = 6.75.$

d. Yes.

5. a.



$A = 4$

b. $\Delta x = \frac{2}{5} = 0.4$, so $x_1 = 0, x_2 = 0.4, x_3 = 0.8, x_4 = 1.2,$ $x_5 = 1.6$. Thus,

$$\begin{aligned} A &\approx 0.4\{[4 - 2(0)] + [4 - 2(0.4)] + [4 - 2(0.8)] \\ &\qquad + [4 - 2(1.2)] + [4 - 2(1.6)]\} = 4.8. \end{aligned}$$

c. $\Delta x = \frac{2}{10} = 0.2$, so $x_1 = 0, x_2 = 0.2, x_3 = 0.4, \ldots, x_{10} = 1.8$. Thus,

$$\begin{aligned} A &\approx 0.2\{[4 - 2(0)] + [4 - 2(0.2)] + [4 - 2(0.4)] \\ &\quad + [4 - 2(0.6)] + [4 - 2(0.8)] + [4 - 2(1.0)] + [4 - 2(1.2)] \\ &\qquad + [4 - 2(1.4)] + [4 - 2(1.6)] + [4 - 2(1.8)]\} = 4.4. \end{aligned}$$

d. Yes.

6. a. $A = 4$. See the graph in the solution to Exercise 5.

b. $\Delta x = 0.4$, so $x_1 = 0.4$, $x_2 = 0.8$, $x_3 = 1.2$, $x_4 = 1.6$, $x_5 = 2$. Thus, $A \approx 0.4\{[4 - 2(0.4)] + [4 - 2(0.8)] + \cdots + [4 - 2(2)]\} \approx 3.2.$

c. $\Delta x = 0.2$, so $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $\ldots$, $x_{10} = 2$. Thus, $A \approx 0.2\{[4 - 2(0.2)] + [4 - 2(0.4)] + \cdots + [4 - 2(2)]\} \approx 3.6.$

d. Yes.

7. a. $\Delta x = \frac{4-2}{2} = 1$, so $x_1 = 2.5, x_2 = 3.5$. The Riemann sum is $1\left(2.5^2 + 3.5^2\right) = 18.5.$

b. $\Delta x = \frac{4-2}{5} = 0.4$, so $x_1 = 2.2$, $x_2 = 2.6$, $x_3 = 3.0$, $x_4 = 3.4$, $x_5 = 3.8$. The Riemann sum is $0.4\left(2.2^2 + 2.6^2 + 3.0^2 + 3.4^2 + 3.8^2\right) = 18.64.$

c. $\Delta x = \frac{4-2}{10} = 0.2$, so $x_1 = 2.1$, $x_2 = 2.3$, $x_2 = 2.5$, $\ldots$, $x_{10} = 3.9$. The Riemann sum is $0.2\left(2.1^2 + 2.3^2 + 2.5^2 + 2.7^2 + 2.9^2 + 3.1^2 + 3.3^2 + 3.5^2 + 3.7^2 + 3.9^2\right) = 18.66.$

d. The area appears to be $18\frac{2}{3}$.

8. **a.** $\Delta x = \frac{4-2}{2} = 1$, so $x_1 = 2, x_2 = 3$. The Riemann sum is $1\left(2^2 + 3^2\right) = 13$.

b. $\Delta x = \frac{4-2}{5} = 0.4$, so $x_1 = 2, x_2 = 2.4, x_3 = 2.8, x_4 = 3.2, x_5 = 3.6$. The Riemann sum is $0.4\left(2^2 + 2.4^2 + 2.8^2 + 3.2^2 + 3.6^2\right) = 16.32$.

c. $\Delta x = \frac{4-2}{10} = 0.2$, so $x_1 = 2, x_2 = 2.2, x_3 = 2.4, \ldots, x_{10} = 3.8$. The Riemann sum is $0.2\left(2^2 + 2.2^2 + 2.4^2 + \cdots + 3.8^2\right) = 17.48$.

d. The area appears to be $17\frac{1}{2}$.

9. **a.** $\Delta x = \frac{4-2}{2} = 1$, so $x_1 = 3, x_2 = 4$. The Riemann sum is $(1)\left(3^2 + 4^2\right) = 25$.

b. $\Delta x = \frac{4-2}{5} = 0.4$, so $x_1 = 2.4, x_2 = 2.8, x_3 = 3.2, x_4 = 3.6, x_5 = 4$. The Riemann sum is $0.4\left(2.4^2 + 2.8^2 + \cdots + 4^2\right) = 21.12$.

c. $\Delta x = \frac{4-2}{10} = 0.2$, so $x_1 = 2.2, x_2 = 2.4, x_3 = 2.6, \ldots, x_{10} = 4$. The Riemann sum is $0.2\left(2.2^2 + 2.4^2 + 2.6^2 + 2.8^2 + 3.0^2 + 3.2^2 + 3.4^2 + 3.6^2 + 3.8^2 + 4^2\right) = 19.88$.

d. The area appears to be 19.9.

10. **a.** $\Delta x = \frac{1-0}{2} = \frac{1}{2}$, so $x_1 = \frac{1}{4}, x_2 = \frac{3}{4}$. The Riemann sum is

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x = \left[\left(\tfrac{1}{4}\right)^3 + \left(\tfrac{3}{4}\right)^3\right]\tfrac{1}{2} = \left(\tfrac{1}{64} + \tfrac{27}{64}\right)\tfrac{1}{2} = \tfrac{7}{32} = 0.21875.$$

b. $\Delta x = \frac{1-0}{5} = \frac{1}{5}$, so $x_1 = \frac{1}{10}, x_2 = \frac{3}{10}, x_3 = \frac{5}{10}, x_4 = \frac{7}{10}, x_5 = \frac{9}{10}$. The Riemann sum is

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_5)\,\Delta x = \left[\left(\tfrac{1}{10}\right)^3 + \left(\tfrac{3}{10}\right)^3 + \cdots + \left(\tfrac{9}{10}\right)^3\right]\tfrac{1}{5} = \tfrac{1}{5000}(1 + 27 + \cdots + 729)$$

$$= \tfrac{1225}{5000} = 0.245.$$

c. $\Delta x = \frac{1-0}{10} = \frac{1}{10}$, so $x_1 = \frac{1}{20}, x_2 = \frac{3}{20}, x_3 = \frac{5}{20}, \ldots, x_{10} = \frac{19}{20}$. The Riemann sum is

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_{10})\,\Delta x = \left[\left(\tfrac{1}{20}\right)^3 + \left(\tfrac{3}{20}\right)^3 + \cdots + \left(\tfrac{19}{20}\right)^3\right]\tfrac{1}{10} = \tfrac{19{,}900}{80{,}000} \approx 0.24875.$$

d. The Riemann sums seem to approach $\frac{1}{4}$.

11. **a.** $\Delta x = \frac{1}{2}$, so $x_1 = 0, x_2 = \frac{1}{2}$. The Riemann sum is $f(x_1)\,\Delta x + f(x_2)\,\Delta x = \left[(0)^3 + \left(\frac{1}{2}\right)^3\right]\frac{1}{2} = \frac{1}{16} = 0.0625$.

b. $\Delta x = \frac{1}{5}$, so $x_1 = 0, x_2 = \frac{1}{5}, x_3 = \frac{2}{5}, x_4 = \frac{3}{5}, x_5 = \frac{4}{5}$. The Riemann sum is

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_5)\,\Delta x = \left[\left(\tfrac{1}{5}\right)^3 + \left(\tfrac{2}{5}\right)^3 + \cdots + \left(\tfrac{4}{5}\right)^3\right]\tfrac{1}{5} = \tfrac{100}{625} = 0.16.$$

c. $\Delta x = \frac{1}{10}$, so $x_1 = 0, x_2 = \frac{1}{10}, x_3 = \frac{2}{10}, \ldots, x_{10} = \frac{9}{10}$. The Riemann sum is

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_{10})\,\Delta x$$

$$= \left[0^3 + \left(\tfrac{1}{10}\right)^3 + \left(\tfrac{2}{10}\right)^3 + \left(\tfrac{3}{10}\right)^3 + \left(\tfrac{4}{10}\right)^3 + \left(\tfrac{5}{10}\right)^3 + \left(\tfrac{6}{10}\right)^3 + \left(\tfrac{7}{10}\right)^3 + \left(\tfrac{8}{10}\right)^3 + \left(\tfrac{9}{10}\right)^3\right]\tfrac{1}{10}$$

$$= \tfrac{2025}{10{,}000} = 0.2025 \approx 0.2.$$

d. The Riemann sums seem to approach 0.2.

12. **a.** $\Delta x = \frac{1}{2}$, so $x_1 = \frac{1}{2}, x_2 = 1$. The Riemann sum is $f(x_1)\,\Delta x + f(x_2)\,\Delta x = \left[\left(\frac{1}{2}\right)^3 + 1^3\right]\frac{1}{2} = 0.5625$.

b. $\Delta x = \frac{1}{5}$, so $x_1 = \frac{1}{5}$, $x_2 = \frac{2}{5}$, $x_3 = \frac{3}{5}$, $x_4 = \frac{4}{5}$, $x_5 = 1$. The Riemann sum is

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots f(x_5)\,\Delta x = \left[\left(\tfrac{1}{5}\right)^3 + \left(\tfrac{2}{5}\right)^3 + \cdots + \left(\tfrac{4}{5}\right)^3 + 1\right]\tfrac{1}{5} = \tfrac{225}{625} = 0.36.$$

c. $\Delta x = \frac{1}{10}$, so $x_1 = \frac{1}{10}$, $x_2 = \frac{2}{10}$, …, $x_{10} = 1$. The Riemann sum is

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_{10})\,\Delta x = \left[\left(\tfrac{1}{10}\right)^3 + \left(\tfrac{2}{10}\right)^3 + \cdots + 1\right]\tfrac{1}{10} = \tfrac{3025}{10{,}000} = 0.3025.$$

d. The Riemann sums seem to approach 0.3.

13. $\Delta x = \frac{2-0}{5} = \frac{2}{5}$, so $x_1 = \frac{1}{5}$, $x_2 = \frac{3}{5}$, $x_3 = \frac{5}{5}$, $x_4 = \frac{7}{5}$, $x_5 = \frac{9}{5}$. Thus,

$$A \approx \left\{\left[\left(\tfrac{1}{5}\right)^2 + 1\right] + \left[\left(\tfrac{3}{5}\right)^2 + 1\right] + \left[\left(\tfrac{5}{5}\right)^2 + 1\right] + \left[\left(\tfrac{7}{5}\right)^2 + 1\right] + \left[\left(\tfrac{9}{5}\right)^2 + 1\right]\right\}\left(\tfrac{2}{5}\right) = \tfrac{580}{125} = 4.64.$$

14. $\Delta x = \frac{2-(-1)}{6} = \frac{1}{2}$, so $x_1 = -1$, $x_2 = -\frac{1}{2}$, $x_3 = 0$, $x_4 = \frac{1}{2}$, $x_5 = 1$, $x_6 = \frac{3}{2}$. Thus,

$$A \approx \left\{\left[4 - (-1)^2\right] + \left[4 - \left(\tfrac{1}{2}\right)^2\right] + \left[4 - 0^2\right] + \left[4 - \left(\tfrac{1}{2}\right)^2\right] + \left[4 - 1^2\right] + \left[4 - \left(\tfrac{3}{2}\right)^2\right]\right\}\left(\tfrac{1}{2}\right) = \tfrac{77}{8} = 9.625.$$

15. $\Delta x = \frac{3-1}{4} = \frac{1}{2}$, so $x_1 = \frac{3}{2}$, $x_2 = \frac{4}{2} = 2$, $x_3 = \frac{5}{2}$, $x_4 = 3$. Thus, $A \approx \left(\frac{1}{3/2} + \frac{1}{2} + \frac{1}{5/2} + \frac{1}{3}\right)\frac{1}{2} \approx 0.95$.

16. $\Delta x = \frac{3-0}{5} = \frac{3}{5}$, so $x_1 = \frac{3}{10}$, $x_2 = \frac{9}{10}$, $x_3 = \frac{15}{10} = \frac{3}{2}$, $x_4 = \frac{21}{10}$, $x_5 = \dfrac{27}{10}$. Thus,

$$A \approx \left(e^{3/10} + e^{9/10} + e^{3/2} + e^{21/10} + e^{27/10}\right)\left(\tfrac{3}{5}\right) \approx 18.8.$$

17. $A \approx 20\left[f(10) + f(30) + f(50) + f(70) + f(90)\right] = 20(80 + 100 + 110 + 100 + 80) = 9400\text{ ft}^2$.

18. $A \approx 20\left[f(10) + f(30) + f(50) + f(70)\right] = 20(100 + 75 + 80 + 82.5) = 6750\text{ ft}^2$.

19. False. Take $f(x) = x$, $a = -1$, and $b = 2$. Then $\int_{-1}^{2} f(x)\,dx = \int_{-1}^{2} x\,dx = \frac{1}{2}x^2\Big|_{-1}^{2} = \frac{1}{2}(4-1) = \frac{3}{2} > 0$, but $f(-1) = -1 < 0$.

20. True. Suppose $f(c) \neq 0$, where $a \le c \le b$. If $c = a$ or $c = b$, then there is an interval $[a, a+s)$ or $(b-t, b]$ where $\left[f(x)\right]^2 > 0$; if $a < c < b$, then there is an interval $(c-u, c+u)$ in (a, b) where $\left[f(x)\right]^2 > 0$. In any case, $\int_I \left[f(x)\right]^2 dx > 0$ on each of these subintervals of $[a, b]$.

6.4 The Fundamental Theorem of Calculus

Concept Questions page 452

1. See the Fundamental Theorem of Calculus on page 444 of the text.

2. See page 448 of the text.

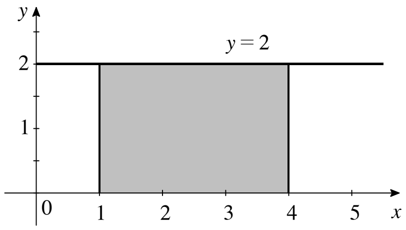
a. It measures the total income generated over $(b-a)$ days.

b. $\int_a^b R(t)\,dt$

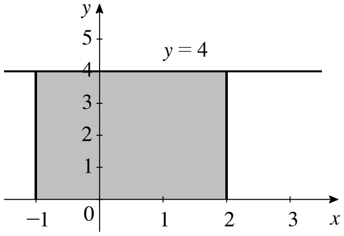
Exercises

page 453

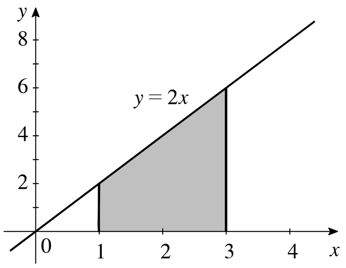
1. $A=\int_1^4 2\,dx=2x\big|_1^4=2\,(4-1)=6$. The region is a rectangle with area $3\cdot 2=6$.



2. $A=\int_{-1}^2 4\,dx=4x\big|_{-1}^2=8-(-4)=12$. The region is a rectangle with area $4\,[2-(-1)]=12$.

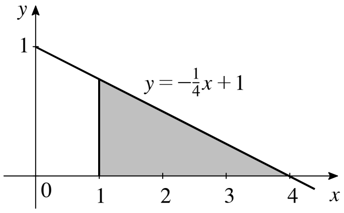


3. $A=\int_1^3 2x\,dx=x^2\big|_1^3=9-1=8$. The region is a parallelogram with area $\frac{1}{2}\,(3-1)\,(2+6)=8$.



4. $A=\int_1^4\left(-\frac{1}{4}x+1\right)dx=\left(-\frac{1}{8}x^2+x\right)\Big|_1^4$
$=(-2+4)-\left(-\frac{1}{8}+1\right)=\frac{9}{8}$.

The region is a triangle with area $\frac{1}{2}\,(3)\left(\frac{3}{4}\right)=\frac{9}{8}$.



5. $A=\int_{-1}^2 (2x+3)\,dx=\left(x^2+3x\right)\big|_{-1}^2=(4+6)-(1-3)=12$.

6. $A=\int_2^4 (4x-1)\,dx=\left(2x^2-x\right)\big|_2^4=(32-4)-(8-2)=22$.

7. $A=\int_{-1}^2\left(-x^2+4\right)dx=\left(-\frac{1}{3}x^3+4x\right)\Big|_{-1}^2=\left(-\frac{8}{3}+8\right)-\left(\frac{1}{3}-4\right)=9$.

8. $A=\int_0^4\left(4x-x^2\right)dx=\left(2x^2-\frac{1}{3}x^3\right)\Big|_0^4=32-\frac{64}{3}=\frac{32}{3}$.

9. $A=\displaystyle\int_1^2\frac{1}{x}\,dx=\ln|x|\big|_1^2=\ln 2-\ln 1=\ln 2$.

10. $A=\displaystyle\int_2^4\frac{1}{x^2}\,dx=\int_2^4 x^{-2}\,dx=-\frac{1}{x}\bigg|_2^4=-\frac{1}{4}+\frac{1}{2}=\frac{1}{4}$.

11. $A=\int_1^9\sqrt{x}\,dx=\frac{2}{3}x^{3/2}\Big|_1^9=\frac{2}{3}\,(27-1)=\frac{52}{3}$.

12. $A=\int_1^3 x^3\,dx=\frac{1}{4}x^4\Big|_1^3=\frac{1}{4}\,(81-1)=20$.

13. $A=\int_{-8}^{-1}\left(1-x^{1/3}\right)dx=\left(x-\frac{3}{4}x^{4/3}\right)\Big|_{-8}^{-1}=\left(-1-\frac{3}{4}\right)-(-8-12)=\frac{73}{4}$.

14. $A = \int_1^9 x^{-1/2}\,dx = 2x^{1/2}\big|_1^9 = 2\,(3-1) = 4.$

15. $A = \int_0^2 e^x\,dx = e^x\big|_0^2 = e^2 - 1 \approx 6.39.$

16. $A = \int_1^2 (e^x - x)\,dx = \left(e^x - \frac{1}{2}x^2\right)\Big|_1^2 = (e^2 - 2) - \left(e - \frac{1}{2}\right) = e^2 - e - \frac{3}{2} \approx 3.17.$

17. $\int_2^4 3\,dx = 3x\big|_2^4 = 3\,(4-2) = 6.$

18. $\int_{-1}^2 (-2)\,dx = -2x\big|_{-1}^2 = -4 - 2 = -6.$

19. $\int_1^4 (2x+3)\,dx = \left(x^2 + 3x\right)\big|_1^4 = (16+12) - (1+3) = 24.$

20. $\int_{-1}^0 (4-x)\,dx = \left(4x - \frac{1}{2}x^2\right)\Big|_{-1}^0 = 0 - \left(-4 - \frac{1}{2}\right) = \frac{9}{2}.$

21. $\int_{-1}^3 2x^2\,dx = \frac{2}{3}x^3\Big|_{-1}^3 = \frac{2}{3}\,(27) - \frac{2}{3}\,(-1) = \frac{56}{3}.$

22. $\int_0^2 8x^3\,dx = 2x^4\big|_0^2 = 32.$

23. $\int_{-2}^2 \left(x^2 - 1\right)dx = \left(\frac{1}{3}x^3 - x\right)\Big|_{-2}^2 = \left(\frac{8}{3} - 2\right) - \left(-\frac{8}{3} + 2\right) = \frac{4}{3}.$

24. $\int_1^4 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_1^4 = \frac{2}{3}\,(8) - \frac{2}{3}\,(1) = \frac{14}{3}.$

25. $\int_1^8 2x^{1/3}\,dx = 2\cdot\frac{3}{4}x^{4/3}\Big|_1^8 = \frac{3}{2}\,(16-1) = \frac{45}{2}.$

26. $\int_1^4 2x^{-3/2}\,dx = 2\left(-2x^{-1/2}\right)\big|_1^4 = -4\left(\frac{1}{2} - 1\right) - 2.$

27. $\int_0^1 \left(x^3 - 2x^2 + 1\right)dx = \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 + x\right)\Big|_0^1 = \frac{1}{4} - \frac{2}{3} + 1 = \frac{7}{12}.$

28. $\int_1^2 \left(t^5 - t^3 + 1\right)dt = \left(\frac{1}{6}t^6 - \frac{1}{4}t^4 + t\right)\Big|_1^2 = \left[\frac{1}{6}\,(64) - \frac{1}{4}\,(16) + 2\right] - \left(\frac{1}{6} - \frac{1}{4} + 1\right) = \frac{31}{4}.$

29. $\displaystyle\int_1^4 \frac{1}{x}\,dx = \ln|x|\big|_1^4 = \ln 4 - \ln 1 = \ln\frac{4}{1} = \ln 4.$

30. $\displaystyle\int_1^3 \frac{2}{x}\,dx = 2\ln|x|\big|_1^3 = 2\ln 3.$

31. $\int_0^4 x\left(x^2 - 1\right)dx = \int_0^4 \left(x^3 - x\right)dx = \left(\frac{1}{4}x^4 - \frac{1}{2}x^2\right)\Big|_0^4 = 64 - 8 = 56.$

32. $\int_0^2 (x-4)\,(x-1)\,dx = \int_0^2 \left(x^2 - 5x + 4\right)dx = \left(\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x\right)\Big|_0^2 = \frac{8}{3} - 10 + 8 = \frac{2}{3}.$

33. $\int_1^3 \left(t^2 - t\right)^2 dt = \int_1^3 \left(t^4 - 2t^3 + t^2\right)dt = \left(\frac{1}{5}t^5 - \frac{1}{2}t^4 + \frac{1}{3}t^3\right)\Big|_1^3 = \left(\frac{243}{5} - \frac{81}{2} + \frac{27}{3}\right) - \left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right) = \frac{512}{30} = \frac{256}{15}.$

34. $\int_{-1}^{1}\left(x^2-1\right)^2 dx = \int_{-1}^{1}\left(x^4-2x^2+1\right)dx = \left(\frac{1}{5}x^5-\frac{2}{3}x^3+x\right)\Big|_{-1}^{1} = \left(\frac{1}{5}-\frac{2}{3}+1\right)-\left(-\frac{1}{5}+\frac{2}{3}-1\right)=\frac{16}{15}.$

35. $\int_{-3}^{-1} x^{-2}dx = -\frac{1}{x}\Big|_{-3}^{-1} = 1-\frac{1}{3}=\frac{2}{3}.$

36. $\int_{1}^{2} 2x^{-3}\,dx = -\frac{1}{x^2}\Big|_{1}^{2} = -\frac{1}{4}+1=\frac{3}{4}.$

37. $\displaystyle\int_{1}^{4}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx = \int_{1}^{4}\left(x^{1/2}-x^{-1/2}\right)dx = \left(\frac{2}{3}x^{3/2}-2x^{1/2}\right)\Big|_{1}^{4} = \left(\frac{16}{3}-4\right)-\left(\frac{2}{3}-2\right)=\frac{8}{3}.$

38. $\int_{0}^{1}\sqrt{2x}\left(\sqrt{x}+\sqrt{2}\right)dx = \int_{0}^{1}\left(\sqrt{2}x+2\sqrt{x}\right)dx = \left(\frac{\sqrt{2}}{2}x^2+\frac{4}{3}x^{3/2}\right)\Big|_{0}^{1} = \frac{\sqrt{2}}{2}+\frac{4}{3}.$

39. $\displaystyle\int_{1}^{4}\frac{3x^3-2x^{2.}+4}{x^2}\,dx = \int_{1}^{4}\left(3x-2+4x^{-2}\right)dx = \left(\frac{3}{2}x^2-2x-\frac{4}{x}\right)\Bigg|_{1}^{4}$

$= \left(24-8-1\right)-\left(\frac{3}{2}-2-40\right)=\frac{39}{2}.$

40. $\displaystyle\int_{1}^{2}\left(1+\frac{1}{u}+\frac{1}{u^2}\right)du = \left(u+\ln u-\frac{1}{u}\right)\Bigg|_{1}^{2} = \left(2+\ln 2-\frac{1}{2}\right)-(1+0-1)=\frac{3}{2}+\ln 2.$

41. a. The change in the number of annual personal bankruptcy filings between September 30, 2010 and September 2012 was $N(t)=\int_0^2\left[-R(t)\right]dt = -\int_0^2(0.077t+0.0825)\,dt = \left[-0.0385t^2+0.0825t\right]_0^2 \approx -0.319$, a decline of approximately 319,000.

b. The approximate number of personal bankruptcy filings in 2012 was $N(2)=N(0)+\int_0^2 N'(t)\,dt = 1.538-0.319=1.219$, or 1,219,000.

42. Projected spending in 2016 is $A(6)=A(0)+\int_0^6 R(t)\,dt = 317+\int_0^6(1.0952t+17.357)\,dt = 317+\left[0.5476t^2+17.357t\right]_0^6 \approx 440.856$, or approximately \$440.9 million.

43. a. $C(300)-C(0)=\int_0^{300}\left(0.0003x^2-0.12x+20\right)dx = \left(0.0001x^3-0.06x^2+20x\right)\big|_0^{300}$

$= 0.0001(300)^3-0.06(300)^2+20(300)=3300.$

Therefore, $C(300)=3300+C(0)=3300+800=\$4100$.

b. $\int_{200}^{300} C'(x)\,dx = \left(0.0001x^3-0.06x^2+20x)\right)\big|_{200}^{300}$

$= \left[0.0001(300)^3-0.06(300)^2+20(300)\right]-\left[0.0001(200)^3-0.06(200)^2+20(200)\right]=\$900.$

44. a. $R(200)=\int_0^{200}(-0.1x+40)\,dx = \left(-0.05x^2+40x\right)\big|_0^{200} = 6000$, or \$6000.

b. $R(300)-R(200)=\int_{200}^{300}(-0.1x+40)dx = \left(-0.05x^2+40x\right)\big|_{200}^{300} = 7500-6000=1500$, or \$1500.

45. a. The profit is

$\int_0^{200}\left(-0.0003x^2+0.02x+20\right)dx+P(0) = \left(-0.0001x^3+0.01x^2+20x\right)\big|_0^{200}+P(0)$

$= 3600+P(0)=3600-800$, or \$2800.

b. $\int_{200}^{220} P'(x)\,dx = P(220)-P(200) = \left(-0.0001x^3+0.01x^2+20x\right)\big|_{200}^{220} = 219.20$, or \$219.20.

46. **a.** $N(4) - N(0) = \int_0^4 N'(t)\,dt = \int_0^4 \left(-\frac{3}{2}t^2 + 6t + 20\right)dt = \left(-\frac{1}{2}t^3 + 3t^2 + 20t\right)\Big|_0^4$

$= -\frac{1}{2}(64) + 3(16) + 20(4) = 96.$

b. $N(1) - N(0) = \left(-\frac{1}{2}t^3 + 3t^2 + 20t\right)\Big|_0^1 = \frac{45}{2}$ and $N(2) - N(1) = \left(-\frac{1}{2}t^3 + 3t^2 + 20t\right)\Big|_1^2 = \frac{51}{2}$.

47. The distance is $\int_0^{20} v(t)\,dt = \int_0^{20} \left(-t^2 + 20t + 440\right)dt = \left(-\frac{1}{3}t^3 + 10t^2 + 440t\right)\Big|_0^{20} \approx 10{,}133.3$ ft.

48. The number is given by $\int_0^6 \left(0.18t^2 + 0.16t + 2.64\right)dt = \left(0.06t^3 + 0.08t^2 + 2.74t\right)\big|_0^6 = 0.06(216) + 0.08(36) + 2.64(6) = 31.68$, or approximately 31.68 million units.

49. The average U.S. household credit card debt at the beginning of 2012 was $D(4) = D(0) + \int_0^4 \left(258t^2 - 680t - 316\right)dt = 8382 + \left[86t^3 - 340t^2 - 316t\right]_0^4 = 7182$, or \$7182.

50. The U.S. national debt in 2012 was

$D(4) = D(0) + \int_0^4 \left(0.070251t^2 - 0.51548t + 2.1667\right)dt$

$\approx 10.025 + \left[0.0234167t^3 - 0.25774t^2 + 2.1667t\right]_0^4 \approx 16.0666,$

or approximately \$16.067 trillion.

51. The amount of smoke left after 5 minutes is

$100 - \int_0^5 R(t)\,dt = 100 - \int_0^5 \left(0.00032t^4 - 0.01872t^3 + 0.3948t^2 - 3.83t + 17.63\right)dt$

$= 100 - \left(0.000064t^5 - 0.00468t^4 + 0.1316t^3 - 1.915t^2 + 17.63t)\right)\big|_0^5 = 46$, or 46%.

The amount of smoke left after 10 minutes is $100 - \int_0^{10} R(t)\,dt = 100 - \left(0.000064t^5 - 0.00468t^4 + 0.1316t^3 - 1.915t^2 + 17.63t\right)\big|_0^{10} = 24$, or 24%.

52. The number of solar panels produced during the second year is approximately $N = \int_{12}^{24} \left(\frac{4t}{1+t^2} + 3t^{1/2}\right)dt$. To find $I = \int \frac{4t}{1+t^2}\,dt$, let $u = 1 + t^2$, so $du = 2t\,dt$. Then $I = 2\int \frac{du}{1+u} = \ln|1+u| + C = \ln\left(1+t^2\right)^2 + C$. Using this result, we find $N = \left[2\ln\left(1+t^2\right) + 2t^{3/2}\right]_{12}^{24} \approx 154.77$, or 15,477 panels.

53. **a.** $f(t) = \int R(t)\,dt = \int 0.8256t^{-0.04}\,dt = \frac{0.8256}{0.96}t^{0.96} + C = 0.86t^{0.96} + C$. $f(1) = 0.9$, and so $0.86 + C = 0.9$ and $C = 0.04$. Thus, $f(t) = 0.86t^{0.96} + 0.04$.

b. In 2014, mobile phone ad spending is projected to be $f(8) = 0.86(8)^{0.96} + 0.04 \approx 6.37$, or approximately \$6.37 billion.

54. The credit card delinquency rate at the beginning of 2012 was $D(24) = D(0) + \int_0^{24} \left(0.150975e^{-0.0275t}\right)dt = 5.7 - \left[\frac{0.150975}{0.0275}e^{-0.0275t}\right]_0^{24} \approx 3.04751$, or approximately 3.048%.

55. **a.** The revenue of the company increased by $\int_0^6 R'(t)\,dt = \int_0^6 0.545043e^{0.291t}\,dt = 1.873e^{0.291t}\big|_0^6 \approx 8.862$, or approximately \$8.86 billion.

b. The revenue of the company in 2012 was $R(0) + \int_0^6 R'(t)\,dt \approx 10.71 + 8.86 \approx 19.57$, or approximately \$19.57 billion.

56. The total production is $P(t) = \displaystyle\int_0^{15} \frac{4.76}{1+4.11e^{-0.22t}}\,dt = \int_0^{15} \frac{4.76e^{0.22t}}{e^{0.22t}+4.11}\,dt$ (multiply numerator and denominator by $e^{0.22t}$). Put $I = \displaystyle\int \frac{4.76e^{0.22t}}{e^{0.22t}+4.11}\,dt$ and let $u = e^{0.22t} + 4.11$, so $du = 0.22e^{0.22t}\,dt$. Then $I = \dfrac{4.76}{0.22}\displaystyle\int \frac{du}{u} = \frac{4.76}{0.22}\ln|u| + C = \frac{4.76}{0.22}\ln\left(e^{0.22t}+4.11\right) + C$, so $P(t) = \left[\dfrac{4.76}{0.22}\ln\left(e^{0.22t}+4.11\right)\right]_0^{15} = \dfrac{4.76}{0.22}\left[\ln\left(e^{3.3}+4.11\right) - \ln(1+4.11)\right] \approx 39.16$, or approximately 39.16 million barrels.

57. The increase in the senior population over the period in question is $\displaystyle\int_0^3 f(t)\,dt = \int_0^3 \frac{85}{1+1.859e^{-0.66t}}\,dt$. Multiplying the numerator and denominator of the integrand by $e^{0.66t}$ gives $\displaystyle\int_0^3 f(t)\,dt = 85\int_0^3 \frac{e^{0.66t}}{e^{0.66t}+1.859}\,dt$. Now let $u = 1.859 + e^{0.66t}$, so $du = 0.66e^{0.66t}\,dt$ and $e^{0.66t}dt = \dfrac{du}{0.66}$. If $t = 0$, then $u = 2.859$, and if $t = 3$, then $u = 9.1017$. Substituting, we have

$$\int_0^3 f(t)\,dt = 85\int_{2.859}^{9.1017} \frac{du}{0.66u} = \frac{85}{0.66}\ln u\Big|_{2.859}^{9.1017} = \frac{85}{0.66}(\ln 9.1017 - \ln 2.859) = \frac{85}{0.66}\ln\frac{9.1017}{2.859} \approx 149.135,$$ or approximately 149.14 million people.

58. $V = \frac{k}{L}\int_0^R x\left(R^2 - x^2\right)dx$. Use substitution with $u = R^2 - x^2$, so that $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}\,du$. If $x = 0$, then $u = R^2$, and if $x = R$, then $u = 0$. Substituting, we have

$$V = \frac{k}{L}\int_{R^2}^0 u\left(-\tfrac{1}{2}du\right) = -\frac{k}{2L}\int_{R^2}^0 u\,du = -\frac{k}{4L}u^2\Big|_{R^2}^0 = 0 - \left(-\frac{k}{4L}R^4\right) = \frac{kR^4}{4L}.$$

59. $f(x) = x^4 - 2x^2 + 2$, so $f'(x) = 4x^3 - 4x = 4x\left(x^2-1\right) = 4x(x+1)(x-1)$. Setting $f'(x) = 0$ gives $x = -1$, 0, and 1 as critical numbers. Now calculate $f''(x) = 12x^2 - 4 = 4\left(3x^2-1\right)$ and use the second derivative test: $f''(-1) = 8 > 0$, so $(-1, 1)$ is a relative minimum; $f''(0) = -4 < 0$, so $(0, 2)$ is a relative maximum; and $f''(1) = 8 > 0$, so $(1, 1)$ is a relative minimum. The graph of f is symmetric with respect to the y-axis because $f(-x) = (-x)^4 - 2(-x)^2 + 2 = x^4 - 2x^2 + 2 = f(x)$. Thus, the required area is the area under the graph of f between $x = 0$ and $x = 1$, that is, $A = \int_0^1 \left(x^4 - 2x^2 + 2\right)dx = \left(\frac{1}{5}x^5 - \frac{2}{3}x^3 + 2x\right)\Big|_0^1 = \frac{1}{5} - \frac{2}{3} + 2 = \frac{23}{15}$.

60. $f(x) = \dfrac{x+1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$, so $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2}x^{-3/2}(x-1)$. Setting $f'(x) = 0$ gives $x = 1$ as the only critical number of f. Now calculate $f''(x) = -\frac{1}{4}x^{-3/2} + \frac{3}{4}x^{-5/2} = -\frac{1}{4}x^{-5/2}(x-3)$. Because $f''(1) = \frac{1}{2} > 0$, we see that $(1, 2)$ is a relative minimum of f. $f''(x) = 0$ gives $x = 3$, and because $f''(x) > 0$ if $x < 3$ and $f''(x) < 0$ if $x > 3$, we see that $\left(3, \frac{4}{\sqrt{3}}\right)$ is an inflection point of f. The required area is

$$A = \int_1^3 \left(x^{1/2} + x^{-1/2}\right)dx = \left(\tfrac{2}{3}x^{3/2} + 2x^{1/2}\right)\Big|_1^3 = \left[\tfrac{2}{3}\left(3^{3/2}\right) + 2\left(3^{1/2}\right)\right] - \left(\tfrac{2}{3} + 2\right) = 4\sqrt{3} - \tfrac{8}{3}$$
$$= \tfrac{12\sqrt{3}-8}{3}.$$

61. False. The integrand $f(x) = 1/x^3$ is discontinuous at $x = 0$.

62. False. The integrand $f(x) = 1/x$ is not defined at $x = 0$, which lies in the interval $[-1, 1]$.

63. False. $f(x)$ is not nonnegative on $[0, 2]$.

64. True.

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1. 6.1787
2. 3.3279
3. 0.7873
4. 0.2024
5. -0.5888
6. 737,038.44
7. 2.7044
8. 0.4251
9. 3.9973
10. 0.4182
11. 46%, 24%
12. 149.14 million
13. 60,156
14. 3,761,490

6.5 Evaluating Definite Integrals

Concept Questions page 463

1. *Approach I:* We first find the indefinite integral. Let $u = x^3 + 1$, so that $du = 3x^2dx$ and or $x^2dx = \frac{1}{3}du$. Then $\int x^2\left(x^3+1\right)^2 dx = \frac{1}{3}\int u^2\,du = \frac{1}{9}u^3 + C = \frac{1}{9}\left(x^3+1\right)^3 + C$. Therefore, $\int_0^1 x^2\left(x^3+1\right)^2 dx = \frac{1}{9}\left(x^3+1^3\right)\Big|_0^1 = \frac{1}{9}(8-1) = \frac{7}{9}$.

Approach II: Transform the definite integral in x into an integral in u: Let $u = x^3 + 1$, so that $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$. Next, find the limits of integration with respect to u. If $x = 0$, then $u = 0^3 + 1 = 1$ and if $x = 1$, then $u = 1^3 + 1 = 2$. Therefore, $\int_0^1 x^2\left(x^3+1\right)^2 dx = \frac{1}{3}\int_1^2 u^2\,du = \frac{1}{9}u^3\Big|_1^2 = \frac{1}{9}(8-1) = \frac{7}{9}$.

2. See the definition and interpretation on pages 447 and 448 of the text.

Exercises page 463

1. Let $u = x^2 - 1$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. If $x = 0$, then $u = -1$ and if $x = 2$, then $u = 3$, so $\int_0^2 x\left(x^2-1\right)^3 dx = \frac{1}{2}\int_{-1}^3 u^3\,du = \frac{1}{8}u^4\Big|_{-1}^3 = \frac{1}{8}(81) - \frac{1}{8}(1) = 10$.

2. Let $u = 2x^3 - 1$, so $du = 6x^2\,dx$ and $x^2\,dx = \frac{1}{6}\,du$. If $x = 0$, then $u = -1$ and if $x = 1$, then $u = 1$, so $\int_0^1 x^2\left(2x^3-1\right)^4 dx = \frac{1}{6}\int_{-1}^1 u^4\,du = \frac{1}{30}u^5\Big|_{-1}^1 = \frac{1}{30} - \left(-\frac{1}{30}\right) = \frac{1}{15}$.

3. Let $u = 5x^2 + 4$, so $du = 10x\,dx$ and $x\,dx = \frac{1}{10}\,du$. If $x = 0$, then $u = 4$ and if $x = 1$, then $u = 9$, so $\int_0^1 x\sqrt{5x^2+4}\,dx = \frac{1}{10}\int_4^9 u^{1/2}\,du = \frac{1}{15}u^{3/2}\Big|_4^9 = \frac{1}{15}(27) - \frac{1}{15}(8) = \frac{19}{15}$.

4. Let $u = 3x^2 - 2$, so $du = 6x\,dx$ and $x\,dx = \frac{1}{6}\,du$. If $x = 1$, then $u = 1$ and if $x = 3$, then $u = 25$, so $\int_1^3 x\sqrt{3x^2-2}\,dx = \frac{1}{6}\int_1^{25} u^{1/2}\,du = \frac{1}{9}u^{3/2}\Big|_1^{25} = \frac{1}{9}(125) - \frac{1}{9}(1) = \frac{124}{9}$.

5. Let $u = x^3 + 1$, so $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$. If $x = 0$, then $u = 1$ and if $x = 2$, then $u = 9$, so $\int_0^2 x^2\left(x^3+1\right)^{3/2} dx = \frac{1}{3}\int_1^9 u^{3/2}\,du = \frac{2}{15}u^{5/2}\Big|_1^9 = \frac{2}{15}(243) - \frac{2}{15}(1) = \frac{484}{15}$.

6. Let $u = 2x - 1$, so $du = 2\,dx$ and $dx = \frac{1}{2}\,du$. If $x = 1$, then $u = 1$ and if $x = 5$ then $u = 9$, so $\int_1^5 (2x-1)^{5/2}\,dx = \frac{1}{2}\int_1^9 u^{5/2}\,du = \frac{1}{7}u^{7/2}\Big|_1^9 = \frac{1}{7}(2187) - \frac{1}{7}(1) = \frac{2186}{7}$.

7. Let $u = 2x + 1$, so $du = 2\,dx$ and $dx = \frac{1}{2}\,du$. If $x = 0$, then $u = 1$ and if $x = 1$ then $u = 3$, so

$$\int_0^1 \frac{1}{\sqrt{2x+1}}\,dx = \frac{1}{2}\int_1^3 \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int_1^3 u^{-1/2}\,du = u^{1/2}\big|_1^3 = \sqrt{3} - 1.$$

8. Let $u = x^2 + 5$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. If $x = 0$, then $u = 5$ and if $x = 2$, then $u = 9$, so

$$\int_0^2 \frac{x}{\sqrt{x^2+5}}\,dx = \frac{1}{2}\int_5^9 \frac{du}{\sqrt{u}} = u^{1/2}\big|_5^9 = 3 - \sqrt{5}.$$

9. Let $u = 2x - 1$, so $du = 2\,dx$ and $dx = \frac{1}{2}\,du$. If $x = 1$, then $u = 1$ and if $x = 3$, then $u = 5$, so $\int_1^3 (2x-1)^4\,dx = \frac{1}{2}\int_1^5 u^4\,du = \frac{1}{10}u^5\Big|_1^5 = \frac{1}{10}(3125 - 1) = \frac{1562}{5}$.

10. Let $u = x^2 + 4x - 8$, so $du = (2x+4)\,dx$. If $x = 1$ then $u = -3$ and if $x = 2$, then $u = 4$, so $\int_1^2 (2x+4)\left(x^2+4x-8\right)^3 dx = \int_{-3}^4 u^3\,du = \frac{1}{4}u^4\Big|_{-3}^4 = \frac{1}{4}(256) - \frac{1}{4}(81) = \frac{175}{4}$.

11. Let $u = x^3 + 1$, so $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$. If $x = -1$, then $u = 0$ and if $x = 1$, then $u = 2$, so $\int_{-1}^1 x^2\left(x^3+1\right)^4 dx = \frac{1}{3}\int_0^2 u^4\,du = \frac{1}{15}u^5\Big|_0^2 = \frac{32}{15}$.

12. Let $u = x^4 + 3x$, so $du = \left(4x^3+3\right)dx = 4\left(x^3+\frac{3}{4}\right)dx$ and $dx = \frac{1}{4}\left(x^3+\frac{3}{4}\right)^{-1} du$. If $x = 1$, then $u = 4$ and if $x = 2$, then $u = 22$, so

$$\int_1^2 \left(x^3+\frac{3}{4}\right)\left(x^4+3x\right)^{-2} dx = \frac{1}{4}\int_4^{22} u^{-2}\,du = -\frac{1}{4u}\bigg|_4^{22} = -\frac{1}{88} + \frac{1}{16} = \frac{-2+11}{176} = \frac{9}{176}.$$

13. Let $u = x - 1$, so $du = dx$. If $x = 1$, then $u = 0$ and if $x = 5$, then $u = 4$, so $\int_1^5 x\sqrt{x-1}\,dx = \int_0^4 (u+1)\,u^{1/2}\,du = \int_0^4 \left(u^{3/2} + u^{1/2}\right) du = \left(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right)\Big|_0^4 = \frac{2}{5}(32) + \frac{2}{3}(8) = \frac{272}{15}$.

14. Let $u = x + 1$, so $x = u - 1$ and $du = dx$. If $x = 1$, then $u = 2$ and if $x = 4$, then $u = 5$, so

$$\int_1^4 x\sqrt{x+1}\,dx = \int_2^5 (u-1)\sqrt{u}\,du = \int_2^5 \left(u^{3/2} - u^{1/2}\right) du = \left(\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right)\Big|_2^5 = \left[\tfrac{2}{15}u^{3/2}(3u-5)\right]_2^5$$
$$= \tfrac{2}{15}\left(50\sqrt{5} - 2\sqrt{2}\right).$$

15. Let $u = x^2$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. If $x = 0$, then $u = 0$ and if $x = 2$, then $u = 4$, so $\int_0^2 2xe^{x^2}\,dx = \int_0^4 e^u\,du = e^u\big|_0^4 = e^4 - 1$.

16. Let $u = -x$, so $du = -dx$ and $dx = -du$. If $x = 0$, then $u = 0$ and if $x = 1$, then $u = -1$, so $\int_0^1 e^{-x}\,dx = -\int_0^{-1} e^u\,du = -e^u\big|_0^{-1} = -e^{-1} + 1 = 1 - \frac{1}{e}$.

17. $\int_0^1 \left(e^{2x} + x^2 + 1\right) dx = \left(\frac{1}{2}e^{2x} + \frac{1}{3}x^3 + x\right)\Big|_0^1 = \left(\frac{1}{2}e^2 + \frac{1}{3} + 1\right) - \frac{1}{2} = \frac{1}{2}e^2 + \frac{5}{6}$.

18. $\int_0^2 \left(e^t - e^{-t}\right) dt = \left(e^t + e^{-t}\right)\big|_0^2 = \left(e^2 + e^{-2}\right) - (1+1) = e^2 + e^{-2} - 2$.

19. Put $u = x^2 + 1$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. Then $\int_{-1}^{1} xe^{x^2+1}\,dx = \frac{1}{2}\int_{2}^{2} e^u\,du = \frac{1}{2}e^u\Big|_2^2 = 0$ because the upper and lower limits are equal.

20. Let $u = \sqrt{x}$, so $du = \dfrac{1}{2\sqrt{x}}\,dx$. If $x = 1$, then $u = 1$, and if $x = 4$, then $u = 2$, so

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int_1^2 e^u\,du = 2e^u\big|_1^2 = 2\left(e^2 - e\right) = 2e\,(e - 1).$$

21. Let $u = x - 2$, so $du = dx$. If $x = 3$, then $u = 1$ and if $x = 6$, then $u = 4$, so

$$\int_3^6 \frac{1}{x-2}\,dx = \int_1^4 \frac{du}{u} = \ln|u|\big|_1^4 = \ln 4.$$

22. Let $u = 1 + 2x^2$, so $du = 4x\,dx$ and $x\,dx = \frac{1}{4}\,du$. If $x = 0$, then $u = 1$ and if $x = 1$, then $u = 3$, so

$$\int_0^1 \frac{x}{1+2x^2}\,dx = \frac{1}{4}\int_1^3 \frac{du}{u} = \tfrac{1}{4}\ln|u|\Big|_1^3 = \tfrac{1}{4}\ln 3.$$

23. Let $u = x^3 + 3x^2 - 1$, so $du = \left(3x^2 + 6x\right)dx = 3\left(x^2 + 2x\right)dx$. If $x = 1$, then $u = 3$, and if $x = 2$, then $u = 19$, so

$$\int_1^2 \frac{x^2 + 2x}{x^3 + 3x^2 - 1}\,dx = \frac{1}{3}\int_3^{19} \frac{du}{u} = \tfrac{1}{3}\ln u\Big|_3^{19} = \tfrac{1}{3}\left(\ln 19 - \ln 3\right).$$

24. $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\,dx = \ln\left(1 + e^x\right)\big|_0^1 = \ln(1+e) - \ln 2 = \ln\frac{1+e}{2}.$

25. $\displaystyle\int_1^2 \left(4e^{2u} - \frac{1}{u}\right)du = 2e^{2u} - \ln u\Big|_1^2 = \left(2e^4 - \ln 2\right) - \left(2e^2 - 0\right) = 2e^4 - 2e^2 - \ln 2.$

26. $\int_1^2 \left(1 + \dfrac{1}{x} + e^x\right)dx = \left(x + \ln x + e^x\right)\big|_1^2 = \left(2 + \ln 2 + e^2\right) - (1 + e) = 1 + \ln 2 + e^2 - e.$

27. $\int_1^2 \left(2e^{-4x} - x^{-2}\right)dx = \left(-\frac{1}{2}e^{-4x} + \frac{1}{x}\right)\Big|_1^2 = \left(-\frac{1}{2}e^{-8} + \frac{1}{2}\right) - \left(-\frac{1}{2}e^{-4} + 1\right) = -\frac{1}{2}e^{-8} + \frac{1}{2}e^{-4} - \frac{1}{2}$

$= \frac{1}{2}\left(e^{-4} - e^{-8} - 1\right).$

28. Let $u = \ln x$, so $du = \frac{1}{x}\,dx$. If $x = 1$, then $u = 0$ and if $x = 2$, then $u = \ln 2$, so

$$\int_1^2 \frac{\ln x}{x}\,dx = \int_0^{\ln 2} u\,du = \tfrac{1}{2}u^2\Big|_0^{\ln 2} = \tfrac{1}{2}\left(\ln 2\right)^2.$$

29. $A = \int_{-1}^{2}\left(x^2 - 2x + 2\right)dx = \left(\frac{1}{3}x^3 - x^2 + 2x\right)\Big|_{-1}^{2} = \left(\frac{8}{3} - 4 + 4\right) - \left(-\frac{1}{3} - 1 - 2\right) = 6.$

30. $A = \int_0^1 \left(x^3 + x\right)dx = \left(\frac{1}{4}x^4 + \frac{1}{2}x^2\right)\Big|_0^1 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$

31. $\displaystyle A = \int_1^2 \frac{dx}{x^2} = \int_1^2 x^{-2}\,dx = -\frac{1}{x}\Big|_1^2 = \frac{1}{4} - (-1) = \frac{1}{2}.$

32. $A = \int_0^3 \left(2 + \sqrt{x+1}\right)dx = \int_0^3 2\,dx + \int_0^3 \sqrt{x+1}\,dx$. Let $u = x + 1$ in the second integral, so $du = dx$. If $x = 0$, then $u = 1$ and if $x = 3$, then $u = 4$. Thus, $A = 2x\big|_0^3 + \int_1^4 u^{1/2}\,du = 6 + \left(\frac{2}{3}u^{3/2}\right)\Big|_1^4 = 6 + \frac{2}{5}\left(8 - 1\right) = \frac{32}{3}.$

33. $A = \int_{-1}^{2} e^{-x/2}\,dx = -2e^{-x/2}\big|_{-1}^{2} = -2\left(e^{-1} - e^{1/2}\right) = 2\left(\sqrt{e} - 1/e\right)$.

34. The required area is $A = \displaystyle\int_{1}^{2} f(x)\,dx = \frac{1}{4}\int_{1}^{2} \frac{\ln x}{x}\,dx = \tfrac{1}{8}(\ln x)^2\Big|_{1}^{2} = \tfrac{1}{8}\left[(\ln 2)^2 - (\ln 1)^2\right] = \tfrac{1}{8}(\ln 2)^2$.

35. The average value is $\frac{1}{2}\int_{0}^{2}(2x+3)\,dx = \frac{1}{2}\left(x^2+3x\right)\Big|_{0}^{2} = \frac{1}{2}(10) = 5$.

36. The average value is

$$\tfrac{1}{b-a}\int_{a}^{b} f(x)\,dx = \tfrac{1}{4-1}\int_{1}^{4}(8-x)dx = \tfrac{1}{3}\int_{1}^{4}(8-x)\,dx = \tfrac{1}{3}\left(8x - \tfrac{1}{2}x^2\right)\Big|_{1}^{4} = \tfrac{1}{3}\left[(32-8) - \left(8 - \tfrac{1}{2}\right)\right] = \tfrac{11}{2}.$$

37. The average value is $\frac{1}{2}\int_{1}^{3}\left(2x^2-3\right)dx = \frac{1}{2}\left(\frac{2}{3}x^3 - 3x\right)\Big|_{1}^{3} = \frac{1}{2}\left(9 + \frac{7}{3}\right) = \frac{17}{3}$.

38. The average value is $\frac{1}{5}\int_{-2}^{3}\left(4-x^2\right)dx = \frac{1}{5}\left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^{3} = \frac{1}{5}\left[(12-9) - \left(-8 + \frac{8}{3}\right)\right] = \frac{5}{3}$.

39. The average value is

$$\begin{aligned}\tfrac{1}{3}\int_{-1}^{2}\left(x^2+2x-3\right)dx &= \tfrac{1}{3}\left(\tfrac{1}{3}x^3 + x^2 - 3x\right)\Big|_{-1}^{2} = \tfrac{1}{3}\left[\left(\tfrac{8}{3}+4-6\right) - \left(-\tfrac{1}{3}+1+3\right)\right]\\ &= \tfrac{1}{3}\left(\tfrac{8}{3} - 2 + \tfrac{1}{3} - 4\right) = -1.\end{aligned}$$

40. The average value is $\frac{1}{2}\int_{-1}^{1} x^3\,dx = \frac{1}{2}\cdot\frac{1}{4}x^4\Big|_{-1}^{1} = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{4}\right) = 0$.

41. The average value is $\frac{1}{4}\int_{0}^{4}(2x+1)^{1/2}\,dx = \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)(2x+1)^{3/2}\Big|_{0}^{4} = \frac{1}{12}(27-1) = \frac{13}{6}$.

42. The average value is $\frac{1}{4-0}\int_{0}^{4} e^{-x}\,dx = -\frac{1}{4}e^{-x}\Big|_{0}^{4} = -\frac{1}{4}\left(e^{-4} - 1\right) = \frac{1}{4}\left(1 - e^{-4}\right) \approx 0.245$.

43. The average value is $\frac{1}{2}\int_{0}^{2} xe^{x^2}\,dx = \dfrac{1}{4}e^{x^2}\Big|_{0}^{2} = \frac{1}{4}\left(e^4 - 1\right)$.

44. The average value is $\dfrac{1}{2}\displaystyle\int_{0}^{2}\frac{dx}{x+1} = \tfrac{1}{2}\ln(x+1)\Big|_{0}^{2} = \tfrac{1}{2}\ln 3$.

45. The distance traveled is $\int_{0}^{4} 3t\sqrt{16-t^2}\,dt = 3\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(16-t^2\right)^{3/2}\Big|_{0}^{4} = 64$ ft.

46. The amount of oil that the well can be expected to yield is

$$\begin{aligned}\int_{0}^{5}\left(\frac{600t^2}{t^3+32} + 5\right)dt &= 600\int_{0}^{5}\frac{t^2}{t^3+32}\,dt + (5t)|_{0}^{5} = 600\left(\tfrac{1}{3}\right)\ln\left(t^3+32\right)\Big|_{0}^{5} + 25\\ &= 200\,(\ln 157 - \ln 32) + 25 \approx 343, \text{ or 343 thousand barrels.}\end{aligned}$$

47. The amount is $\int_{1}^{2} t\left(\frac{1}{2}t^2 + 1\right)^{1/2} dt$. Let $u = \frac{1}{2}t^2 + 1$, so $du = t\,dt$. Then

$$\int_{1}^{2} t\left(\tfrac{1}{2}t^2+1\right)^{1/2} dt = \int_{3/2}^{3} u^{1/2}\,du = \tfrac{2}{3}u^{3/2}\Big|_{3/2}^{3} = \tfrac{2}{3}\left[(3)^{3/2} - \left(\tfrac{3}{2}\right)^{3/2}\right] \approx \$2.24 \text{ million.}$$

48. Using the substitution $u = 0.05t$, we find that by 7 p.m., the temperature will have dropped $\int_0^3 \left(-18e^{-0.6t}\right) dt = \frac{-18}{-0.6}e^{-0.6t}\Big|_0^3 = 30\left(e^{-1.8} - 1\right) = 25.04$, or approximately 25 degrees. $f(t) = 30e^{-0.6t} + C$ and $f(0) = 30 + C = 68$, so $C = 38$. The temperature of the wine at 7 p.m. is $f(3) = 30e^{-1.8} + 38 \approx 42.96$, or approximately 43°F.

49. Using the substitution $u = 0.05t$, we find that the amount produced was $\int_0^{20} 3.5e^{0.05t}\,dt = \frac{3.5}{0.05}e^u\Big|_0^{20} = 70(e - 1) \approx 120.3$, or 120.3 billion metric tons.

50. The tractor depreciates by

$$\int_0^5 13{,}388.61e^{-0.22314t}\,dt = \frac{13{,}388.61}{-0.22314}e^{-0.22314t}\Big|_0^5 = -60{,}000.94e^{-0.22314t}\Big|_0^5$$
$$= -60{,}000.94(-0.672314) = 40{,}339.47, \text{ or } \$40{,}339.$$

51. The average spending per year between 2005 and 2011 is
$$A = \frac{1}{7-1}\int_1^7 0.86t^{0.96}\,dt = \frac{0.86}{6}\cdot\frac{1}{1.96}t^{1.96}\Big|_1^7 = \frac{0.86}{6(1.96)}\left(7^{1.96} - 1\right) \approx 3.24, \text{ or \$3.24 billion per year.}$$

52. The average rate of increase of the average amount of carbon dioxide in the atmosphere between 1958 and 2016 is given by $A = \frac{1}{59-1}\int_1^{59}\left(0.012414t^2 + 0.7485t + 313.9\right) dt = \frac{1}{58}\left(0.004138t^3 + 0.37425t^2 + 313.9t\right)\Big|_1^{59} \approx 351.01$, or 351.01 ppmv/yr.

53. a. The gasoline consumption in 2017 is given by $A(10) = 0.014\left(10^2\right) + 1.93(10) + 140 = 160.7$, or 160.7 billion gallons per year.

b. The average consumption per year between 2007 and 2017 is given by $A = \frac{1}{10-0}\int_0^{10}\left(0.014t^2 + 1.93t + 140\right) dt = \frac{1}{10}\left(\frac{0.014}{3}t^3 + \frac{1.93}{2}t^2 + 140t\right)\Big|_0^{10} \approx 150.12$, or 150.1 billion gallons per year per year.

54. The average rate of growth between 2000 ($t = 10$) and 2050 ($t = 15$) is

$$\frac{1}{15-10}\int_{10}^{15} R(t)\,dt = \frac{1}{5}\int_{10}^{15}\left(0.063t^2 - 0.48t + 3.87\right) dt = \frac{1}{5}\left(0.021t^3 - 0.24t^2 + 3.87t\right)\Big|_{10}^{15}$$
$$= \frac{1}{5}\left\{\left[(0.021)\left(15^3\right) - 0.24\left(15^2\right) + 3.87(15)\right] - \left[(0.021)\left(10^3\right) - 0.24\left(10^2\right) + 3.87(10)\right]\right\}$$
$$= 7.845, \text{ or } 7.845 \text{ million people/decade.}$$

The average rate between 1950 ($t = 5$) and 2000 ($t = 10$) is

$$\frac{1}{10-5}\int_5^{10} R(t)\,dt = \frac{1}{5}\int_5^{10}\left(0.063t^2 - 0.48t + 3.87\right) dt = \frac{1}{5}\left(0.021t^3 - 0.24t^2 + 3.87t\right)\Big|_5^{10}$$
$$= \frac{1}{5}\left\{\left[(0.021)\left(10^3\right) - 0.24\left(10^2\right) + 3.87(10)\right] - \left[(0.021)\left(5^3\right) - 0.24\left(5^2\right) + 3.87(5)\right]\right\}$$
$$= 3.945, \text{ or } 3.945 \text{ million people/decade.}$$

The conclusion follows.

55. $I = \int_0^{0.02}\left[-9{,}400{,}000\left(t^2 - 0.02t\right)\right] dt = -9{,}400{,}000\left[\frac{1}{3}t^3 - 0.01t^2\right]_0^{0.02} \approx 12.5333$, or approximately 12.53 newton-seconds. The average force acting on the baseball is $F = \frac{1}{0.02-0}\int_0^{0.02} F(t)\,dt \approx \frac{12.53}{0.02} = 626.5$, or 626.5 N.

56. The approximate average number of commercial vehicle registrations per year in the period from 2010 through 2015 is projected to be

$$\frac{1}{5-0}\int_0^5 N(t)\,dt = \frac{1}{5}\int_0^5 \left(1.3926t^3 - 9.2873t^2 + 74.719t + 228.3\right)dt$$
$$= \frac{1}{5}\left[0.34815t^4 - 3.09577t^3 + 37.3595t^2 + 228.3\right]_0^5 \approx 381.22,$$

or approximately 381,000.

57. The average velocity of the blood is

$$\frac{1}{R}\int_0^R k\left(R^2 - r^2\right)dr = \frac{k}{R}\int_0^R \left(R^2 - r^2\right)dr = \frac{k}{r}\left(R^2r - \frac{1}{3}r^3\right)\Big|_0^4 = \frac{k}{R}\left(R^3 - \frac{1}{3}R^3\right) = \frac{k}{R}\cdot\frac{2}{3}R^3 = \frac{2}{3}kR^2 \text{ cm/sec.}$$

58. The projected average number of total knee replacement procedures from 1991 through 2030 is

$$\frac{1}{40}\int_0^{40}\left(0.0000846t^3 - 0.002116t^2 + 0.03897t + 0.16\right)dt = \frac{1}{40}\left(\frac{0.0000846}{4}t^4 - \frac{0.002116}{3}t^3 + \frac{0.03897}{2}t^2 + 0.16t\right)\Big|_0^{40}$$
$$\approx 1.16 \text{ (million/year).}$$

59. The average concentration of the drug is

$$\frac{1}{4}\int_0^4 \frac{0.2t}{t^2+1}\,dt = \frac{0.2}{4}\int_0^4 \frac{t}{t^2+1}\,dt = \frac{0.2}{(4)(2)}\ln\left(t^2+1\right)\Big|_0^4 = 0.025\ln 17 \approx 0.071,$$

or 0.071 milligrams per cm^3.

60. The average percentage of seatbelt use from 2001 through 2009 was

$$A = \frac{1}{9-0}\int_0^9 72.9\,(t+1)^{0.057}\,dt = \frac{72.9}{9}\cdot\frac{1}{1.057}\,(t+1)^{1.057}\Big|_0^9 \approx 79.7, \text{ or approximately } 79.7\%.$$

61. The average yearly sales of the company over its first 5 years of operation is given by

$$\frac{1}{5-0}\int_0^5 t\left(0.2t^2+4\right)^{1/2}dt = \frac{1}{5}\left[\left(\frac{5}{2}\right)\left(\frac{2}{3}\right)\left(0.2t^2+4\right)^{3/2}\right]_0^5 \quad (\text{let } u = -0.2t^2 + 4)$$
$$= \frac{1}{5}\left[\frac{5}{3}(5+4)^{3/2} - \frac{5}{3}(4)^{3/2}\right] = \frac{1}{3}(27-8) = \frac{19}{3}, \text{ or about \$6.33 million.}$$

62. The average value is

$$A = \frac{1}{5}\int_0^5\left(-\frac{40{,}000}{\sqrt{1+0.2t}} + 50{,}000\right)dt = -8000\int_0^5 (1+0.2t)^{-1/2}\,dt + 10{,}000\int_0^5 dt.$$ We use the substitution $u = 1 + 0.2t$ for the first integral, so $du = 0.2\,dt$ and $dt = 5\,du$. Thus,

$$A = -8000\int_1^2 5u^{-1/2}\,du + \int_0^5 10{,}000\,dt = -40{,}000\left(2u^{1/2}\right)\Big|_1^2 + 10{,}000t\Big|_0^5$$
$$= -40{,}000\left(2\sqrt{2}-2\right) + 50{,}000 \approx 16{,}863, \text{ or 16,863 subscribers.}$$

63. The average increase in the number of credit cards issued in China between 2009 and 2012 was

$$\frac{1}{4-1}\int_1^4 153e^{0.21t}\,dt = \frac{1}{3}\left[\frac{153}{0.21}e^{0.21t}\right]_1^4 \approx 262.9, \text{ or approximately 262.9 million cards per year.}$$

64. The average number of e-commerce sales in the U.S. from 2011 through 2014 is

$$\frac{1}{4-1}\int_1^4 157.6e^{0.09t}\,dt = \frac{1}{3}\left[\frac{157.6}{0.09}e^{0.09t}\right]_1^4 \approx 197.97, \text{ or approximately \$198 billion per year.}$$

65. a. The projected number of passengers in year t is $N(t) = \int R(t)\,dt = \int 6.69e^{0.0456t}\,dt \approx 146.711e^{0.0456t} + C$. To find C, we use the condition $N(2) = 160$, giving $C \approx -0.720$. Therefore, $N(t) \approx 146.711e^{0.0456t} - 0.72$.

b. The number of passengers in 2030 is projected to be $N(20) \approx 146.711e^{0.0456(20)} - 0.72 \approx 364.49$, or approximately 364.5 million.

c. The average growth rate from 2012 to 2030 is approximately $\dfrac{1}{20-2}\displaystyle\int_2^{20} R(t)\,dt \approx \dfrac{364.49-160}{18} \approx 11.36$, or approximately 11.4 million per year.

66. $\frac{1}{5}\int_0^5 p\,dt = \frac{1}{5}\int_0^5 \left(18 - 3e^{-2t} - 6e^{-t/3}\right) dt = \frac{1}{5}\left(18t + \frac{3}{2}e^{-2t} + 18e^{-t/3}\right)\Big|_0^5$

$= \frac{1}{5}\left[18(5) + \frac{3}{2}e^{-10} + 18e^{-5/3} - \frac{3}{2} - 18\right] = 14.78$, or \$14.78.

67. $\frac{1}{h}\int_0^h (2gx)^{1/2}\,dx = \frac{1}{3h}(2gx)^{3/2}\Big|_0^h = \frac{2}{3}\sqrt{2gh}$; that is, $\frac{2}{3}\sqrt{2gh}$ ft/sec.

68. The average content of oxygen in the pond over the first 10 days is

$$A = \frac{1}{10-0}\int_0^{10} 100\left(\frac{t^2+10t+100}{t^2+20t+100}\right) dt = \frac{100}{10}\int_0^{10}\left[1 - \frac{10}{t+10} + \frac{100}{(t+10)^2}\right] dt$$

$= 10\int_0^{10}\left[1 - 10(t+10)^{-1} + 100(t+10)^{-2}\right] dt.$

Using the substitution $u = t + 10$ for the third integral, we have

$$A = 10\left[t - 10\ln(t+10) - \tfrac{100}{t+10}\right]\Big|_0^{10} = 10\left\{\left[10 - 10\ln 20 - \tfrac{100}{2p}\right] - [-10\ln 10 - 10]\right\}$$

$= 10(10 - 10\ln 20 - 5 + 10\ln 10 + 10) \approx 80.6853$, or approximately 80.7%.

69. $\int_a^a f(x)\,dx = F(x)|_a^a = F(a) - F(a) = 0$, where $F'(x) = f(x)$.

70. $\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) = -[F(a) - F(b)] = -F(x)|_b^a = -\int_b^a f(x)\,dx.$

71. $\int_1^3 x^2\,dx = \frac{1}{3}x^3\Big|_1^3 = 9 - \frac{1}{3} = \frac{26}{3} = -\int_3^1 x^2\,dx = -\frac{1}{3}x^3\Big|_3^1 = -\frac{1}{3} + 9 = \frac{26}{3}.$

72. $\int_a^b cf(x)\,dx = xF(x)|_a^b = c[F(b) - F(a)] = c\int_a^b f(x)\,dx.$

73. $\int_1^9 2\sqrt{x}\,dx = \frac{4}{3}x^{3/2}\Big|_1^9 = \frac{4}{3}(27-1) = \frac{104}{3}$ and $2\int_1^9 \sqrt{x}\,dx = 2\left(\frac{2}{3}x^{3/2}\right)\Big|_1^9 = \frac{104}{3}.$

74. $\int_0^1 (1 + x - e^x)\,dx = \left(x + \frac{1}{2}x^2 - e^x\right)\Big|_0^1 = \left(1 + \frac{1}{2} - e\right) + 1 = \frac{5}{2} - e$ and

$\int_0^1 dx + \int_0^1 x\,dx - \int_0^1 e^x\,dx = x|_0^1 + \left(\frac{1}{2}x^2\right)\Big|_0^1 - (e^x)|_0^1 = (1-0) + \left(\frac{1}{2} - 0\right) - (e-1) = \frac{5}{2} - e$, demonstrating Property 4.

75. $\int_0^3 \left(1 + x^3\right) dx = x + \frac{1}{4}x^4\Big|_0^3 = 3 + \frac{81}{4} = \frac{93}{4}$ and

$\int_0^1 \left(1 + x^3\right) dx + \int_1^3 \left(1 + x^3\right) dx = \left(x + \frac{1}{4}x^4\right)\Big|_0^1 + \left(x + \frac{1}{4}x^4\right)\Big|_1^3 = \left(1 + \frac{1}{4}\right) + \left(3 + \frac{81}{4}\right) - \left(1 + \frac{1}{4}\right) = \frac{93}{4}$, demonstrating Property 5.

76. $\int_0^3 \left(1+x^3\right) dx = \left(x+\frac{1}{4}x^4\right)\Big|_0^3 = 3+\frac{81}{4} = \frac{93}{4}$ and

$$\int_0^1 \left(1+x^3\right) dx + \int_1^2 \left(1+x^3\right) dx + \int_2^3 \left(1+x^3\right) dx = \left(x+\frac{1}{4}x^4\right)\Big|_0^1 + \left(x+\frac{1}{4}x^4\right)\Big|_1^2 + \left(x+\frac{1}{4}x^4\right)\Big|_2^3$$
$$= \left(1+\frac{1}{4}\right) + (2+4) - \left(1+\frac{1}{4}\right) + \left(3+\frac{81}{4}\right) - (2+4) = \frac{93}{4}.$$

77. $\int_3^3 \left(1+\sqrt{x}\right) e^{-x}\, dx = 0$ by Property 1 of the definite integral.

78. $\int_3^0 f(x)\, dx = -\int_0^3 f(x)\, dx = -4$ by Property 2 of the definite integral.

79. **a.** $\int_{-1}^2 \left[2f(x)+g(x)\right] dx = 2\int_{-1}^2 f(x)\, dx + \int_{-1}^2 g(x)\, dx = 2(-2)+3 = -1.$

b. $\int_{-1}^2 \left[g(x)-f(x)\right] dx = \int_{-1}^2 g(x)\, dx - \int_{-1}^2 f(x)\, dx = 3-(-2) = 5.$

c. $\int_{-1}^2 \left[2f(x)-3g(x)\right] dx = 2\int_{-1}^2 f(x)\, dx - 3\int_{-1}^2 g(x)\, dx = 2(-2)-3(3) = -13.$

80. **a.** $\int_{-1}^0 f(x)\, dx = \int_{-1}^2 f(x) - \int_0^2 f(x)\, dx = 2-3 = -1.$

b. $\int_0^2 f(x)\, dx - \int_{-1}^0 f(x)\, dx = 3-(-1) = 4.$

81. True. This follows from Property 1 of the definite integral.

82. False. The integrand $f(x) = \dfrac{1}{x-2}$ is not defined at $x = 2$.

83. False. Only a constant can be "moved out" of the integral.

84. True. This follows from the fundamental theorem of calculus.

85. True. This follows from Properties 3 and 4 of the definite integral.

86. True. We have $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$, and so $\int_c^b f(x)\, dx = \int_a^b f(x)\, dx - \int_a^c f(x)\, dx$, $-\int_c^b f(x)\, dx = \int_c^b f(x)\, dx = \int_a^b f(x)\, dx - \int_a^c f(x)\, dx$, and $\int_c^b f(x)\, dx = \int_a^c f(x)\, dx - \int_a^b f(x)\, dx$.

Using Technology page 468

1. 7.716667

2. 1.153100

3. 17.564865

4. −14.333333

5. 159/bean stem

6. 60.5 mg/day

7. 0.48 g/cm^3/day

6.6 Area between Two Curves

Concept Questions page 475

1. $\int_a^b \left[f(x)-g(x)\right] dx$

2. $\int_a^b \left[f(x)-g(x)\right] dx + \int_b^c \left[g(x)-f(x)\right] dx + \int_c^d \left[f(x)-g(x)\right] dx$

Exercises page 475

1. $-\int_0^6 \left(x^3-6x^2\right) dx = \left(-\frac{1}{4}x^4+2x^3\right)\Big|_0^6 = -\frac{1}{4}(6)^4 + 2(6)^3 = 108.$

2. $-\int_0^2 \left(x^4 - 2x^3\right) dx = \left(\frac{1}{2}x^4 - \frac{1}{5}x^5\right)\Big|_0^2 = 8 - \frac{32}{5} = \frac{8}{5}.$

3. $A = -\int_{-1}^0 x\sqrt{1-x^2}\,dx + \int_0^1 x\sqrt{1-x^2}dx = 2\int_0^1 x\left(1-x^2\right)^{1/2} dx$ by symmetry. Let $u = 1 - x^2$, so $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}\,du$. If $x = 0$, then $u = 1$ and if $x = 1$, then $u = 0$, so $A = (2)\left(-\frac{1}{2}\right)\int_0^1 u^{1/2}\,du = -\frac{2}{3}u^{3/2}\Big|_1^0 = \frac{2}{3}.$

4. $A = -\int_{-2}^{0} \frac{2x}{x^2+4}\,dx + \int_0^2 \frac{2x}{x^2+4}\,dx = 2\int_0^2 \frac{2x}{x^2+4}\,dx = 2\ln\left(x^2+4\right)\big|_0^2 = (\ln 8 - \ln 4)\,2 = \ln 4.$

5. $A = -\int_0^4 \left(x - 2\sqrt{x}\right) dx = \int_0^4 \left(-x + 2x^{1/2}\right) dx = \left(-\frac{1}{2}x^2 + \frac{4}{3}x^{3/2}\right)\Big|_0^4 = 8 + \frac{32}{3} = \frac{8}{3}.$

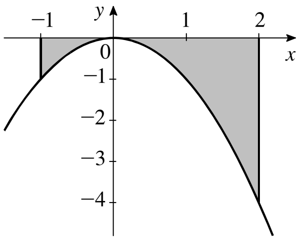
6. $A = \int_0^4 \left[\sqrt{x} - (x-2)\right] dx = \int_0^4 \left(x^{1/2} - x + 2\right) dx = \left(\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right)\Big|_0^4 = \frac{16}{3} - 8 + 8 = \frac{16}{3}.$

7. The required area is given by

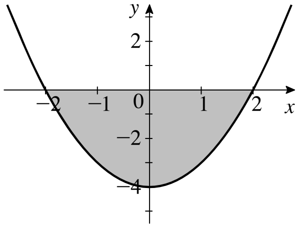
$\int_{-1}^0 \left(x^2 - x^{1/3}\right) dx + \int_0^1 \left(x^{1/3} - x^2\right) dx = \left(\frac{1}{3}x^3 - \frac{3}{4}x^{4/3}\right)\Big|_{-1}^0 + \left(\frac{3}{4}x^{4/3} - \frac{1}{3}x^3\right)\Big|_0^1 = -\left(-\frac{1}{3} - \frac{3}{4}\right) + \left(\frac{3}{4} - \frac{1}{3}\right) = \frac{3}{2}.$

8. $A = \int_{-4}^0 \left[(x+6) - \left(-\frac{1}{2}x\right)\right] dx + \int_0^2 \left[(x+6) - x^3\right] dx = \int_{-4}^0 \left(\frac{3}{2}x + 6\right) dx + \int_0^2 \left[(x+6) - x^3\right] dx$
$= \left(\frac{3}{4}x^2 + 6x\right)\Big|_{-4}^0 + \left(\frac{1}{2}x^2 + 6x - \frac{1}{4}x^4\right)\Big|_0^2 = -(12 - 24) + (2 + 12 - 4) = 22.$

9. The required area is given by $-\int_{-1}^2 -x^2 dx = \frac{1}{3}x^3\Big|_{-1}^2 = \frac{8}{3} + \frac{1}{3} = 3.$

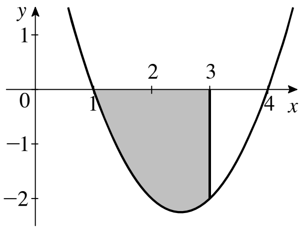


10. $A = -\int_{-2}^2 \left(x^2 - 4\right) dx = -2\int_0^2 \left(x^2 - 4\right) dx$
$= 2\left(-\frac{1}{3}x^3 + 4x\right)\Big|_0^2 = 2\left(-\frac{8}{3} + 8\right) = \frac{32}{3}.$



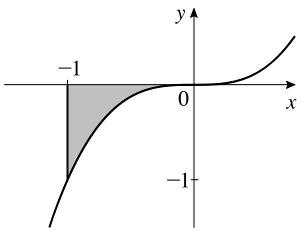
11. $y = x^2 - 5x + 4 = (x-4)(x-1) = 0$ if $x = 1$ or 4, the x-intercepts of the graph of f. Thus,

$A = -\int_1^3 \left(x^2 - 5x + 4\right) dx = \left(-\frac{1}{3}x^3 + \frac{5}{2}x^2 - 4x\right)\Big|_1^3$
$= \left(-9 + \frac{45}{2} - 12\right) - \left(-\frac{1}{3} + \frac{5}{2} - 4\right) = \frac{10}{3}.$



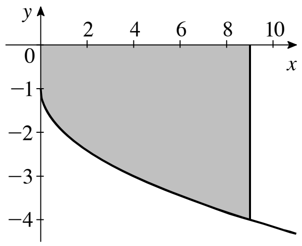
12. The required area is given by

$$-\int_{-1}^{0} x^3\,dx = -\tfrac{1}{4}x^4\Big|_{-1}^{0} = -\tfrac{1}{4}(0) + \tfrac{1}{4}(1) = \tfrac{1}{4}.$$



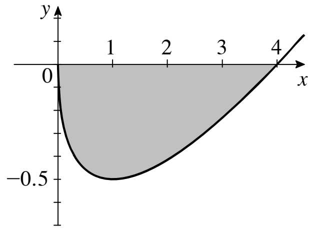
13. The required area is given by

$$-\int_0^9 -\left(1+\sqrt{x}\right)dx = \left(x + \tfrac{2}{3}x^{3/2}\right)\Big|_0^9 = 9 + 18 = 27.$$

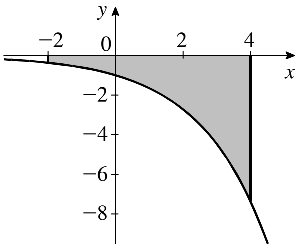


14. $A = -\int_0^4 \left(\tfrac{1}{2}x - x^{1/2}\right)dx = \left(-\tfrac{1}{4}x^2 + \tfrac{2}{3}x^{3/2}\right)\Big|_0^4$

$= \left(-4 + \tfrac{16}{3}\right) = \tfrac{4}{3}.$

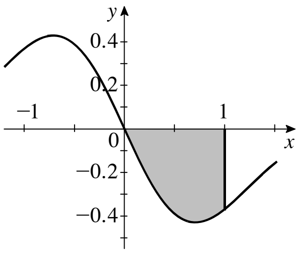


15. $-\int_{-2}^{4} \left(-e^{x/2}\right)dx = 2e^{x/2}\Big|_{-2}^{4} = 2\left(e^2 - e^{-1}\right).$



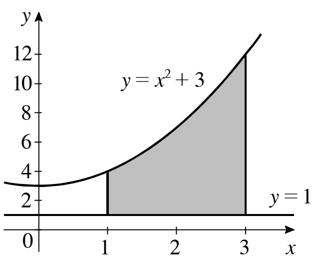
16. $A = -\int_0^1 \left(-xe^{-x^2}\right)dx = \int_0^1 xe^{-x^2}\,dx$. Let $u = -x^2$, so $du = -2x\,dx$ and $x\,dx = -\tfrac{1}{2}\,du$. If $x = 0$, then $u = 0$ and if $x = 1$ then $u = -1$, so

$$A = -\tfrac{1}{2}\int_0^{-1} e^u\,du = -\tfrac{1}{2}e^u\Big|_0^{-1} = -\tfrac{1}{2}e^{-1} + \tfrac{1}{2} = \tfrac{1}{2}\left(1 - e^{-1}\right).$$



17. $A = \int_1^3 \left[\left(x^2+3\right) - 1\right]dx = \int_1^3 \left(x^2+2\right)dx = \left(\tfrac{1}{3}x^3 + 2x\right)\Big|_1^3$

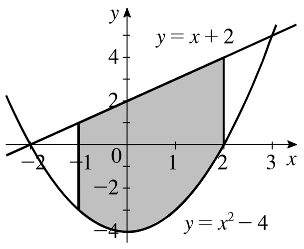
$= (9+6) - \left(\tfrac{1}{3} + 2\right) = \tfrac{38}{3}.$



18. $A = \int_{-1}^{2} \left[(x+2) - \left(x^2-4\right)\right] dx = \int_{-1}^{2} \left(-x^2+x+6\right) dx$

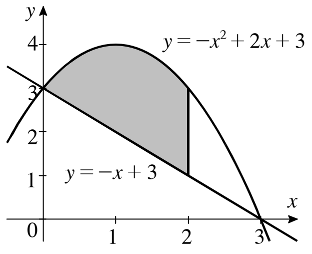
$= \left(-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x\right)\Big|_{-1}^{2}$

$= \left(-\frac{8}{3} + 2 + 12\right) - \left(\frac{1}{3} + \frac{1}{2} - 6\right) = \frac{33}{2}.$



19. $A = \int_{0}^{2} \left(-x^2+2x+3+x-3\right) dx = \int_{0}^{2} \left(-x^2+3x\right) dx$

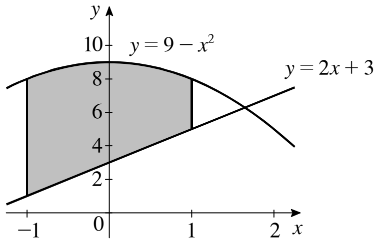
$= \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2\right)\Big|_{0}^{2} = -\frac{1}{3}\,(8) + \frac{3}{2}\,(4) = 6 - \frac{8}{3} = \frac{10}{3}.$



20. $A = \int_{-1}^{1} \left[\left(9-x^2\right) - (2x+3)\right] dx = \int_{-1}^{1} \left(-x^2-2x+6\right) dx$

$= \left(-\frac{1}{3}x^3 - x^2 + 6x\right)\Big|_{-1}^{1} = \left(-\frac{1}{3} - 1 + 6\right) - \left(\frac{1}{3} - 1 - 6\right)$

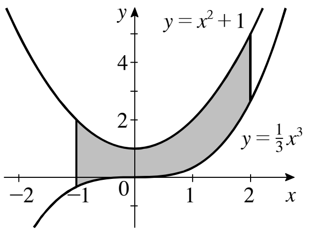
$= \frac{34}{3}.$



21. $A = \int_{-1}^{2} \left[\left(x^2+1\right) - \frac{1}{3}x^3\right] dx = \int_{-1}^{2} \left(-\frac{1}{3}x^3 + x^2 + 1\right) dx$

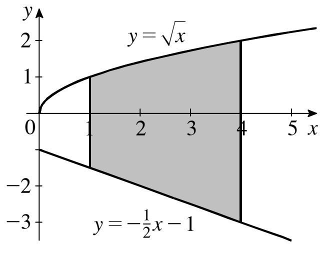
$= \left(-\frac{1}{12}x^4 + \frac{1}{3}x^3 + x\right)\Big|_{-1}^{2}$

$= \left(-\frac{4}{3} + \frac{8}{3} + 2\right) - \left(-\frac{1}{12} - \frac{1}{3} - 1\right) = \frac{19}{4}.$



22. $A = \int_{1}^{4} \left(x^{1/2} + \frac{1}{2}x + 1\right) dx = \left(\frac{2}{3}x^{3/2} + \frac{1}{4}x^2 + x\right)\Big|_{1}^{4}$

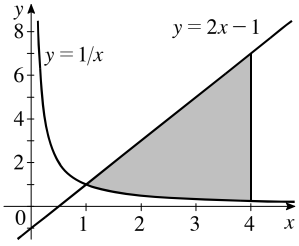
$= \left(\frac{16}{3} + 4 + 4\right) - \left(\frac{2}{3} + \frac{1}{4} + 1\right) = \frac{137}{12}.$



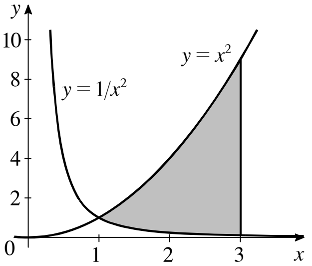
23. $A = \int_{1}^{4} \left[(2x-1) - \frac{1}{x}\right] dx = \int_{1}^{4} \left(2x - 1 - \frac{1}{x}\right) dx$

$= \left(x^2 - x - \ln x\right)\Big|_{1}^{4} = (16 - 4 - \ln 4) - (1 - 1 - \ln 1)$

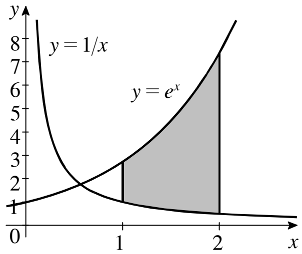
$= 12 - \ln 4 \approx 10.6.$



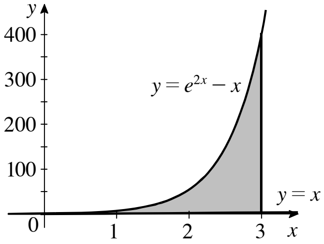
24. $A = \int_1^3 \left(x^2 - \frac{1}{x^2}\right) dx = \int_1^3 (x^2 - x^{-2}) dx = \left(\frac{x^3}{3} + \frac{1}{x}\right)\Big|_1^3$
$= \left(9 + \frac{1}{3}\right) - \left(\frac{1}{3} + 1\right) = 8.$



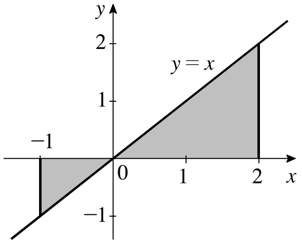
25. $A = \int_1^2 \left(e^x - \frac{1}{x}\right) dx = (e^x - \ln x)|_1^2 = \left(e^2 - \ln 2\right) - e = e^2 - e - \ln 2.$



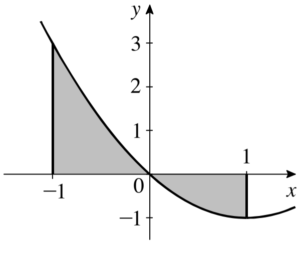
26. $A = \int_1^3 \left(e^{2x} - x\right) dx = \left(\frac{1}{2}e^{2x} - \frac{1}{2}x^2\right)\Big|_1^3$
$= \left(\frac{1}{2}e^6 - \frac{9}{2}\right) - \left(\frac{1}{2}e^2 - \frac{1}{2}\right) = \frac{1}{2}\left(e^6 - e^2 - 8\right) \approx 194.$



27. $A = -\int_{-1}^0 x\,dx + \int_0^2 x\,dx = -\frac{1}{2}x^2\Big|_{-1}^0 + \frac{1}{2}x^2\Big|_0^2 = \frac{1}{2} + 2 = \frac{5}{2}.$

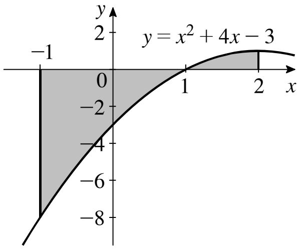


28. $A = \int_{-1}^0 \left(x^2 - 2x\right) dx - \int_0^1 \left(x^2 - 2x\right) dx$
$= \left(\frac{1}{3}x^3 - x^2\right)\Big|_{-1}^0 - \left(\frac{1}{3}x^3 - x^2\right)\Big|_0^1$
$= -\left(-\frac{1}{3} - 1\right) - \left(\frac{1}{3} - 1\right) = 2.$

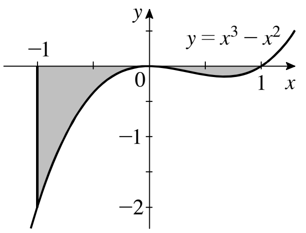


29. The x-intercepts are found by solving $x^2 - 4x + 3 = (x-3)(x-1) = 0$, giving $x = 1$ or 3. Thus,

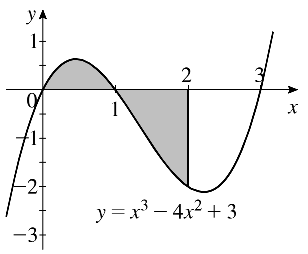
$$\begin{aligned} A &= -\int_{-1}^{1}\left(-x^2+4x-3\right)dx + \int_{1}^{2}\left(-x^2+4x-3\right)dx \\ &= \left(\tfrac{1}{3}x^3-2x^2+3x\right)\Big|_{-1}^{1} + \left(-\tfrac{1}{3}x^3+2x^2-3x\right)\Big|_{1}^{2} \\ &= \left(\tfrac{1}{3}-2+3\right)-\left(-\tfrac{1}{3}-2-3\right) \\ &\qquad + \left(-\tfrac{8}{3}+8-6\right)-\left(-\tfrac{1}{3}+2-3\right) \\ &= \tfrac{22}{3}. \end{aligned}$$



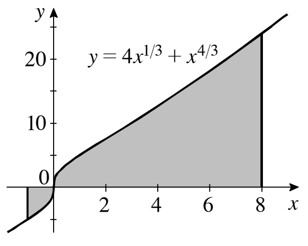
30.
$$\begin{aligned} A &= -\int_{-1}^{1}\left(x^3-x^2\right)dx = \left(-\tfrac{1}{4}x^4+\tfrac{1}{3}x^3\right)\Big|_{-1}^{1} \\ &= \left(-\tfrac{1}{4}+\tfrac{1}{3}\right)-\left(-\tfrac{1}{4}-\tfrac{1}{3}\right)=\tfrac{2}{3}. \end{aligned}$$



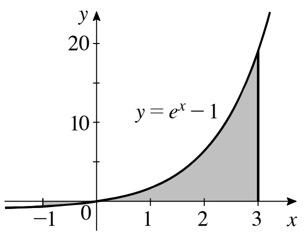
31.
$$\begin{aligned} A &= \int_{0}^{1}\left(x^3-4x^2+3x\right)dx - \int_{1}^{2}\left(x^3-4x^2+3x\right)dx \\ &= \left(\tfrac{1}{4}x^4-\tfrac{4}{3}x^3+\tfrac{3}{2}x^2\right)\Big|_{0}^{1} - \left(\tfrac{1}{4}x^4-\tfrac{4}{3}x^3+\tfrac{3}{2}x^2\right)\Big|_{1}^{2} = \tfrac{3}{2}. \end{aligned}$$



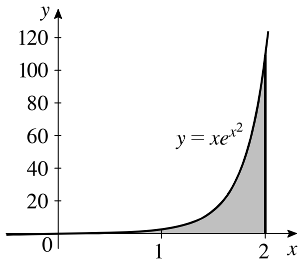
32.
$$\begin{aligned} A &= \int_{-1}^{0}\left(4x^{1/3}+x^{4/3}\right)dx + \int_{0}^{8}\left(4x^{1/3}+x^{4/3}\right)dx \\ &= \left(-3x^{4/3}-\tfrac{3}{7}x^{7/3}\right)\Big|_{-1}^{0} + \left(3x^{4/3}+\tfrac{3}{7}x^{7/3}\right)\Big|_{0}^{8} \\ &= \tfrac{18}{7}+\tfrac{720}{7}=\tfrac{738}{7}. \end{aligned}$$



33.
$$\begin{aligned} A &= -\int_{-1}^{0}\left(e^x-1\right)dx + \int_{0}^{3}\left(e^x-1\right)dx \\ &= \left(-e^x+x\right)\big|_{-1}^{0} + \left(e^x-x\right)\big|_{0}^{3} \\ &= -1-\left(-e^{-1}-1\right)+\left(e^3-3\right)-1 = e^3-4+\tfrac{1}{e} \approx 16.5. \end{aligned}$$

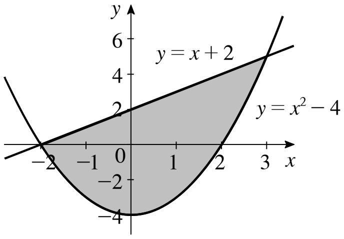


34. $A = \int_0^2 xe^{x^2}\,dx = \frac{1}{2}e^{x^2}\Big|_0^2 = \frac{1}{2}\left(e^4 - 1\right).$



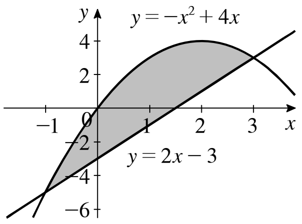
35. To find the points of intersection of the two curves, we solve the equation $x^2 - 4 = x + 2$, obtaining $x^2 - x - 6 = (x - 3)(x + 2) = 0$, so $x = -2$ or $x = 3$. Thus,

$$A = \int_{-2}^{3} \left[(x + 2) - \left(x^2 - 4\right)\right] dx = \int_{-2}^{3} \left(-x^2 + x + 6\right) dx$$
$$= \left(-\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 6x\right)\Big|_{-2}^{3}$$
$$= \left(-9 + \tfrac{9}{2} + 18\right) - \left(\tfrac{8}{3} + 2 - 12\right) = \tfrac{125}{6}.$$

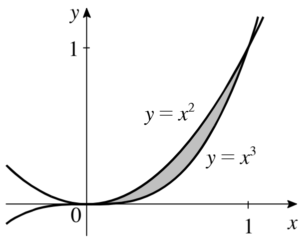


36. To find the intersection of the two curves, we solve $y = -x^2 + 4x$ and $y = 2x - 3$, obtaining $-x^2 + 4x = 2x - 3$, $x^2 - 2x - 3 = 0$, $(x - 3)(x + 1) = 0$, and so $x = -1$ or 3. The required area is given by

$$A = \int_{-1}^{3} \left[\left(-x^2 + 4x\right) - (2x - 3)\right] dx = \int_{-1}^{3} \left(-x^2 + 2x + 3\right) dx$$
$$= \left(-\tfrac{1}{3}x^3 + x^2 + 3x\right)\Big|_{-1}^{3} = (-9 + 9 + 9) - \left(\tfrac{1}{3} + 1 - 3\right) = \tfrac{32}{3}.$$

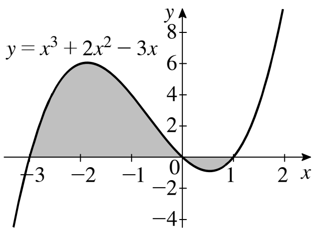


37. To find the points of intersection of the two curves, we solve the equation $x^3 = x^2$, obtaining $x^3 - x^2 = x^2(x - 1) = 0$, so $x = 0$ or 1. Thus, $A = -\int_0^1 \left(x^2 - x^3\right) dx = \left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$



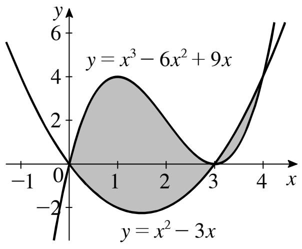
38. $A = \int_{-3}^{0} \left(x^3 + 2x^2 - 3x\right) dx + \int_0^1 \left(x^3 + 2x^2 - 3x\right) dx$

$$= \left(\tfrac{1}{4}x^4 + \tfrac{2}{3}x^3 - \tfrac{3}{2}x^2\right)\Big|_{-3}^{0} - \left(\tfrac{1}{4}x^4 + \tfrac{2}{3}x^3 - \tfrac{3}{2}x^2\right)\Big|_{0}^{1}$$
$$= 0 - \left[\tfrac{1}{4}(81) + \tfrac{2}{3}(-27) - \tfrac{3}{2}(9)\right] - \left(\tfrac{1}{4} + \tfrac{2}{3} - \tfrac{3}{2}\right) + 0$$
$$= -\tfrac{81}{4} + 18 + \tfrac{27}{2} - \tfrac{1}{4} - \tfrac{2}{3} + \tfrac{3}{2} = \tfrac{71}{6}.$$



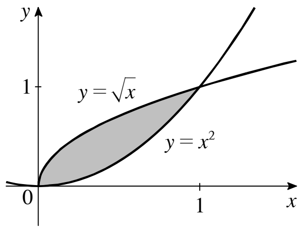
39. To find the points of intersection of the two curves, we solve the equation $x^3 - 6x^2 + 9x = x^2 - 3x$, obtaining $x^3 - 7x^2 + 12x = x(x-4)(x-3) = 0$, so $x = 0$, 3, or 4. Thus,

$$\begin{aligned} A &= \int_0^3 \left[\left(x^3 - 6x^2 + 9x\right) - \left(x^2 + 3x\right)\right] dx \\ &\qquad + \int_3^4 \left[\left(x^2 - 3x\right) - \left(x^3 - 6x^2 + 9x\right)\right] dx \\ &= \int_0^3 \left(x^3 - 7x^2 + 12x\right) dx - \int_3^4 \left(x^3 - 7x^2 + 12x\right) dx \\ &= \left(\tfrac{1}{4}x^4 - \tfrac{7}{3}x^3 + 6x^2\right)\Big|_0^3 - \left(\tfrac{1}{4}x^4 - \tfrac{7}{3}x^3 + 6x^2\right)\Big|_3^4 \\ &= \left(\tfrac{81}{4} - 63 + 54\right) - \left(64 - \tfrac{448}{3} + 96\right) + \left(\tfrac{81}{4} - 63 + 54\right) = \tfrac{71}{6}. \end{aligned}$$



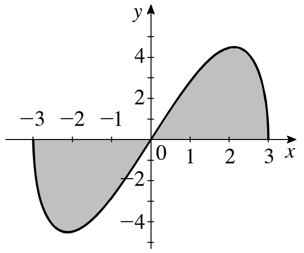
40. The graphs intersect at the points where $\sqrt{x} = x^2$. Solving, we find $x = x^4$, $x\left(x^3 - 1\right) = 0$, and so $x = 0$ or 1. The required area is

$$A = \int_0^1 \left(x^{1/2} - x^2\right) dx = \tfrac{2}{3}x^{3/2} - \tfrac{1}{3}x^3\Big|_0^1 = \tfrac{2}{3} - \tfrac{1}{3} = \tfrac{1}{3}.$$

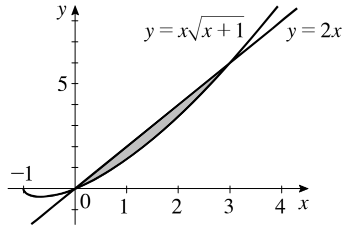


41. By symmetry, $A = 2\int_0^3 x\left(9 - x^2\right)^{1/2} dx$. We integrate using the substitution $u = 9 - x^2$, so $du = -2x\,dx$. If $x = 0$, then $u = 9$ and if $x = 3$, then $u = 0$, so

$$A = 2\int_9^0 -\tfrac{1}{2}u^{1/2}\,du = -\int_9^0 u^{1/2}\,du = -\tfrac{2}{3}u^{3/2}\Big|_9^0 = \tfrac{2}{3}(9)^{3/2} = 18.$$



42. To find the points of intersection of the two graphs we solve $x\sqrt{x+1} = 2x$. Squaring, we obtain $x^2(x+1) = 4x^2$, so $x^3 - 3x^2 = x^2(x-3) = 0$, giving $x = 0$ or 3. Thus, $A = \int_0^3 \left(2x - x\sqrt{x-1}\right) dx = 2\int_0^3 x\,dx - \int_0^3 x\sqrt{x+1}\,dx$. Now $2\int_0^3 x\,dx = x^2\big|_0^3 = 9$, and to evaluate the second integral, let $u = x + 1$, so $du = dx$ and $x = u - 1$. If $x = 0$, then $u = 1$ and if $x = 3$, then $u = 4$, so



$$\int_0^3 x\sqrt{x+1}dx = \int_1^4 (u-1)\sqrt{u}\,du = \int_1^4 \left(u^{3/2} - u^{1/2}\right) du = \left(\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right)\Big|_1^4 = \left(\tfrac{64}{5} - \tfrac{16}{3}\right) - \left(\tfrac{2}{5} - \tfrac{2}{3}\right) = \tfrac{116}{15}.$$

Thus, $A = 9 - \frac{116}{15} = \frac{19}{15}$.

43. $S = \int_0^b \left[g(x) - f(x)\right] dx$ gives the additional revenue that the company would realize if it used a different advertising agency.

44. $S = \int_0^b \left[f(t) - g(t)\right] dt$ gives the difference between the total number of pulse beats between the present and that of six months ago.

45. The shortfall is $\int_{2010}^{2050}\left[f(t)-g(t)\right]dt$.

46. a. S gives the difference in the amount of smoke removed by the two brands over the same time interval $[a,b]$.

b. $S=\int_a^b\left[f(t)-g(t)\right]dt$.

47. a. $\int_{T_1}^{T}\left[g(t)-f(t)\right]dt-\int_0^{T_1}\left[f(t)-g(t)\right]dt=A_2-A_1$.

b. The number A_2-A_1 gives the distance car 2 is ahead of car 1 after T seconds.

48. $\int_{T_1}^{T}\left[g(t)-f(t)\right]dt-\int_0^{T_1}\left[f(t)-g(t)\right]dt$.

49. Mexican oil profits from hedging in 2009 are given by $P=70\cdot 8-\int_0^8 f(t)\,dt-\left[\int_8^{12}f(t)\,dt-70\cdot 4\right]=840-\int_0^{12}f(t)\,dt$ (dollars).

50. The turbocharged model is moving at $A=\int_0^{10}\left[\left(4+1.2t+0.03t^2\right)-(4+0.8t)\right]dt=\int_0^{10}\left(0.4t+0.03t^2\right)dt=\left(0.2t^2+0.1t^3\right)\Big|_0^{10}=20+10$, or 30 ft/sec faster than the standard model.

51. The additional amount of coal that will be produced is

$$\int_0^{20}\left(3.5e^{0.05t}-3.5e^{0.01t}\right)dt=3.5\int_0^{20}\left(e^{0.05t}-e^{0.01t}\right)dt=3.5\left(20e^{0.05t}-100e^{0.01t}\right)\Big|_0^{20}$$
$$=3.5\left[\left(20e-100e^{0.2}\right)-(20-100)\right]=42.8 \text{ billion metric tons.}$$

52. The additional number of cars is given by

$$\int_0^5\left(5e^{0.3t}-5-0.5t^{3/2}\right)dt=\left(\frac{5}{0.3}e^{0.3t}-5t-0.2t^{5/2}\right)\Bigg|_0^5=\frac{5}{0.3}e^{1.5}-25-0.2\,(5)^{5/2}-\frac{5}{0.3}$$
$$=74.695-25-0.2\,(5)^{5/2}-\frac{50}{3}\approx 21.85\text{, or 21,850 cars.}$$

53. If the campaign is mounted, there will be

$\int_0^5\left(60e^{0.02t}+t^2-60\right)dt=\left(3000e^{0.02t}+\frac{1}{3}t^3-60t\right)\Big|_0^5=3315.5+\frac{125}{3}-300-3000\approx 57.179$, or 57,179 fewer people.

54. The projected deficit for the period in question is

$$D=\int_0^4\left(692-\frac{292.6t+134.4}{t+6.9}-500\right)dt=\int_0^4\left(192-\frac{292.6t+134.4}{t+6.9}\right)dt.$$

From the long division at right, we see that

$$\frac{292.6t+134.4}{t+6.9}=292.6-\frac{1884.54}{t+6.9}\text{, so}$$
$$D=\int_0^4\left(192-292.6+\frac{1884.54}{t+6.9}\right)dt$$
$$=\left[-100.6t+1884.54\ln(t+6.9)\right]_0^4\approx 459.29.$$

$$\begin{array}{r|l} & 292.6 \\ \hline t+6.9 & 292.6t+\ \ 134.4 \\ & 292.6t+2018.94 \\ \hline & \quad\ \ -1884.54 \end{array}$$

Thus, the deficit is approximately \$459 billion.

55. True. If $f(x)\geq g(x)$ on $[a,b]$, then the area of the region is $\int_a^b\left[f(x)-g(x)\right]dx=\int_a^b|f(x)-g(x)|\,dx$. If $f(x)\leq g(x)$ on $[a,b]$, then the area of the region is $\int_a^b\left[g(x)-f(x)\right]dx=\int_a^b\left\{-\left[f(x)-g(x)\right]\right\}dx=\int_a^b|f(x)-g(x)|\,dx$.

56. False. The area is given by $\int_0^2 \left[g(x) - f(x)\right] dx$ because $g(x) \ge f(x)$ on [0, 2].

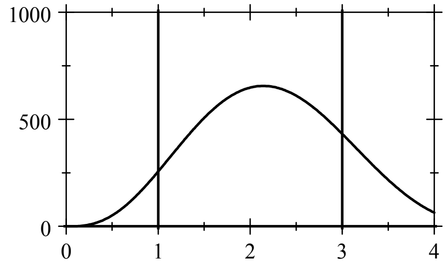
57. False. Take $f(x) = x$ and $g(x) = 0$ on [0, 1]. Then the area bounded by the graphs of f and g on [0, 1] is $A = \int_0^1 (x - 0)\, dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$ and so $A^2 = \frac{1}{4}$. However, $\int_0^1 \left[f(x) - g(x)\right]^2 dx = \int_0^1 x^2\, dx = \frac{1}{3}$.

58. False. Take $f(t) = t^2$ and $g(t) = 1$ on [0, 2]. Then $\int_0^2 \left[f(t) - g(t)\right] dt = \int_0^2 \left(t^2 - 1\right) dt = \left(\frac{1}{3}t^3 - t\right)\Big|_0^2 = \frac{8}{3} - 2 = \frac{2}{3} > 0$, but $f(t)$ is not greater than or equal to $g(t)$ for all t in [0, 2]. For instance, $f\left(\frac{1}{2}\right) = \frac{1}{4} < 1 = g\left(\frac{1}{2}\right)$.

59. The area of R' is $A = \int_a^b \left\{\left[f(x) + C\right] - \left[g(x) + C\right]\right\} dx = \int_a^b \left[f(x) + C - g(x) - C\right] dx = \int_a^b \left[f(x) - g(x)\right] dx$.

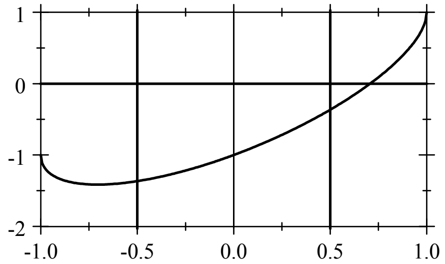
Using Technology page 480

1. a.



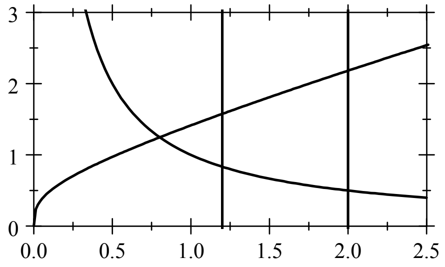
b. $A \approx 1074.2857$.

2. a.



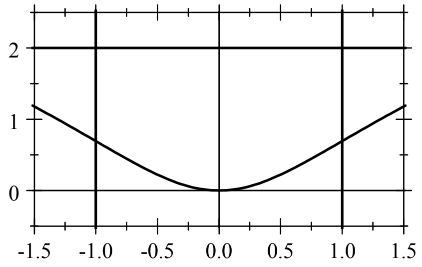
b. $A \approx 0.9566$.

3. a.



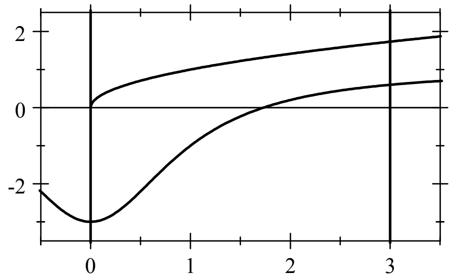
b. $A \approx 0.9961$.

4. a.



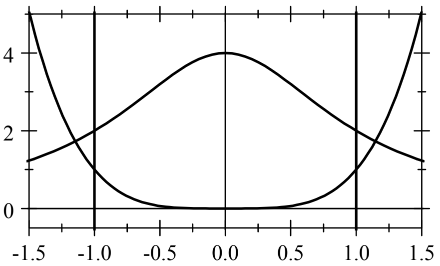
b. $A \approx 3.4721$.

5. a.



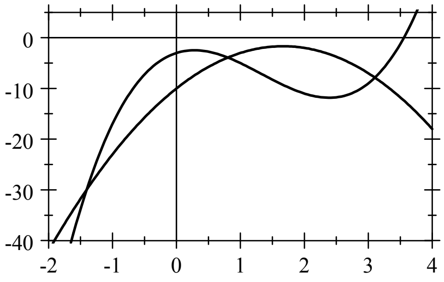
b. $A \approx 5.4603$.

6. a.



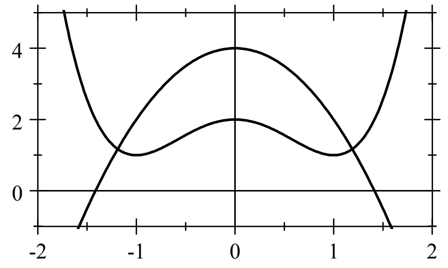
b. $A \approx 5.8832$.

7. a.



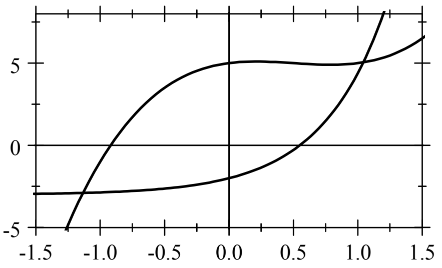
b. $A \approx 25.8549$.

8. a.



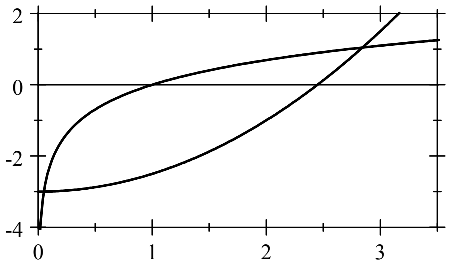
b. $A \approx 3.8055$.

9. a.



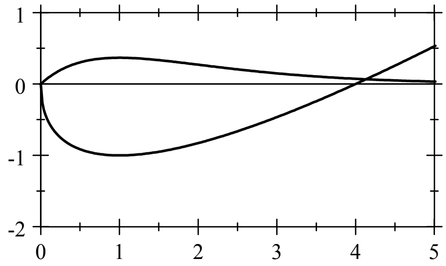
b. $A \approx 10.5144$.

10. a.



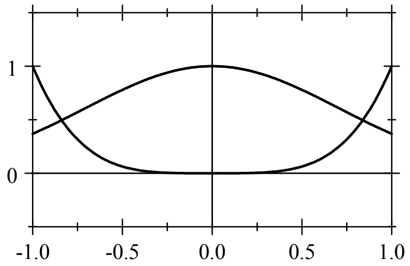
b. $A \approx 4.8765$.

11. a.



b. $A \approx 3.5799$.

12. a.



b. $A \approx 1.1889$.

13. The area of the larger region is 207.43.

6.7 Applications of the Definite Integral to Business and Economics

Concept Questions

page 490

1. a. See the definition on page 482 of the text. **b.** See the definition on page 482 of the text.

2. a. See the definition on page 484 of the text. **b.** See the definition on page 485 of the text.

3. See the definition on page 487 of the text.

4. a. See the definition on page 488 of the text. **b.** See the definition on page 489 of the text.

Exercises page 490

1. When $p = 4$, $-0.01x^2 - 0.1x + 6 = 4$, so $x^2 + 10x - 200 = 0$, and therefore $(x - 10)(x + 20) = 0$, giving $x = 10$ or -20. We reject the root $x = -20$ and find that the equilibrium price occurs at $x = 10$. The consumers' surplus is thus $CS = \int_0^{10} \left(-0.01x^2 - 0.1x + 6\right) dx - (4)(10) = \left(-\frac{0.01}{3}x^3 - 0.05x^2 + 6x\right)\Big|_0^{10} - 40 \approx 11.667$, or \$11,667.

2. Setting $p = 5$, we find that the demand equation is $-0.01x^2 - 0.2x + 8 = 5$, so $-0.01x^2 - 0.2x + 3 = 0$, and therefore $x^2 + 20x - 300 = (x + 30)(x - 10) = 0$, giving $x = -30$ or 10. Thus, the equilibrium price occurs at $x = 10$ and the consumers' surplus is

$$CS = \int_0^{10} \left(-0.01x^2 - 0.2x + 8\right) dx - 5(10) = \left[-0.01\left(\tfrac{1}{3}x^3\right) - 0.1x^2 + 8x\right]_0^{10} - 50$$
$$= -\tfrac{10}{3} - 10 + 80 - 50 = \tfrac{50}{3}\text{, or approximately \$16,667.}$$

3. Setting $p = 10$, we have $\sqrt{225 - 5x} = 10$, $225 - 5x = 100$, and so $x = 25$. Then $CS = \int_0^{25} \sqrt{225 - 5x}\, dx - (10)(25) = \int_0^{25} (225 - 5x)^{1/2}\, dx - 250$. To evaluate the integral, let $u = 225 - 5x$, so $du = -5\, dx$ and $dx = -\frac{1}{5}\, du$. If $x = 0$, then $u = 225$ and if $x = 25$, then $u = 100$, so $CS = -\frac{1}{5}\int_{225}^{100} u^{1/2}\, du - 250 = -\frac{2}{15}u^{3/2}\Big|_{225}^{100} - 250 = -\frac{2}{15}(1000 - 3375) - 250 = 66.667$, or \$6,667.

4. When $p = 9$, $\sqrt{36 + 1.8x} = 9$, $36 + 1.8x = 81$, $1.8x = 45$, and so $x = 25$. The producers' surplus is

$$PS = (9)(25) - \int_0^{25} (36 + 1.8x)^{1/2}\, dx = 225 - \left[\tfrac{1}{1.8}\left(\tfrac{2}{3}\right)(36 + 1.8x)^{3/2}\right]_0^{25}$$
$$= 225 - \tfrac{1}{2.7}\left[(36 + 45)^{3/2} - 36^{3/2}\right] = 35\text{, or \$3500.}$$

5. To find the equilibrium point, we solve $0.01x^2 + 0.1x + 3 = -0.01x^2 - 0.2x + 8$, finding $0.02x^2 + 0.3x - 5 = 0$, $2x^2 + 30x - 500 = (2x - 20)(x + 25) = 0$, and so $x = -25$ or 10. Thus, the equilibrium point is (10, 5). Then $PS = (5)(10) - \int_0^{10} \left(0.01x^2 + 0.1x + 3\right) dx = 50 - \left(\dfrac{0.01}{3}x^3 + 0.05x^2 + 3x\right)\Bigg|_0^{10} = 50 - \frac{10}{3} - 5 - 30 = \frac{35}{3}$, or approximately \$11,667.

6. a. If $p = 400$, then we have $600e^{-0.04x} = 400$, so $e^{-0.04x} = \frac{2}{3}$, $\ln e^{-0.04x} = \ln \frac{2}{3}$, $-0.04x = \ln \frac{2}{3}$, and $x = -\dfrac{\ln \frac{2}{3}}{0.04} \approx 10.137$. Thus, the demand for matresses is approximately 1040 per month.

b. Taking $\overline{p} = 400$ and $\overline{x} = 10.137$ and using Formula (16), we have $CS \approx \int_0^{10.137} 600e^{-0.04x}\, dx - 400 \cdot 10.137 = -\frac{600}{0.04}e^{-0.04x}\Big|_0^{10.137} - 400 \cdot 10.137 \approx 945.35$, or \$94,535.

7. a. Setting $p = 250$, we have $100 + 80e^{0.05x} = 250$, $e^{0.05x} = \frac{150}{80} = \frac{15}{8}$, $\ln e^{0.05x} = \ln \frac{15}{8}$, $0.05x = \ln \frac{15}{8}$, and $x \approx 12.572$. The number of matresses the supplier will make available in the market per month is approximately 1257.

b. Taking $\overline{p} = 250$ and $\overline{x} = 12.572$ and using Formula (17), we find $PS \approx 12.572 \cdot 250 - \int_0^{12.572} \left(100 + 80e^{0.05x}\right) dx = 3143 - \left(100x + \frac{80}{0.05}e^{0.05x}\right)\Big|_0^{12.572} \approx 485.826$, and so the producers' surplus is approximately \$48,583.

8. If $p = 200$, then we have $\dfrac{600}{0.5x+2} = 200$, so $200\,(0.5x+2) = 600$, $0.5x + 2 = 3$, and $x = 2$. Using Formula (16) with $\overline{p} = 200$ and $\overline{x} = 2$, we have $CS = \displaystyle\int_0^2 \frac{600}{0.5x+2}\,dx - 200 \cdot 2 = \tfrac{600}{0.5} \ln(0.5x+2)\Big|_0^2 - 400 = 1200\,(\ln 3 - \ln 2) - 400 \approx 86.558$, or approximately \$86,558.

9. If $p = 160$, then we have $100\left(0.5x + \dfrac{0.4}{1+x}\right) = 160$, so $50x + \dfrac{40}{1+x} = 160$, $50x^2 + 50x + 40 = 160 + 160x$, $50x^2 - 110x - 120 = 0$, $5x^2 - 11x - 12 = 0$, and $(5x+4)\,(x-3) = 0$. Thus, $x = -\frac{4}{5}$ or $x = 3$. We reject the negative root, and using Formula (17) with $\overline{p} = 160$ and $\overline{x} = 3$, we have $PS = 3 \cdot 160 - \displaystyle\int_0^3 100\left(0.5x + \frac{0.4}{1+x}\right)dx = 480 - 100\left[0.25x^2 + 0.4\ln(1+x)\right]_0^3 \approx 199.548$. Therefore, the producers' surplus is approximately \$199,548.

10. To determine the market equilibrium, we solve $p = 144 - x^2$ and $p = 48 + \frac{1}{2}x^2$ simultaneously, obtaining $\frac{1}{2}x^2 + 48 = 144 - x^2$, $\frac{3}{2}x^2 = 96$, and $x = \pm 8$. Because x must be nonnegative, we take $x = 8$, and so $p = 80$. The consumers' surplus is $CS = \int_0^8 \left(144 - x^2\right)dx - (8)\,(80) = \left(144x - \frac{1}{3}x^3\right)\Big|_0^8 - 640 = 144\,(8) - \frac{1}{3}\,(8)^3 - 640 \approx 341.333$, or approximately \$341,333. The producers' surplus is $PS = 640 - \left(48x + \frac{1}{6}x^3\right)\Big|_0^8 = 640 - 48\,(8) - \frac{1}{6}\,(8)^3 \approx 170.667$, or approximately \$170,667.

11. To find the market equilibrium, we solve $-0.2x^2 + 80 = 0.1x^2 + x + 40$, obtaining $0.3x^2 + x - 40 = 0$, $3x^2 + 10x - 400 = 0$, $(3x+40)\,(x-10) = 0$, and so $x = -\frac{40}{3}$ or $x = 10$. We reject the negative root. The corresponding equilibrium price is \$60, the consumers' surplus is $CS = \int_0^{10} \left(-0.2x^2 + 80\right)dx - (60)\,(10) = \left(-\frac{0.2}{3}x^3 + 80x\right)\Big|_0^{10} - 600 \approx 133.33$, or \$13,333, and the producers' surplus is $PS = 600 - \int_0^{10} \left(0.1x^2 + x + 40\right)dx = 600 - \left(\frac{0.1}{3}x^3 + \frac{1}{2}x^2 + 40x\right)\Big|_0^{10} \approx 116.67$, or \$11,667.

12. Let $u = -0.1t$, so $du = -0.1\,dt$ and $dt = -\frac{1}{0.1}\,du = -10\,du$. If $t = 0$, then $u = 0$ and if $t = 3$, then $u = -0.3$, so

$$\int_0^3 580{,}000e^{-0.1t}\,dt - 180{,}000 = 580{,}000\,(-10)\int_0^{-0.3} e^u\,du - 180{,}000 = -5{,}800{,}000e^u\big|_0^{-0.3} - 180{,}000$$
$$= -5{,}800{,}000\left(e^{-0.3} - 1\right) - 180{,}000 \approx 1{,}323{,}254.$$

13. Here $R(t) = 120{,}000$, $r = 0.035$, and $T = 4$, so the required future value is $A = e^{0.035(4)} \int_0^4 (120{,}000)\,e^{-0.035t}\,dt = 120{,}000e^{0.14}\left[-\frac{1}{0.035}e^{-0.035t}\right]_0^4 \approx 515{,}224.453$, or approximately \$515,224.45.

14. Here $R(t) = 30{,}000e^{0.03t}$, $r = 0.045$, and $T = 3$, so the accumulated value is $A = e^{0.045(3)} \int_0^3 R(t)\,e^{-0.045t}\,dt = e^{0.135} \int_0^3 30{,}000e^{-0.015}\,dt = 30{,}000e^{0.135}\left[-\frac{1}{0.015}e^{-0.015t}\right]_0^3 \approx 100{,}725.001$, or approximately \$100,725.

15. Here $P = 200{,}000$, $r = 0.08$, and $T = 5$, so $PV = \int_0^5 200{,}000e^{-0.08t}\,dt = -\frac{200{,}000}{0.08}e^{-0.08t}\Big|_0^5 = -2{,}500{,}000\left(e^{-0.4} - 1\right) \approx 824{,}199.85$, or \$824,200.

16. $PV = \int_0^{15} 400{,}000e^{-0.08t}\,dt = -\frac{400{,}000}{0.08}e^{-0.08t}\Big|_0^{15} = -5{,}000{,}000\left(e^{-1.2}-1\right) \approx 3{,}494{,}028.94$, or approximately \$3,494,029.

17. Here $P = 250$, $m = 12$, $T = 20$, and $r = 0.04$, so $A = \dfrac{mP}{r}\left(e^{rT}-1\right) = \dfrac{12\,(250)}{0.04}\left(e^{0.8}-1\right) \approx 91{,}915.57$, or approximately \$91,916.

18. Here $P = 400$, $m = 12$, $T = 20$, and $r = 0.05$, so $A = \dfrac{12\,(400)}{0.05}\left(e^{1.0}-1\right) \approx 164{,}955.06$, or approximately \$164,955.

19. Here $R(t) = 20e^{0.08t}$, $r = 0.03$, and $T = 5$.

a. The future value is $A = e^{0.03(5)}\int_0^5 20e^{0.08t}e^{-0.03t}\,dt = 20e^{0.15}\int_0^5 e^{0.05t}\,dt = 20e^{0.15}\left[\frac{1}{0.05}e^{0.05t}\right]_0^5 \approx 131.9962$, or approximately \$131,996.

b. The present value is $PV = \int_0^5 20e^{0.08t}e^{-0.03t}\,dt = 20\int_0^5 e^{0.05t}\,dt = \left[\frac{20}{0.05}e^{0.05t}\right]_0^5 \approx 113.61017$, or approximately \$113,610.

20. The present value of Sharon's net income under plan A is

$$PV_A = \int_0^5 3{,}050{,}000e^{0.02t}e^{-0.03t}\,dt - 600{,}000 = 3{,}050{,}000\int_0^5 e^{-0.01t}\,dt - 600{,}000$$
$$= \left[\frac{3{,}050{,}000}{-0.01}e^{-0.01t}\right]_0^5 - 600{,}000 \approx 14{,}275{,}026 \text{ (dollars).}$$

The present value under plan B is

$$PV_B = \int_0^5 3{,}200{,}000e^{-0.03t}\,dt - 400,000 = \left[\frac{3{,}200{,}000}{-0.03}e^{-0.03t}\right]_0^5 - 400{,}000 \approx 14{,}457{,}816 \text{ (dollars).}$$

Comparing the present values under the two plans, we conclude that plan B will yield a higher net income over five years.

21. Here $P = 150$, $m = 12$, $T = 15$, and $r = 0.05$, so $A = \dfrac{12\,(150)}{0.05}\left(e^{0.75}-1\right) \approx 40{,}212.00$, or approximately \$40,212.

22. Here $P = 200$, $m = 12$, $T = 10$, and $r = 0.04$, so $A = \dfrac{12\,(200)}{0.04}\left(e^{0.4}-1\right) \approx 29{,}509.48$, or approximately \$29,509.

23. Here $P = 2000$, $m = 1$, $T = 15.75$, and $r = 0.05$, so $A = \dfrac{1\,(2000)}{0.05}\left(e^{0.7875}-1\right) \approx 47{,}915.79$, or approximately \$47,916.

24. Here $P = 800$, $m = 12$, $T = 12$, and $r = 0.05$, so $PV = \dfrac{12\,(800)}{0.05}\left(1-e^{-0.6}\right) \approx 86{,}628.17$, or approximately \$86,628.

25. Here $P = 1200$, $m = 12$, $T = 15$, and $r = 0.06$, so $PV = \dfrac{12\,(1200)}{0.06}\left(1-e^{-0.9}\right) \approx 142{,}423.28$, or approximately \$142,423.

26. $PV = \dfrac{mP}{r}\left(1-e^{-rT}\right) = \dfrac{1\,(50{,}000)}{0.06}\left(1-e^{-0.06\cdot 20}\right) = 833{,}333.33\left(1-e^{-1.2}\right) \approx 582{,}338.16$, or approximately \$582,338.

27. We want the present value of an annuity with $P = 300$, $m = 12$, $T = 10$, and $r = 0.05$, so

$$PV = \frac{12\,(300)}{0.05}\left(1 - e^{-0.5}\right) \approx 28{,}329.79, \text{ or approximately \$28,330.}$$

28. We want the present value of an annuity with $P = 400$, $m = 12$, $T = 15$, and $r = 0.06$, so

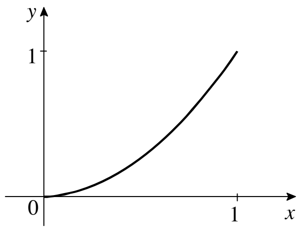
$$PV = \frac{12\,(400)}{0.06}\left(1 - e^{-0.9}\right) \approx 47{,}474.43, \text{ or approximately \$47,474.}$$

29. $L = 2\int_0^1 \left[x - \left(0.3x^{1.5} + 0.7x^{2.5}\right)\right] dx = 2\int_0^1 \left(x - 0.3x^{1.5} - 0.7x^{2.5}\right) dx = 2\left[\frac{1}{2}x^2 - \frac{0.3}{2.5}x^{2.5} - \frac{0.7}{3.5}x^{3.5}\right]_0^1 \approx 0.36.$

30. $L = 2\int_0^1 \left(x - \frac{e^{0.1x} - 1}{e^{0.1} - 1}\right) dx = 2\int_0^1 x\,dx - \frac{2}{e^{0.1} - 1}\int \left(e^{0.1x} - 1\right) dx$

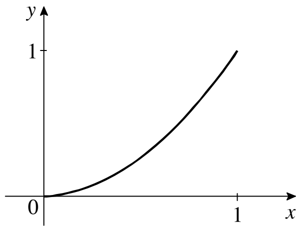
$= 2\left[\frac{1}{2}x^2\right]_0^1 - \left[\frac{2}{e^{0.1} - 1}\left(\frac{e^{0.1x}}{0.1} - x\right)\right]_0^1 \approx 0.017.$

31. a.



b. $f(0.4) = \frac{15}{16}(0.4)^2 + \frac{1}{16}(0.4) \approx 0.175$ and $f(0.9) = \frac{15}{16}(0.9)^2 + \frac{1}{16}(0.9) \approx 0.816$. Thus, the lowest 40% of earners receive 17.5% of the total income and the lowest 90% of earners receive 81.6%.

32. a.



b. $f(0.3) = \frac{14}{15}(0.03)^2 + \frac{1}{15}(0.3) = 0.104$ and $f(0.7) = \frac{14}{15}(0.7)^2 + \frac{1}{15}(0.7) \approx 0.504$.

33. a. $L_1 = 2\int_0^1 \left[x - f(x)\right] dx = 2\int_0^1 \left(x - \frac{13}{14}x^2 - \frac{1}{14}x\right) dx = 2\int_0^1 \left(\frac{13}{14}x - \frac{13}{14}x^2\right) dx = \frac{13}{7}\int_0^1 \left(x - x^2\right) dx$

$= \frac{13}{7}\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{13}{7}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{13}{7} \cdot \frac{1}{6} = \frac{13}{42} = 0.3095$ and

$L_2 = 2\int_0^1 \left(x - \frac{9}{11}x^4 - \frac{2}{11}x\right) dx = 2\int_0^1 \left(\frac{9}{11}x - \frac{9}{11}x^4\right) dx = 2\left(\frac{9}{11}\right)\int_0^1 \left(x - x^4\right) dx$

$= \frac{18}{11}\left(\frac{1}{2}x^2 - \frac{1}{5}x^5\right)\Big|_0^1 = \frac{18}{11}\left(\frac{1}{2} - \frac{1}{5}\right) = 0.4909.$

b. College teachers have a more equitable income distribution.

34. a. The coefficient of inequality for stockbrokers is

$2\int_0^1 \left[x - \left(\frac{11}{12}x^2 + \frac{1}{12}x\right)\right] dx = 2\left(\frac{11}{12}\right)\int_0^1 \left(x - x^2\right) dx = \frac{11}{6}\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 \approx 0.31.$

The coefficient of inequality for high school teachers is

$2\int_0^1 \left[x - \left(\frac{5}{6}x^2 + \frac{1}{6}x\right)\right] dx = 2\left(\frac{5}{6}\right)\int_0^1 \left(x - x^2\right) dx = \frac{5}{3}\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 \approx 0.28.$

b. The results of part (a) suggest that the teaching profession has a more equitable income distribution.

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1. The consumer's surplus is \$18,000,000 and the producer's surplus is \$11,700,000.

2. The consumer's surplus is \$13,333 and the producer's surplus is \$11,667.

3. The consumer's surplus is \$33,120 and the producer's surplus is \$2880.

4. The consumer's surplus is \$55,104 and the producer's surplus is \$141,669.

5. Investment A will generate a higher net income.

6. Investment B will generate a higher net income.

CHAPTER 6 Concept Review Questions page 496

1. a. $F'(x) = f(x)$ **b.** $F(x) + C$

2. a. $c \int f(x)\,dx$ **b.** $\int f(x)\,dx \pm \int g(x)\,dx$

3. a. unknown **b.** function

4. $g'(x)\,dx$, $\int f(u)\,du$

5. a. $\int_a^b f(x)\,dx$ **b.** minus

6. a. $F(b) - F(a)$, antiderivative **b.** $\int_a^b f'(x)\,dx$

7. a. $\frac{1}{b-a}\int_a^b f(x)\,dx$ **b.** area, area

8. $\int_a^b [f(x) - g(x)]\,dx$

9. a. $\int_0^{\overline{x}} D(x)\,dx - \overline{p}\,\overline{x}$ **b.** $\overline{p}\,\overline{x} - \int_0^{\overline{x}} S(x)\,dx$

10. a. $e^{rT}\int_0^T R(t)e^{-rt}\,dt$ **b.** $\int_0^T R(t)e^{-rt}\,dt$

11. $\frac{mP}{r}\left(e^{rT} - 1\right)$ **12.** $2\int_0^1 [x - f(x)]\,dx$

CHAPTER 6 Review Exercises page 496

1. $\int \left(x^3 + 2x^2 - x\right)dx = \frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{1}{2}x^2 + C.$

2. $\int \left(\frac{1}{3}x^3 - 2x^2 + 8\right)dx = \frac{1}{12}x^4 - \frac{2}{3}x^3 + 8x + C.$

3. $\int\left(x^4-2x^3+\frac{1}{x^2}\right)dx=\frac{x^5}{5}-\frac{x^4}{2}-\frac{1}{x}+C.$

4. $\int\left(x^{1/3}-x^{1/2}+4\right)dx=\frac{3}{4}x^{4/3}-\frac{2}{3}x^{3/2}+4x+C.$

5. $\int x\left(2x^2+x^{1/2}\right)dx=\int\left(2x^3+x^{3/2}\right)dx=\frac{1}{2}x^4+\frac{2}{5}x^{5/2}+C.$

6. $\int\left(x^2+1\right)\left(\sqrt{x}-1\right)dx=\int\left(x^{5/2}-x^2+x^{1/2}-1\right)dx=\frac{2}{7}x^{7/2}-\frac{1}{3}x^3+\frac{2}{3}x^{3/2}-x+C.$

7. $\int\left(x^2-x+\frac{2}{x}+5\right)dx=\int x^2\,dx-\int x\,dx+2\int\frac{dx}{x}+5\int dx=\frac{1}{3}x^3-\frac{1}{2}x^2+2\ln|x|+5x+C.$

8. Let $u=2x+1$, so $du=2\,dx$ and $dx=\frac{1}{2}\,du$. Then $\int\sqrt{2x+1}\,dx=\frac{1}{2}\int u^{1/2}\,du=\frac{1}{3}u^{3/2}=\frac{1}{3}\left(2x+1\right)^{3/2}+C.$

9. Let $u=3x^2-2x+1$, so $du=\left(6x-2\right)dx=2\left(3x-1\right)dx$ or $\left(3x-1\right)dx=\frac{1}{2}du$. So $\int\left(3x-1\right)\left(3x^2-2x+1\right)^{1/3}dx=\frac{1}{2}\int u^{1/3}du=\frac{3}{8}u^{4/3}+C=\frac{3}{8}\left(3x^2-2x+1\right)^{4/3}+C.$

10. Put $u=x^3+2$, so $du=3x^2\,dx$ and $x^2\,dx=\frac{1}{3}\,du$. Then $\int x^2\left(x^3+2\right)^{10}dx=\frac{1}{3}\int u^{10}\,du=\frac{1}{33}u^{11}+C=\frac{\left(x^3+2\right)^{11}}{33}+C.$

11. Let $u=x^2-2x+5$, so $du=2\left(x-1\right)dx$ and $\left(x-1\right)dx=\frac{1}{2}\,du$. Then $\int\frac{x-1}{x^2-2x+5}\,dx=\frac{1}{2}\int\frac{du}{u}=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln\left(x^2-2x+5\right)+C.$

12. Let $u=-2x$, so $du=-2\,dx$. Then $\int 2e^{-2x}\,dx=-\int e^u\,du=-e^u+C=-e^{-2x}+C.$

13. Put $u=x^2+x+1$, so $du=\left(2x+1\right)dx=2\left(x+\frac{1}{2}\right)dx$ and $\left(x+\frac{1}{2}\right)dx=\frac{1}{2}du$. Then $\int\left(x+\frac{1}{2}\right)e^{x^2+x+1}\,dx=\frac{1}{2}\int e^u\,du=\frac{1}{2}e^u+C=\frac{1}{2}e^{x^2+x+1}+C.$

14. Let $u=e^{-x}+x$, so $du=\left(-e^{-x}+1\right)dx$ and $\left(e^{-x}-1\right)dx=-du$. Then $\int\frac{e^{-x}-1}{\left(e^{-x}+x\right)^2}\,dx=-\int\frac{du}{u^2}=\frac{1}{u}+C=\frac{1}{e^{-x}+x}+C.$

15. Let $u=\ln x$, so $du=\frac{1}{x}\,dx$. Then $\int\frac{\left(\ln x\right)^5}{x}\,dx=\int u^5\,du=\frac{1}{6}u^6+C=\frac{1}{6}\left(\ln x\right)^6+C.$

16. $\int\frac{\ln x^2}{x}\,dx=2\int\frac{\ln x}{x}\,dx$. Now put $u=\ln x$, so $du=\frac{1}{x}\,dx$. Then $\int\frac{\ln x^2}{x}\,dx=2\int u\,du=u^2+C=\left(\ln x\right)^2+C.$

17. Let $u=x^2+1$, so $x^2=u-1$, $du=2x\,dx$, $x\,dx=\frac{1}{2}\,du$. Then

$$\int x^3\left(x^2+1\right)^{10}dx=\frac{1}{2}\int\left(u-1\right)u^{10}\,du=\frac{1}{2}\int\left(u^{11}-u^{10}\right)du=\frac{1}{2}\left(\frac{1}{12}u^{12}-\frac{1}{11}u^{11}\right)+C$$

$$=\frac{1}{264}u^{11}\left(11u-12\right)+C=\frac{1}{264}\left(x^2+1\right)^{11}\left(11x^2-1\right)+C.$$

18. Let $u = x + 1$, so $du = dx$. Then $x = u - 1$, so

$$\int x\sqrt{x+1}\,dx = \int (u-1)\,u^{1/2}\,du = \int \left(u^{3/2} - u^{1/2}\right) du = \tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2} + C$$
$$= \tfrac{2}{15}u^{3/2}\,(3u-5) + C = \tfrac{2}{15}\,(3x-2)\,(x+1)^{3/2} + C.$$

19. Put $u = x - 2$, so $du = dx$. Then $x = u + 2$ and

$$\int \frac{x}{\sqrt{x-2}}\,dx = \int \frac{u+2}{\sqrt{u}}\,du = \int \left(u^{1/2} + 2u^{-1/2}\right) du = \int u^{1/2}\,du + 2\int u^{-1/2}\,du = \tfrac{2}{3}u^{3/2} + 4u^{1/2} + C$$
$$= \tfrac{2}{3}u^{1/2}\,(u+6) + C = \tfrac{2}{3}\sqrt{x-2}\,(x-2+6) + C = \tfrac{2}{3}\,(x+4)\,\sqrt{x-2} + C.$$

20. Let $u = x + 1$, so $x = u - 1$ and $du = dx$. Then

$$\int \frac{3x}{\sqrt{x+1}}\,dx = 3\int \frac{u-1}{\sqrt{u}}\,du = 3\int \left(u^{1/2} - u^{-1/2}\right) du = 3\left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) + C = 2u^{1/2}\,(u-3) + C$$
$$= 2\,(x-2)\,\sqrt{x+1} + C.$$

21. $\int_0^1 \left(2x^3 - 3x^2 + 1\right) dx = \left(\tfrac{1}{2}x^4 - x^3 + x\right)\Big|_0^1 = \tfrac{1}{2} - 1 + 1 = \tfrac{1}{2}.$

22. $\int_0^2 \left(4x^3 - 9x^2 + 2x - 1\right) dx = \left(x^4 - 3x^3 + x^2 - x\right)\Big|_0^2 = 16 - 24 + 4 - 2 = -6.$

23. $\int_1^4 \left(x^{1/2} + x^{-3/2}\right) dx = \left(\tfrac{2}{3}x^{3/2} - 2x^{-1/2}\right)\Big|_1^4 = \left(\tfrac{2}{3}x^{3/2} - \tfrac{2}{\sqrt{x}}\right)\Big|_1^4 = \left(\tfrac{16}{3} - 1\right) - \left(\tfrac{2}{3} - 2\right) = \tfrac{17}{3}.$

24. Let $u = 2x^2 + 1$, so $du = 4x\,dx$ and $x\,dx = \tfrac{1}{4}du$. If $x = 0$, then $u = 1$ and if $x = 1$, then $u = 3$, so

$\int_0^1 20x\left(2x^2+1\right)^4\,dx = \tfrac{20}{4}\int_1^3 u^4\,du = u^5\Big|_1^3 = 243 - 1 = 242.$

25. Put $u = x^3 - 3x^2 + 1$, so $du = \left(3x^2 - 6x\right) dx = 3\left(x^2 - 2x\right) dx$ and $\left(x^2 - 2x\right) dx = \tfrac{1}{3}\,du$. If $x = -1$, $u = -3$ and if $x = 0$, $u = 1$, so $\int_{-1}^0 12\left(x^2 - 2x\right)\left(x^3 - 3x^2 + 1\right)^3\,dx = (12)\left(\tfrac{1}{3}\right)\int_{-3}^1 u^3\,du = 4\left(\tfrac{1}{4}\right)u^4\Big|_{-3}^1 = 1 - 81 = -80.$

26. Let $u = x - 3$, so $du = dx$. If $x = 4$, then $u = 1$ and if $x = 7$, then $u = 4$, so

$$\int_4^7 x\sqrt{x-3}\,dx = \int_1^4 (u+3)\sqrt{u}\,du = \int_1^4 \left(u^{3/2} + 3u^{1/2}\right) du = \left(\tfrac{2}{5}u^{5/2} + 2u^{3/2}\right)\Big|_1^4$$
$$= \left(\tfrac{64}{5} + 16\right) - \left(\tfrac{2}{5} + 2\right) = \tfrac{132}{5}.$$

27. Let $u = x^2 + 1$, so $du = 2x\,dx$ and $x\,dx = \tfrac{1}{2}du$. If $x = 0$, then $u = 1$, and if $x = 2$, then $u = 5$, so

$$\int_0^2 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_1^5 \frac{du}{u} = \tfrac{1}{2}\ln u\Big|_1^5 = \tfrac{1}{2}\ln 5.$$

28. Let $u = 5 - 2x$, so $du = -2\,dx$, or $dx = -\tfrac{1}{2}du$. If $x = 0$, then $u = 5$ and if $x = 1$, then $u = 3$, so

$$\int_0^1 (5-2x)^{-2}\,dx = \int_5^3 \left(-\frac{1}{2}\right)\frac{du}{u^2} = \tfrac{1}{2}u^{-1}\Big|_5^3 = \tfrac{1}{6} - \tfrac{1}{10} = \tfrac{1}{15}.$$

29. Let $u = 1 + 2x^2$, so $du = 4x\,dx$ and $x\,dx = \tfrac{1}{4}\,du$. If $x = 0$, then $u = 1$ and if $x = 2$, then $u = 9$, so

$$\int_0^2 \frac{4x}{\sqrt{1+2x^2}}\,dx = \int_1^9 \frac{du}{u^{1/2}} = 2u^{1/2}\Big|_1^9 = 2\,(3-1) = 4.$$

30. Let $u = -\frac{1}{2}x^2$, so $du = -x\,dx$ and $x\,dx = -du$. If $x = 0$, then $u = 0$ and if $x = 2$, then $u = -2$, so $\int_0^2 xe^{-x^2/2}\,dx = -\int_0^{-2} e^u\,du = -e^u\big|_0^{-2} = -e^{-2} + 1 = 1 - \dfrac{1}{e^2}$.

31. Let $u = 1 + e^{-x}$, so $du = -e^{-x}\,dx$ and $e^{-x}\,dx = -du$. Then $\displaystyle\int_{-1}^{0} \frac{e^{-x}}{\left(1+e^{-x}\right)^2}\,dx = -\int_{1+e}^{2} \frac{du}{u^2} = \frac{1}{u}\bigg|_{1+e}^{2} = \frac{1}{2} - \frac{1}{1+e} = \frac{e-1}{2\,(1+e)}$.

32. Let $u = \ln x$, so $du = \dfrac{dx}{x}$. If $x = 1$, then $u = 0$, and if $x = e$, then $u = \ln e = 1$, so $\displaystyle\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \tfrac{1}{2}u^2\Big|_0^1 = \tfrac{1}{2}$.

33. $f(x) = \int f'(x)\,dx = \int \left(3x^2 - 4x + 1\right)dx = 3\int x^2\,dx - 4\int x\,dx + \int dx = x^3 - 2x^2 + x + C$. The given condition implies that $f(1) = 1$, so $1 - 2 + 1 + C = 1$, and thus $C = 1$. Therefore, the required function is $f(x) = x^3 - 2x^2 + x + 1$.

34. $\displaystyle f(x) = \int f'(x)\,dx = \int \frac{x}{\sqrt{x^2+1}}\,dx$. Let $u = x^2 + 1$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. Then $\displaystyle f(x) = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2+1} + C$. Now $f(0) = 1$ implies $\sqrt{0+1} + C = 1$, so $C = 0$. Thus, $f(x) = \sqrt{x^2+1}$.

35. $f(x) = \int f'(x)\,dx = \int \left(1 - e^{-x}\right)dx = x + e^{-x} + C$. Now $f(0) = 2$ implies $0 + 1 + C = 2$, so $C = 1$ and the required function is $f(x) = x + e^{-x} + 1$.

36. $\displaystyle f(x) = \int f'(x)\,dx = \int \frac{\ln x}{x}\,dx$. Let $u = \ln x$, so $du = \dfrac{dx}{x}$. Then $f(x) = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$. $f(1) = 0 + C = -2$ implies that $C = -2$, so the required function is $f(x) = \frac{1}{2}(\ln x)^2 - 2$.

37. **a.** The integral $\int_0^T \left[f(t) - g(t)\right]dt$ represents the distance in feet between Car A and Car B at time T. If Car A is ahead, the integral is positive and if Car B is ahead, it is negative.

b. The distance is greatest at $t = 10$, at which point Car B's velocity exceeds that of Car A and Car B starts catching up. At that instant, the distance between the cars is $\int_0^{10} \left[f(t) - g(t)\right]dt$.

38. **a.** The integral $\int_0^T \left[f(t) - g(t)\right]dt$ represents the difference in the total revenues of Branch A and Branch B at time T. If Branch A has greater revenues, the integral is positive, and if Branch B has greater revenues, it is negative.

b. The difference is greatest at $t = 10$, at which point Branch B's revenues begin growing faster than those of Branch A. At that instant, the difference in revenues is $\int_0^{10} \left[f(t) - g(t)\right]dt$.

39. $\Delta x = \frac{2-1}{5} = \frac{1}{5}$, so $x_1 = \frac{6}{5}$, $x_2 = \frac{7}{5}$, $x_3 = \frac{8}{5}$, $x_4 = \frac{9}{5}$, $x_5 = \frac{10}{5}$. The Riemann sum is

$$\begin{aligned} f(x_1)\,\Delta x + \cdots + f(x_5)\,\Delta x &= \left\{\left[-2\left(\tfrac{6}{5}\right)^2 + 1\right] + \left[-2\left(\tfrac{7}{5}\right)^2 + 1\right] + \cdots + \left[-2\left(\tfrac{10}{5}\right)^2 + 1\right]\right\}\left(\tfrac{1}{5}\right) \\ &= \tfrac{1}{5}\left(-1.88 - 2.92 - 4.12 - 5.48 - 7\right) = -4.28. \end{aligned}$$

40. The percentage of mobile phone users with smartphones is $P(t) = \int R(t)\,dt = \int 10.8\,dt = 10.8t + C$. To find C, we use the condition $P(0) = 38.5$, obtaining $C = 38.5$. Thus, $P(t) = 10.8t + 38.5$. The projected percentage of mobile phone users with smartphones in October 2013 is $P(2) = 10.8(2) + 38.5 = 60.1$, or 60.1%.

41. $V(t) = \int V'(t)\,dt = 3800\int(t-10)\,dt = 1900(t-10)^2 + C$. The initial condition implies that $V(0) = 200{,}000$, that is, $190{,}000 + C = 200{,}000$, and so $C = 10{,}000$. Therefore, $V(t) = 1900(t-10)^2 + 10{,}000$. The resale value of the computer after 6 years is given by $V(6) = 1900(-4)^2 + 10{,}000 = 40{,}400$, or \$40,400.

42. $C(x) = \int C'(x)\,dx = \int\left(0.00003x^2 - 0.03x + 20\right)dx = 0.00001x^3 - 0.015x^2 + 20x + k$. $C(0) = k = 500$, so the required total cost function is $C(x) = 0.00001x^3 - 0.015x^2 + 20x + 500$. The total cost of producing the first 400 coffeemakers per day is $C(400) = 0.00001(400)^3 - 0.015(400)^2 + 20(400) + 500 = 6740$, or \$6740.

43. a. $R(x) = \int R'(x)\,dx = \int(-0.03x + 60)\,dx = -0.015x^2 + 60x + C$. $R(0) = 0$ implies that $C = 0$, so $R(x) = -0.015x^2 + 60x$.

b. From $R(x) = px$, we have $-0.015x^2 + 60x = px$, and so $p = -0.015x + 60$.

44. a. We have the initial-value problem $T'(t) = 0.15t^2 - 3.6t + 14.4$ with $T(0) = 24$. Integrating, we find $T(t) = \int T'(t)\,dt = \int\left(0.15t^2 - 3.6t + 14.4\right)dt = 0.05t^3 - 1.8t^2 + 14.4t + C$. Using the initial condition, we find $T(0) = 24 = 0 + C$, so $C = 24$. Therefore, $T(t) = 0.05t^3 - 1.8t^2 + 14.4t + 24$.

b. The temperature at 10 a.m. was $T(4) = 0.05(4)^3 - 1.8(4)^2 + 14.4(4) + 24 = 56$, or 56°F.

45. a. The total number of DVDs sold as of year t is $T(t) = \int R(t)\,dt = \int\left(-0.03t^2 + 0.218t - 0.032\right)dt = -0.01t^3 + 0.109t^2 - 0.032t + C$. Using the condition $T(0) = 0.1$, we find $T(0) = C = 0.1$. Therefore, $T(t) = -0.01t^3 + 0.109t^2 - 0.032t + 0.1$.

b. The total number of DVDs sold in 2003 is $T(4) = -0.01(4)^3 + 0.109(4)^2 - 0.032(4) + 0.1 = 1.076$, or 1.076 billion.

46. $C(t) = \int C'(t)\,dt = \int\left(0.003t^2 + 0.06t + 0.1\right)dt = 0.001t^3 + 0.03t^2 + 0.1t + k$. But $C(0) = 2$, so $C(0) = k = 2$. Therefore, $C(t) = 0.001t^3 + 0.03t^2 + 0.1t + 2$. The pollution five years from now will be $C(5) = 0.001(5)^3 + 0.03(5)^2 + 0.1(5) + 2 = 3.375$, or 3.375 parts per million.

47. $C(x) = \int C'(x)\,dx = \int\left(0.00003x^2 - 0.03x + 10\right)dx = 0.00001x^3 - 0.015x^2 + 10x + k$. Now $C(0) = 600$ implies that $k = 600$, so $C(x) = 0.00001x^3 - 0.015x^2 + 10x + 600$. The total cost incurred in producing the first 500 corn poppers is $C(500) = 0.00001(500)^3 - 0.015(500)^2 + 10(500) + 600 = 3100$, or \$3100.

48. The number is $\int_0^{10}\left(0.00933t^3 + 0.019t^2 - 0.10833t + 1.3467\right)dt = \left(0.0023325t^4 + 0.0063333t^3 - 0.054165t^2 + 1.3467t\right)\Big|_0^{10} = 37.7$, or approximately 37.7 million Americans.

49. Using the substitution $u = 1 + 0.4t$, we find that $N(t) = \int 3000(1+0.4t)^{-1/2}\,dt = \frac{3000}{0.4}\cdot 2(1+0.4t)^{1/2} + C = 15{,}000\sqrt{1+0.4t} + C$. $N(0) = 100{,}000$ implies $15{,}000 + C = 100{,}000$, so $C = 85{,}000$. Therefore, $N(t) = 15{,}000\sqrt{1+0.4t} + 85{,}000$. The number using the subway six months from now will be $N(6) = 15{,}000\sqrt{1+2.4} + 85{,}000 \approx 112{,}659$.

50. Let $u = 5 - x$, so $du = -x\,dx$. Then $p(x) = \displaystyle\int \frac{240}{(5-x)^2}\,dx = 240\int (5-x)^{-2}\,dx = 240\int \left(-u^{-2}\right) du = 240u^{-1} + C = \frac{240}{5-x} + C$. Next, the condition $p(2) = 50$ gives $\frac{240}{3} + C = 80 + C = 50$, so $C = -30$. Therefore, $p(x) = \dfrac{240}{5-x} - 30$.

51. **a.** The online retail sales will be
$S(t) = \int R(t)\,dt = 15.82 \int e^{-0.176t}\,dt = -\frac{15.82}{0.176}e^{-0.176t} + C = -89.89e^{-0.176t} + C$.
$S(0) = 116$ implies that $-89.89 + C = 116$, so $C = 205.89$. Therefore, $S(t) = 205.89 - 89.89e^{-0.176t}$.

b. The sales will be $S(4) = 205.89 - 89.89e^{-0.176(4)} \approx 161.43$, or \$161.43 billion.

52. The total number of systems that Vista may expect to sell t months from the time they are put on the market is given by $f(t) = 3000t - 50{,}000\left(1 - e^{-0.04t}\right)$. The number is

$$\int_0^{12}\left(3000 - 2000e^{-0.04t}\right)dt = \int\left(3000 - 2000e^{-0.04t}\right)dt = \left(3000t - \tfrac{2000}{-0.04}e^{-0.04t}\right)\Big|_0^{12}$$
$$= 3000(12) + 50{,}000e^{-0.48} - 50{,}000 = 16{,}939.$$

53. The number of speakers sold at the end of 5 years is
$f(t) = \int f'(t)\,dt = \int_0^5 2000\left(3 - 2e^{-t}\right)dt = 2000\left[3(5) - 2e^{-5}\right] - 2000\left[3 - 2(1)\right] = 26{,}027$.

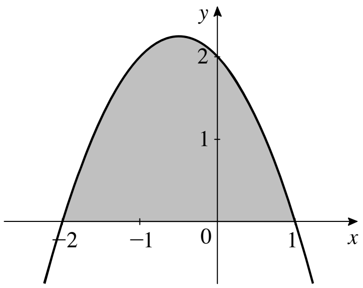
54. $A = \int_{-1}^{2}\left(3x^2 + 2x + 1\right)dx = \left(x^3 + x^2 + x\right)\Big|_{-1}^{2} = \left(2^3 + 2^2 + 2\right) - \left[(-1)^3 + 1 - 1\right] = 14 - (-1) = 15$.

55. $A = \int_0^2 e^{2x}\,dx = \frac{1}{2}e^{2x}\Big|_0^2 = \frac{1}{2}\left(e^4 - 1\right)$.

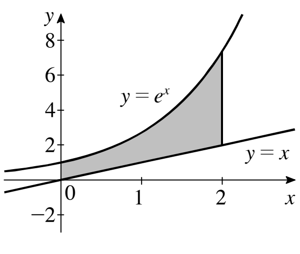
56. $A = \displaystyle\int_1^3 \frac{1}{x^2}\,dx = \int_1^3 x^{-2}\,dx = -\frac{1}{x}\Big|_1^3 = -\frac{1}{3} + 1 = \frac{2}{3}$.

57. The graph of y intersects the x-axis at $x = -2$ and $x = 1$, so

$$A = \int_{-2}^{1}\left(-x^2 - x + 2\right)dx = \left(-\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + 2x\right)\Big|_{-2}^{1}$$
$$= \left(-\tfrac{1}{3} - \tfrac{1}{2} + 2\right) - \left(\tfrac{8}{3} - \tfrac{4}{2} - 4\right) = \tfrac{7}{6} + \tfrac{10}{3} = \tfrac{9}{2}.$$

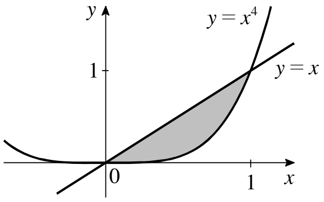


58. $A = \int_a^b \left[f(x) - g(x)\right]dx = \int_0^2 \left(e^x - x\right)dx = \left(e^x - \tfrac{1}{2}x^2\right)\Big|_0^2$
$= \left(e^2 - 2\right) - (1 - 0) = e^2 - 3$.

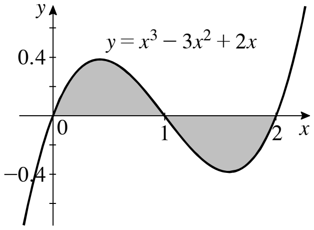


59. To find the points of intersection of the two curves, we solve $x^4 = x$, obtaining $x\left(x^3 - 1\right) = 0$, and so $x = 0$ or 1. Thus,

$$A = \int_0^1 \left(x - x^4\right) dx = \left(\tfrac{1}{2}x^2 - \tfrac{1}{5}x^5\right)\Big|_0^1 = \tfrac{1}{2} - \tfrac{1}{5} = \tfrac{3}{10}.$$



60.
$$\begin{aligned} A &= \int_0^1 \left(x^3 - 3x^2 + 2x\right) dx - \int_1^2 \left(x^3 - 3x^2 + 2x\right) dx \\ &= \left(\tfrac{1}{4}x^4 - x^3 + x^2\right)\Big|_0^1 - \left(\tfrac{1}{4}x^4 - x^3 + x^2\right)\Big|_1^2 \\ &= \tfrac{1}{4} - 1 + 1 - \left[(4 - 8 + 4) - \left(\tfrac{1}{4} - 1 + 1\right)\right] \\ &= \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}. \end{aligned}$$



61. The amount of additional oil that will be produced over the next ten years is given by

$$\begin{aligned} \int_0^{10} \left[R_2(t) - R_1(t)\right] dt &= \int_0^{10} \left(100e^{0.08t} - 100e^{0.05t}\right) dt = 100 \int_0^{10} \left(e^{0.08t} - e^{0.05t}\right) dt \\ &= \left(\tfrac{100}{0.08}e^{0.08t} - \tfrac{100}{0.05}e^{0.05t}\right)\Big|_0^{10} = 1250e^{0.8} - 2000e^{0.5} - 1250 + 2000 \\ &= 2781.9 - 3297.4 - 1250 + 2000 = 234.5, \text{ or } 234{,}500 \text{ barrels.} \end{aligned}$$

62. $A = \dfrac{1}{3}\displaystyle\int_0^3 \frac{x}{\sqrt{x^2 + 16}}\,dx = \tfrac{1}{3} \cdot \tfrac{1}{2} \cdot 2\left(x^2 + 16\right)^{1/2}\Big|_0^3 = \tfrac{1}{3}\left(x^2 + 16\right)^{1/2}\Big|_0^3 = \tfrac{1}{3}(5 - 4) = \tfrac{1}{3}$.

63. The average temperature is

$$\tfrac{1}{12}\int_0^{12} \left(-0.05t^3 + 0.4t^2 + 3.8t + 5.6\right) dt = \tfrac{1}{12}\left(-\tfrac{0.05}{4}t^4 + \tfrac{0.4}{3}t^3 + 1.9t^2 + 5.6t\right)\Big|_0^{12} = 26°\text{F}.$$

64. $\overline{A} = \tfrac{1}{5}\int_0^5 \left(\tfrac{1}{12}t^2 + 2t + 44\right) dt = \tfrac{1}{5}\left(\tfrac{1}{36}t^3 + t^2 + 44t\right)\Big|_0^5 = \tfrac{1}{5}\left(\tfrac{125}{36} + 25 + 220\right) = \dfrac{125 + 900 + 7920}{5(36)}$

≈ 49.69, or 49.7 ft/sec.

65. The average rate of growth between $t = 0$ and $t = 9$ is

$$\begin{aligned} \tfrac{1}{9-0}\int_0^9 R(t)\,dt &= \tfrac{1}{9}\int_0^9 \left(-0.0039t^2 + 0.0374t + 0.0046\right) dt = \tfrac{1}{9}\left(-0.0013t^3 + 0.0187t^2 + 0.0046t\right)\Big|_0^9 \\ &= \tfrac{1}{9}\left[-0.0013\left(9^3\right) + 0.0187\left(9^2\right) + 0.0046(9)\right] = 0.0676, \text{ or } 67{,}600/\text{yr.} \end{aligned}$$

66. Setting $p = 8$, we have $-0.01x^2 - 0.2x + 23 = 8$, $-0.01x^2 - 0.2x + 15 = 0$, and so $x^2 + 20x - 1500 = (x - 30)(x + 50) = 0$, giving $x = -50$ or 30. Thus,

$$\begin{aligned} CS &= \int_0^{30} \left(-0.01x^2 - 0.2x + 23\right) dx - 8(30) = \left(-\tfrac{0.01}{3}x^3 - 0.1x^2 + 23x\right)\Big|_0^{30} - 240 \\ &= -\tfrac{0.01}{3}(30)^3 - 0.1(900) + 23(30) - 240 = 270, \text{ or } \$270{,}000. \end{aligned}$$

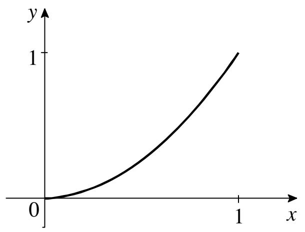
67. To find the equilibrium point, we solve $0.1x^2 + 2x + 20 = -0.1x^2 - x + 40$, obtaining $0.2x^2 + 3x - 20 = 0$, $x^2 + 15x - 100 = 0$, $(x + 20)(x - 5) = 0$, and so $x = 5$. Therefore, $p = -0.1(25) - 5 + 40 = 32.5$, and $CS = \int_0^5 \left(-0.1x^2 - x + 40\right) dx - (5)(32.5) = \left(-\frac{0.1}{3}x^3 - \frac{1}{2}x^2 + 40x\right)\Big|_0^5 - 162.5 = 20.833$, or \$2083. Also, $PS = 5(32.5) - \int_0^5 \left(0.1x^2 + 2x + 20\right) dx = 162.5 - \left(\frac{0.1}{3}x^3 + x^2 + 20x\right)\Big|_0^5 \approx 33.33$, or \$3333.

68. Use Equation (17) with $P = 4000$, $r = 0.08$, $T = 20$, and $m = 1$ to get $A = \frac{1 \cdot 4000}{0.08}\left(e^{1.6} - 1\right) \approx 197{,}651.62$. That is, Chi-Tai will have approximately \$197,652 in his account after 20 years.

69. Use Equation (18) with $P = 925$, $m = 12$, $T = 30$, and $r = 0.06$ to get $PV = \frac{mP}{r}\left(1 - e^{-rT}\right) = \frac{12 \cdot 925}{0.06}\left(1 - e^{-0.06 \cdot 30}\right) = 154{,}419.71$. We conclude that the present value of the purchase price of the house is $154{,}419.71 + 20{,}000$, or approximately \$174,420.

70. Here $P = 80{,}000$, $m = 1$, $T = 10$, and $r = 0.1$, so $PV = \frac{1 \cdot 80{,}000}{0.1}\left(1 - e^{-1}\right) \approx 505{,}696$, or approximately \$505,696.

71. a.



b. $f(0.3) = \frac{17}{18}(0.3)^2 + \frac{1}{18}(0.3) \approx 0.1$. Thus, 30% of the people receive 10% of the total income.
$f(0.6) = \frac{17}{18}(0.6)^2 + \frac{1}{18}(0.6) \approx 0.37$, so 60% of the people receive 37% of the total income.

c. The coefficient of inequality for this curve is
$L = 2\int_0^1 \left(x - \frac{17}{18}x^2 - \frac{1}{18}x\right) dx = \frac{17}{9}\int_0^1 \left(x - x^2\right) dx = \frac{17}{9}\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{17}{54} \approx 0.315$.

72. The average population will be $\frac{1}{5}\int 80{,}000e^{0.05t}\,dt = \frac{80{,}000}{5}\left(\frac{1}{0.05}\right)e^{0.05t}\Big|_0^5 = 320{,}000\left(e^{0.25} - 1\right) \approx 90{,}888$.

CHAPTER 6 Before Moving On... page 500

1. $\int \left(2x^3 + \sqrt{x} + \frac{2}{x} - \frac{2}{\sqrt{x}}\right) dx = 2\int x^3\,dx + \int x^{1/2}\,dx + 2\int \frac{1}{x}\,dx - 2\int x^{-1/2}\,dx$
$= \frac{1}{2}x^4 + \frac{2}{3}x^{3/2} + 2\ln|x| - 4x^{1/2} + C.$

2. $f(x) = \int f'(x)\,dx = \int (e^x + x)\,dx = e^x + \frac{1}{2}x^2 + C$. $f(0) = 2$ implies $f(0) = e^0 + 0 + C = 2$, so $C = 1$. Therefore, $f(x) = e^x + \frac{1}{2}x^2 + 1$.

3. Let $u = x^2 + 1$, so $du = 2x\,dx$ or $x\,dx = \frac{1}{2}du$. Then
$\int \frac{x}{\sqrt{x^2 + 1}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\left(2u^{1/2}\right) + C = \sqrt{u} + C = \sqrt{x^2 + 1} + C.$

4. Let $u = 2 - x^2$, so $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}\,du$. If $x = 0$, then $u = 2$ and if $x = 1$, then $u = 1$. Therefore, $\int_0^1 x\sqrt{2-x^2}\,dx = -\frac{1}{2}\int_2^1 u^{1/2}\,du = -\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\Big|_2^1 = -\frac{1}{3}u^{3/2}\Big|_2^1 = -\frac{1}{3}\left(1 - 2^{3/2}\right) = \frac{1}{3}\left(2\sqrt{2} - 1\right)$.

5. To find the points of intersection, we solve $x^2 - 1 = 1 - x$, obtaining $x^2 + x - 2 = 0$, $(x+2)(x-1) = 0$, and so $x = -2$ or $x = 1$. The points of intersection are $(-2, 3)$ and $(1, 0)$. Thus, the required area is
$$A = \int_{-2}^1 \left[(1-x) - \left(x^2-1\right)\right] dx = \int_{-2}^1 \left(2 - x - x^2\right) dx = \left(2x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_{-2}^1$$
$$= \left(2 - \tfrac{1}{2} - \tfrac{1}{3}\right) - \left(4 - 2 + \tfrac{8}{3}\right) = \tfrac{9}{2}.$$

CHAPTER 6 Explore & Discuss

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$F(2) = \frac{2}{15}(8+1)^{5/2} = \frac{162}{5}$. To find $F(2)$ using the function G, we have to compute $G(u)$, where $u = 2^3 + 1 = 9$, obtaining $G(9) = \frac{2}{15} \cdot 9^{5/2} = \frac{162}{5}$. We use the value 9 for u because $u = x^3 + 1$ and when $x = 2$, $u = 2^3 + 1 = 9$.

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1. Let $u = ax + b$, so that $du = a\,dx$ and $dx = \frac{1}{a}\,du$. Then
$$\int f(ax+b)\,dx = \int f(u) \cdot \tfrac{1}{a}du = \tfrac{1}{a}\int f(u)\,du = \tfrac{1}{a}F(u) + C = \tfrac{1}{a}F(ax+b) + C.$$

2. In order to evaluate $\int (2x+3)^5\,dx$, we write $f(u) = u^5$ so that $F(u) = \int f(u)\,du = \int u^5\,du = \frac{1}{6}u^6 + C$. Next, identifying $a = 2$ and $b = 3$, we obtain $\int (2x+3)^5\,dx = \frac{1}{2}F(2x+3)^6 + C = \frac{1}{2} \cdot \frac{1}{6}(2x+3)^6 + C = \frac{1}{12}(2x+3)^6 + C$. In order to evaluate $\int e^{3x-2}\,dx$, we let $f(x) = e^u$, $a = 3$, and $b = -2$. We find $\int e^{3x-2}\,dx = \frac{1}{3}e^{3x-2} + C$.

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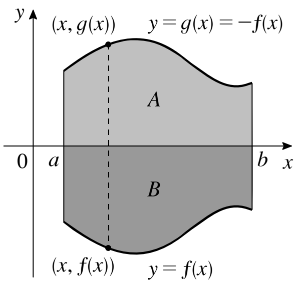
Observe that for each x in $[a, b]$, the point $(x, -f(x))$ is the mirror image of the point $(x, f(x))$ with respect to the x-axis. Therefore, the graph of the function $g = -f$ is symmetric to that of f with respect to the x-axis. Therefore, the area of A is equal to the area of B. But $g(x) = -f(x) \geq 0$ for all x in $[a, b]$, so that area is equal to $\int g(x)\,dx = \int \left[-f(x)\right] dx = -\int f(x)\,dx$, as was to be shown.



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1. A formal application of Equation (9) would seem to yield $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^1 = -1 - 1 = -2.$

2. The indicated observation would appear to follow.

3. Because $f(x) = 1/x^2$ is not continuous on the interval $[-1, 1]$, the fundamental theorem of calculus is not applicable. Thus, the result obtained in Exercise 1 is not valid. Furthermore, the fact that this result is suspect is suggested by the observation made in Exercise 2.

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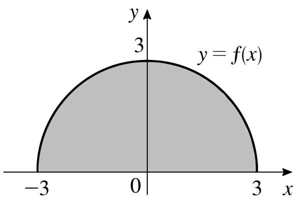

The graph of the integrand $y = f(x) = \sqrt{9 - x^2}$ is the upper semicircle with radius 3 centered at the origin. Interpreting the given integral as the area under the graph of f we find $\int_{-3}^{3} \sqrt{9 - x^2}\, dx = \frac{1}{2}(\pi)(3)^2 = \frac{9\pi}{2}$.

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The required area is

$$\int_0^2 f(x)\, dx = \int_0^1 f(x)\, dx + \int_1^2 f(x)\, dx = \int_0^1 x^{1/2}\, dx + \int_1^2 (1/x)\, dx = \left(\tfrac{2}{3}x^{3/2}\right)\Big|_0^1 + (\ln x)|_1^2$$
$$= \tfrac{2}{3} + (\ln 2 - \ln 1) = \tfrac{2}{3} + \ln 2.$$

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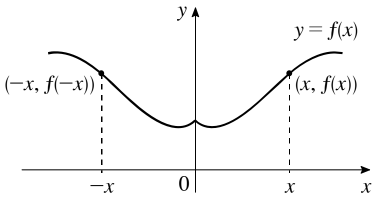
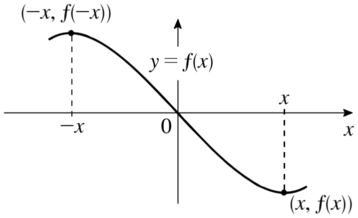


f is even



f is odd

Suppose f is even, so that $f(-x) = f(x)$. If $(x, f(x))$ is any point lying on the graph of f, then $(-x, f(-x)) = (-x, f(x))$, and thus the graph of f is symmetric with respect to the y-axis. If f is odd, then $f(-x) = -f(x)$, and so f is symmetric with respect to the x-axis. Finally, if f is even, then $\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx$.Let $u = -x$ in the first integral on the right-hand side. Then $du = -dx$, if $x = -a$, then $u = a$, and if $x = 0$, then $u = 0$. Using Property 2 of the definite integral, we have $\int_{-a}^{a} f(x)\, dx = \int_a^0 f(-u)(-du) + \int_0^a f(x)\, dx = \int_0^a f(u)\, du + \int_0^a f(x)\, dx$ because f is even. But u is a dummy variable and can be replaced with x, so the expression is equal to $2\int_0^a f(x)\, dx$. If f is odd, a similar argument gives

$$\int_{-a}^{a} f(x)\, dx = \int_a^0 f(x)\, dx + \int_0^a f(x)\, dx = -\int_0^a f(-u)(-du) + \int_0^a f(x)\, dx$$
$$= -\int_0^a f(u)\, du + \int_0^a f(x)\, dx = 0.$$

CHAPTER 6 Exploring with Technology

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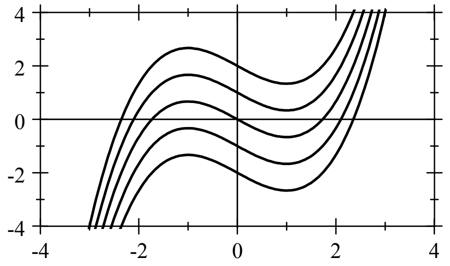
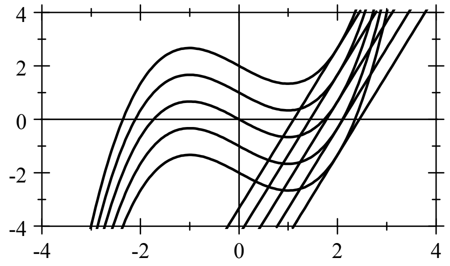
1. $F'(x) = \frac{1}{3}\left(3x^2\right) - 1 = x^2 - 1 = f(x)$, and so F is an antiderivative of f.

2.



3.



4. The slope of the tangent line is $f(2) = 4 - 1 = 3$.

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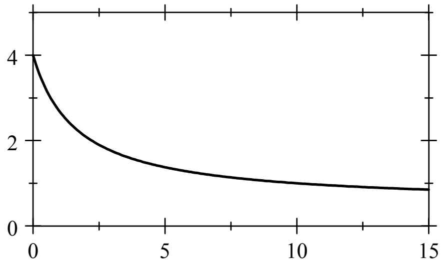
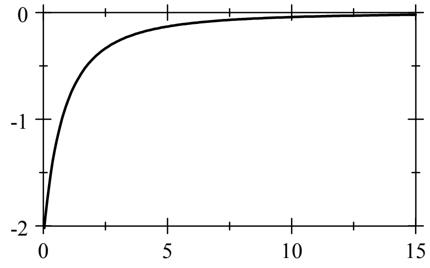
1.



$C'(12) \approx -0.0312314$.

2.



$C'(12) \approx -0.0312314$. Because C' is the derivative of C, we can find $C'(12)$ either by taking the derivative of C at $t = 12$ or by evaluating $C'(t)$ at $t = 12$.

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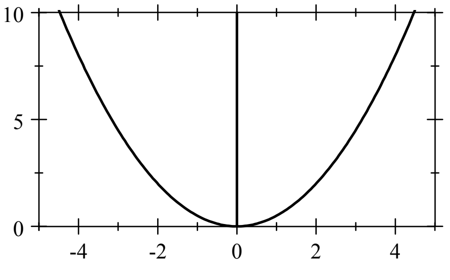
1.



2. The two graphs are identical.

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1. $\int_0^4 x\sqrt{9+x^2}\,dx \approx 32.6666666667$.

2. $\frac{1}{2}\int_9^{25} \sqrt{u}\,du = \frac{1}{2} \cdot \frac{2}{3}u^{3/2}\Big|_9^{25} = \frac{1}{3}\left(25^{3/2} - 9^{3/2}\right) = \frac{98}{3}$.

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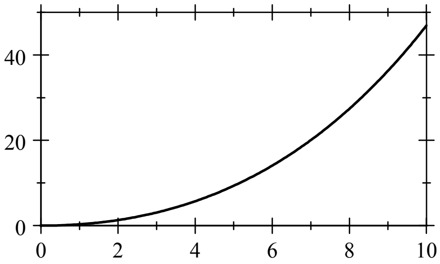
1. $F(x) = \int_0^x [R_1(t) - R(t)]\,dt = 20\left(\dfrac{e^{0.08t}}{0.08} - \dfrac{e^{0.05t}}{0.05}\right)\Bigg|_0^x = \left(250e^{0.08t} - 400e^{0.05t}\right)\Big|_0^x$

$= 250e^{0.08x} - 400e^{0.05x} + 150.$

2.



3. $F(5) = 250e^{0.08(5)} - 400e^{0.05(5)} + 150 = 9.346.$

4. The advantage of this model is that we can easily find the amount of oil saved by evaluating the function at the appropriate value of x.

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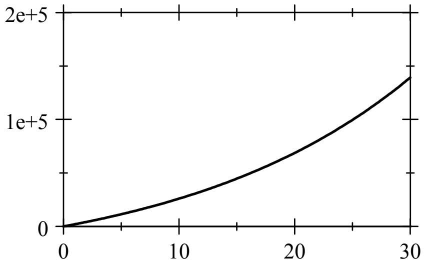
1. With $P = 2000$, $r = 0.05$, $m = 1$, and $T = x$, we obtain $A = f(x) = \frac{2000}{0.05}\left(e^{0.05x} - 1\right) = 40{,}000\left(e^{0.05x} - 1\right)$.

2.



3. The advantage of the model is that we can compute the amount that Marcus will have in his IRA at any time T simply by evaluating the function f at $x = T$.

7 ADDITIONAL TOPICS IN INTEGRATION

7.1 Integration by Parts

Concept Questions page 507

1. $\int u\,dv = uv - \int v\,du$

2. See page 502 of the text. For the integral $\int x^2e^{-x}\,dx$, we let $u = x^2$ and $dv = e^{-x}\,dx$ so that $du = 2x\,dx$ and $v = -e^{-x}$. This leads to $\int x^2e^{-x}\,dx = -x^2e^{-x} + 2\int xe^{-x}\,dx$. The integral on the right can be evaluated by parts with $u = x$ and $dv = e^{-x}\,dx$. If we had chosen the substitution $u = e^{-x}$ and $dv = x^2\,dx$, for example, then $du = -e^{-x}\,dx$ and $v = \frac{1}{3}x^3$. In this case $\int x^2e^{-x}\,dx = \frac{1}{3}x^3e^{-x} + \frac{1}{3}\int x^3e^{-x}\,dx$, and the integral on the right is more difficult to evaluate than the original integral.

Exercises page 507

1. $I = \int xe^{2x}\,dx$. Let $u = x$ and $dv = e^{2x}\,dx$, so $du = dx$ and $v = \frac{1}{2}e^{2x}$. Then $I = uv - \int v\,du = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} = \frac{1}{4}e^{2x}(2x-1) + C$.

2. $I = \int xe^{-x}\,dx$. Let $u = x$ and $dv = e^{-x}\,dx$, so $du = dx$ and $v = -e^{-x}$. Then $\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C = -(x+1)e^{-x} + C$.

3. $I = \int \frac{1}{2}xe^{x/4}\,dx$. Let $u = \frac{1}{2}x$ and $dv = e^{x/4}\,dx$, so $du = \frac{1}{2}dx$ and $v = 4e^{x/4}$. Then $\int \frac{1}{2}xe^{x/4}\,dx = uv - \int v\,du = 2xe^{x/4} - 2\int e^{x/4}\,dx = 2xe^{x/4} - 8e^{x/4} + C = 2(x-4)e^{x/4} + C$.

4. $I = \int 6xe^{3x}\,dx$. Let $u = 6x$ and $dv = e^{3x}\,dx$, so $du = 6\,dx$ and $v = \frac{1}{3}e^{3x}$. Then $I = 2xe^{3x} - \int 2e^{3x}\,dx = 2xe^{3x} - \frac{2}{3}e^{3x} + C = \frac{2}{3}e^{3x}(3x-1) + C$.

5. $\int (e^x - x)^2\,dx = \int \left(e^{2x} - 2xe^x + x^2\right)dx = \int e^{2x}\,dx - 2\int xe^x\,dx + \int x^2\,dx$. Using the result $\int xe^x\,dx = (x-1)e^x + k$, from Example 1, we see that $\int (e^x - x)^2\,dx = \frac{1}{2}e^{2x} - 2(x-1)e^x + \frac{1}{3}x^3 + C$.

6. Using the result from Example 1, we have $\int \left(e^{-x} + x\right)^2\,dx = \int e^{-2x}\,dx + 2\int xe^{-x}\,dx + \int x^2\,dx = -\frac{1}{2}e^{-2x} + 2\left[-(x+1)e^{-x}\right] + \frac{1}{3}x^3 + C = -\frac{1}{2}e^{-2x} - 2(x+1)e^{-x} + \frac{1}{3}x^3 + C$.

7. $I = \int (x+1)e^x\,dx$. Let $u = x+1$ and $dv = e^x\,dx$, so $du = dx$ and $v = e^x$. Then $I = (x+1)e^x - \int e^x\,dx = (x+1)e^x - e^x + C = xe^x + C$.

8. Let $u = x - 3$ and $dv = e^{3x}\,dx$, so $du = dx$ and $v = \frac{1}{3}e^{3x}$. Then $\int (x-3)e^{3x}\,dx = uv - \int v\,du = \frac{1}{3}(x-3)e^{3x} - \frac{1}{3}\int e^{3x}\,dx = \frac{1}{3}(x-3)e^{3x} - \frac{1}{9}e^{3x} + C$.

9. Let $u = x$ and $dv = (x+1)^{-3/2}\,dx$, so $du = dx$ and $v = -2(x+1)^{-1/2}$. Then

$$\begin{aligned}\int x(x+1)^{-3/2}\,dx &= uv - \int v\,du = -2x(x+1)^{-1/2} + 2\int (x+1)^{-1/2}\,dx \\ &= -2x(x+1)^{-1/2} + 4(x+1)^{1/2} + C = 2(x+1)^{-1/2}\left[-x + 2(x+1)\right] + C = \frac{2(x+2)}{\sqrt{x+1}} + C.\end{aligned}$$

10. Let $u = x$ and $dv = (x+4)^{-2}\,dx$, so $du = dx$ and $v = -(x+4)^{-1}$. Then

$$\int x(x+4)^{-2}\,dx = uv - \int v\,du = -x(x+4)^{-1} + \int \frac{1}{x+4}\,dx = -\frac{x}{x+4} + \ln|x+4| + C.$$

11. $I = \int x(x-5)^{1/2}\,dx$. Let $u = x$ and $dv = (x-5)^{1/2}\,dx$, so $du = dx$ and $v = \frac{2}{3}(x-5)^{3/2}$. Then

$$\begin{aligned}I &= \tfrac{2}{3}x(x-5)^{3/2} - \int \tfrac{2}{3}(x-5)^{3/2}\,dx = \tfrac{2}{3}x(x-5)^{3/2} - \tfrac{2}{3}\cdot\tfrac{2}{5}(x-5)^{5/2} + C \\ &= \tfrac{2}{3}(x-5)^{3/2}\left[x - \tfrac{2}{5}(x-5)\right] + C = \tfrac{2}{15}(x-5)^{3/2}(5x - 2x + 10) + C = \tfrac{2}{15}(x-5)^{3/2}(3x+10) + C.\end{aligned}$$

12. $\displaystyle\int \frac{3x}{\sqrt{2x+3}}\,dx = \int 3x(2x+3)^{-1/2}\,dx$. Let $u = 3x$ and $dv = (2x+3)^{-1/2}\,dx$, so $du = 3dx$ and $\displaystyle v = \int (2x+3)^{-1/2}\,dx = (2x+3)^{1/2} + C$ (using the substitution $u = 2x+3$.) Then

$$\begin{aligned}\int \frac{3x}{\sqrt{2x+3}}\,dx &= 3x(2x+3)^{1/2} - 3\int (2x+3)^{1/2}\,dx = 3x(2x+3)^{1/2} - (2x+3)^{3/2} + C \\ &= (2x+3)^{1/2}\left[3x - (2x+3)\right] + C = (x-3)\sqrt{2x+3} + C.\end{aligned}$$

13. $I = \int x\ln 2x\,dx$. Let $u = \ln 2x$ and $dv = x\,dx$, so $du = \frac{1}{x}\,dx$ and $v = \frac{1}{2}x^2$. Then
$I = \frac{1}{2}x^2\ln 2x - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2\ln 2x - \frac{1}{4}x^2 + C = \frac{1}{4}x^2(2\ln 2x - 1) + C.$

14. Let $u = \ln 2x$ and $dv = x^2\,dx$, so $du = \frac{1}{x}\,dx$ and $v = \frac{1}{3}x^3$. Then
$\int x^2\ln 2x\,dx = \frac{1}{3}x^3\ln 2x - \frac{1}{3}\int x^2\,dx = \frac{1}{3}x^3\ln 2x - \frac{1}{9}x^3 + C.$

15. Let $u = \ln x$ and $dv = x^3\,dx$, so $du = \frac{1}{x}\,dx$, and $v = \frac{1}{4}x^4$. Then
$\int x^3\ln x\,dx = \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3\,dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C = \frac{1}{16}x^4(4\ln x - 1) + C.$

16. $I = \int x^{1/2}\ln x\,dx$. Let $u = \ln x$ and $dv = x^{1/2}\,dx$, so $du = \frac{1}{x}\,dx$ and $v = \frac{2}{3}x^{3/2}$. Then
$I = \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{1/2}\,dx = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C = \frac{2}{9}x^{3/2}(3\ln x - 2) + C.$

17. Let $u = \ln x^{1/2}$ and $dv = x^{1/2}\,dx$, so $du = \frac{1}{2x}\,dx$ and $v = \frac{2}{3}x^{3/2}$. Then

$$\begin{aligned}\int \sqrt{x}\ln\sqrt{x}\,dx &= uv - \int v\,du = \tfrac{2}{3}x^{3/2}\ln x^{1/2} - \tfrac{1}{3}\int x^{1/2}\,dx = \tfrac{2}{3}x^{3/2}\ln x^{1/2} - \tfrac{2}{9}x^{3/2} + C \\ &= \tfrac{2}{9}x\sqrt{x}\left(3\ln\sqrt{x} - 1\right) + C.\end{aligned}$$

18. $I = \int x^{-1/2}\ln x\,dx$. Let $u = \ln x$ and $dv = x^{-1/2}\,dx$, so $du = \frac{1}{x}\,dx$ and $v = 2x^{1/2}$. Then
$I = 2x^{1/2}\ln x - \int 2x^{-1/2}\,dx = 2x^{1/2}\ln x - 4x^{1/2} + C = 2\sqrt{x}(\ln x - 2) + C.$

19. Let $u = \ln x$ and $dv = x^{-2}\,dx$, so $du = \frac{1}{x}\,dx$ and $v = -x^{-1}$. Then

$$\int \frac{\ln x}{x^2}\,dx = uv - \int v\,du = -\frac{\ln x}{x} + \int x^{-2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C = -\frac{1}{x}(\ln x + 1) + C.$$

20. Let $u = \ln x$ and $dv = x^{-3}\,dx$, so $du = \dfrac{dx}{x}$ and $v = -\frac{1}{2}x^{-2}$. Then

$$\int \frac{\ln x}{x^3}\,dx = -\frac{\ln x}{2x^2} + \frac{1}{2}\int x^{-3}\,dx = -\frac{\ln x}{2x^2} - \tfrac{1}{4}x^{-2} + C = -\frac{1}{4x^2}\,(2\ln x + 1) + C.$$

21. $\int xe^{(x+1)^2}\,dx = \int xe^{x^2+2x+1}\,dx = \int \left(xe^{x^2} + xe^{2x} + ex\right)dx = \int xe^{x^2}\,dx + \int xe^{2x}\,dx + e\int x\,dx$

$= \frac{1}{2}e^{x^2} + I + \frac{1}{2}ex^2 + C_1.$

To find $I = \int xe^{2x}\,dx$, we integrate by parts with $u = x$ and $dv = e^{2x}\,dx$, so $du = dx$ and $v = \frac{1}{2}e^{2x}$. Then $I = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C_2$. Finally, $\int xe^{(x+1)^2}\,dx = \frac{1}{2}e^{x^2} + \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C_2\right) + \frac{1}{2}ex^2 + C_1 = \frac{1}{2}e^{x^2} + \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + \frac{1}{2}ex^2 + C$, where $C = C_1 + C_2$.

22. $\int \ln\left(xe^{x^2}\right)dx = \int \left(\ln x + \ln e^{x^2}\right)dx = \int \left(\ln x + x^2\right)dx = \int \ln x\,dx + \int x^2\,dx$. To evaluate $\int \ln x\,dx$, let $u = \ln x$ and $dv = dx$, so $du = \frac{1}{x}\,dx$ and $v = x$. Then $\int \ln x\,dx = uv - \int v\,du = x\ln x - \int dx = x\ln x - x + C = x\,(\ln x - 1) + C$. Thus, $\int \ln\left(xe^{x^2}\right)dx = x\ln x - x + \frac{1}{3}x^3 + C$.

23. Let $u = \ln x$ and $dv = dx$, so $du = \frac{1}{x}\,dx$ and $v = x$. Then $\int \ln x\,dx = uv - \int v\,du = x\ln x - \int dx = x\ln x - x + C = x\,(\ln x - 1) + C$.

24. $I = \int \ln(x+1)\,dx$. Let $u = \ln(x+1)$ and $dv = dx$, so $du = \dfrac{dx}{x+1}$ and $v = x$. Then $I = x\,(\ln x + 1) - \displaystyle\int \frac{x}{x+1}\,dx$. Now to evaluate $J = \displaystyle\int \frac{x}{x+1}\,dx$, let $u = x + 1$ so that $du = dx$. Then $J = \displaystyle\int \frac{u-1}{u}\,du = \int \left(1 - \frac{1}{u}\right)du = u - \ln|u| + k$, where k is a constant of integration. Thus, $I = x\ln(x+1) - (x+1) + \ln|x+1| + C = (x+1)\,(\ln|x+1| - 1) + C$.

25. Let $u = x^2$ and $dv = e^{-x}\,dx$, so $du = 2x\;dx$ and $v = -e^{-x}$. Then $\int x^2e^{-x}\,dx = uv - \int v\,du = -x^2e^{-x} + 2\int xe^{-x}\,dx$. We can integrate by parts again or, using the result of Exercise 2, we can write

$\int x^2e^{-x}\,dx = -x^2e^{-x} + 2\left[-(x+1)\,e^{-x}\right] + C = -x^2e^{-x} - 2\,(x+1)\,e^{-x} + C$

$= -\left(x^2 + 2x + 2\right)e^{-x} + C.$

26. Let $u = \sqrt{x} = x^{1/2}$, so that $du = \frac{1}{2}x^{-1/2}\,dx$ and $dx = 2\sqrt{x}\,du = 2u\,du$. Then

$\int e^{-\sqrt{x}}\,dx = 2\int ue^{-u}du = -2\,(u+1)\,e^{-u} + C$ (using the result of Exercise 2)

$= -2\left(\sqrt{x} + 1\right)e^{-\sqrt{x}} + C$ (since $u = \sqrt{x}$).

27. $I = \int x\,(\ln x)^2\;dx$. Let $u = (\ln x)^2$ and $dv = x\,dx$, so $du = 2\,(\ln x)\left(\dfrac{1}{x}\right) = \dfrac{2\ln x}{x}$ and $v = \frac{1}{2}x^2$. Then $I = \frac{1}{2}x^2\,(\ln x)^2 - \int x\ln x\,dx$. Next, we evaluate $\int x\ln x\,dx$ by letting $u = \ln x$ and $dv = x\,dx$, so $du = \frac{1}{x}\,dx$ and $v = \frac{1}{2}x^2$. Then $\int x\ln x\,dx = \frac{1}{2}x^2\,(\ln x) - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$. Therefore, $\int x\,(\ln x)^2\;dx = \frac{1}{2}x^2\,(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C = \frac{1}{4}x^2\left[2\,(\ln x)^2 - 2\ln x + 1\right] + C$.

28. Let $u = x + 1$, so $du = dx$. Then $\int x \ln(x+1)\,dx = \int (u-1)\ln u\,du = \int u\ln u\,du - \int \ln u\,du$. Integrating each integral by parts, or using the results of Example 2 and Exercise 23, we obtain $\int u\ln u\,du - \int \ln u\,du = \frac{1}{4}u^2(2\ln u - 1) - u(\ln u - 1) + C$. Thus, $\int x\ln(x+1)\,dx = \frac{1}{4}(x+1)^2[2\ln(x+1) - 1] - (x+1)[\ln(x+1) - 1] + C$.

29. $\int_0^{\ln 3} xe^x\,dx = (x-1)e^x\big|_0^{\ln 3}$ (using the results of Example 1)

$= (\ln 3 - 1)e^{\ln 3} - \left(-e^0\right) = 3(\ln 3 - 1) + 1 = 3\ln 3 - 2$ (because $e^{\ln 3} = 3$).

30. Using the results of Exercise 2, we have $\int_0^2 xe^{-x}\,dx = -(x+1)e^{-x}\big|_0^2 = -3e^{-2} + 1$.

31. We first integrate $I = \int \ln x\,dx$. Using parts with $u = \ln x$ and $dv = dx$, so $du = \frac{1}{x}\,dx$ and $v = x$, we have $I = x\ln x - \int dx = x\ln x - x + C = x(\ln x - 1) + C$. Therefore, $3\int_1^4 \ln x\,dx = 3x(\ln x - 1)\big|_1^4 = 3[4(\ln 4 - 1) - 1(\ln 1 - 1)] = 12\ln 4 - 9 = 3(4\ln 4 - 3)$.

32. Using the result of Example 2, we find $\int_1^2 x\ln x\,dx = \frac{1}{4}x^2(2\ln x - 1)\big|_1^2 = 2\ln 2 - 1 + \frac{1}{4} = \frac{1}{4}(8\ln 2 - 3)$.

33. Let $u = x$ and $dv = e^{2x}\,dx$. Then $du = dx$ and $v = \frac{1}{2}e^{2x}$, so $\int_0^2 xe^{2x}\,dx = \frac{1}{2}xe^{2x}\Big|_0^2 - \frac{1}{2}\int_0^2 e^{2x}\,dx = e^4 - \left(\frac{1}{4}e^{2x}\right)\Big|_0^2 = e^4 - \frac{1}{4}e^4 + \frac{1}{4} = \frac{1}{4}\left(3e^4 + 1\right)$.

34. We first integrate $I = \int x^2e^{-x}\,dx$. Using parts with $u = x^2$ and $dv = e^{-x}\,dx$, so $du = 2x\,dx$ and $v = -e^{-x}$, we find that $I = -x^2e^{-x} + 2\int xe^{-x}\,dx$. Now let $J = \int xe^{-x}\,dx$. Integrating J by parts with $u = x$ and $dv = -e^{-x}\,dx$, so $du = dx$ and $v = e^{-x}$, we have $J = \int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$. Therefore, $I = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C = -\left(x^2 + 2x + 2\right)e^{-x} + C$, and so $\int_0^1 x^2e^{-x}\,dx = -\left(x^2 + 2x + 2\right)e^{-x}\big|_0^1 = -5e^{-1} + 2$.

35. Let $u = x$ and $dv = e^{-2x}\,dx$, so $du = dx$ and $v = -\frac{1}{2}e^{-2x}$. Then $f(x) = \int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$. Solving $f(0) = -\frac{1}{4} + C = 3$, we find that $C = \frac{13}{4}$, so $y = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + \frac{13}{4}$.

36. Let $u = x$ and $dv = (x+1)^{1/2}$, so $du = dx$ and $v = \frac{2}{3}(x+1)^{3/2}$. Then $f(x) = \int x(x+1)^{1/2}\,dx = \frac{2}{3}x(x+1)^{3/2} - \frac{2}{3}\int (x+1)^{3/2}\,dx = \frac{2}{3}x(x+1)^{3/2} - \frac{4}{15}(x+1)^{5/2} + C$. $f(3) = 2(8) - \frac{4}{15}(32) + C = 6$ implies that $C = -\frac{22}{15}$, so $f(x) = \frac{2}{3}x(x+1)^{3/2} - \frac{4}{15}(x+1)^{5/2} - \frac{22}{15}$.

37. The required area is given by $\int_1^5 \ln x\,dx$. We first find $\int \ln x\,dx$. Using parts with $u = \ln x$ and $dv = dx$, so $du = \frac{1}{x}\,dx$ and $v = x$, we have $\int \ln x\,dx = x\ln x - \int dx = x\ln x - x = x(\ln x - 1) + C$. Therefore, $\int_1^5 \ln x\,dx = x(\ln x - 1)\big|_1^5 = 5(\ln 5 - 1) - 1(\ln 1 - 1) = 5\ln 5 - 4$, and the required area is $5\ln 5 - 4$.

38. $A = \int_0^3 xe^{-x}\,dx$. Using the result of Exercise 2, we have $A = -(x+1)e^{-x}\big|_0^3 = -4e^{-3} + 1 \approx 0.8$.

39. The distance covered is given by $\int_0^{10} 100te^{-0.2t}\,dt = 100\int_0^{10} te^{-0.2t}\,dt$. We integrate by parts with $u = t$ and $dv = e^{-0.2t}\,dt$, so $du = dt$ and $v = -\frac{1}{0.2}e^{-0.2t} = -5e^{-0.2t}$. Therefore,

$$100\int_0^{10} te^{-0.2t}dt = 100\left(-5te^{-0.2t}\right)\Big|_0^{10} + 500\int_0^{10} e^{-0.2t}\,dt = -5000e^{-2} - \left(2500e^{-0.2t}\right)\Big|_0^{10}$$

$$= -5000e^{-2} - \left(2500e^{-2} - 2500\right) = 2500 - 7500e^{-2} \approx 1485, \text{ or } 1485 \text{ feet.}$$

40. Let P denote the required function. Then $P'(t) = 2te^{-0.05t}$, so $P(t) = 2\int te^{-0.05t}dt$. Put $u = t$ and $dv = e^{-0.05t}\,dt$, so $du = dt$ and $v = -\frac{1}{0.05}e^{-0.05t} = -20e^{-0.05t}$. Then $P(t) = -40te^{-0.05t} + 40\int e^{-0.05t}dt = -40te^{-0.05t} - 800e^{-0.05t} + C = -40(t+20)e^{-0.05t} + C$. We now use the initial condition $P(0) = 20$ to obtain the equation $-800 + C = 20$, so $C = 820$. Therefore, $P(t) = -40(t+20)e^{-0.05t} + 820$. The amount of coal produced by the company in the next 20 years is $P(20) = -40(40)e^{-(0.05)(20)} + 820 \approx 231.39$, or 231.39 million metric tons.

41. $N = 2\int te^{-0.1t}dt$. Let $u = t$ and $dv = e^{-0.1t}$, so $du = dt$ and $v = -10e^{-0.1t}$. Then $N(t) = 2\left(-10te^{-0.1t} + 10\int e^{-0.1t}\,dt\right) = 2\left(-10te^{-0.1t} - 100e^{-0.1t}\right) + C = -20e^{-0.1t}(t+10) + C$. Because $N(0) = -20(10) + C = 0$, $C = 200$. Therefore, $N(t) = -20e^{-0.1t}(t+10) + 200$.

42. The average concentration is $C = \frac{1}{12}\int_0^{12} 3te^{-t/3}\,dt = \frac{1}{4}\int_0^{12} te^{-t/3}\,dt$. Let $u = t$ and $dv = e^{-t/3}\,dt$, so $du = dt$ and $v = -3e^{-t/3}$. Then $C = \frac{1}{4}\left[-3te^{-t/3}\Big|_0^{12} + 3\int_0^{12} e^{-t/3}dt\right] = \frac{1}{4}\left[-36e^{-4} - \left(9e^{-t/3}\right)\Big|_0^{12}\right] = \frac{1}{4}\left(-36e^{-4} - 9e^{-4} + 9\right) \approx 2.04$ mg/ml.

43. The number of accidents is expected to be

$$E = 982 + \int_0^{12}\left(-10 - te^{0.1t}\right)dt = 982 - \int_0^{12} 10\,dt - \int_0^{12} te^{0.1t}\,dt = 982 - 120 - \int_0^{12} te^{0.1t}\,dt$$

$$= 862 - \int_0^{12} te^{0.1t}\,dt.$$

To evaluate the last integral, let $u = t$ and $dv = e^{0.1t}\,dt$, so $du = dt$ and $v = 10e^{0.1t}$. Then $\int_0^{12} te^{0.1t}dt = 10te^{0.1t}\Big|_0^{12} - 10\int_0^{12} e^{0.1t}dt = 120e^{1.2} - \left(100e^{0.1t}\right)\Big|_0^{12} = 120e^{1.2} - 100e^{1.2} + 100 \approx 166$. Therefore, $E \approx 862 - 166 \approx 696$, or approximately 696 accidents.

44. The average price is given by $\frac{1}{4}\int_0^4\left(8 + 4e^{-2t} + te^{-2t}\right)dt$. Integrate the last term by parts with $u = t$ and $dv = e^{-2t}\,dt$, so $du = dt$ and $v = -\frac{1}{2}e^{-2t}$, to obtain $\int te^{-2t}\,dt = -\frac{1}{2}te^{-2t} + \frac{1}{2}\int e^{-2t}\,dt = -\frac{1}{4}e^{-2t}(1+2t) + C$. Then

$$\frac{1}{4}\int_0^4\left(8 + 4e^{-2t} + te^{-2t}\right)dt = \frac{1}{4}\left\{8t - 2e^{-2t} - \left[\frac{1}{4}e^{-2t}(1+2t)\right]_0^4\right\} = \frac{1}{4}\left[8t - 2e^{-2t} - \frac{1}{4}e^{-2t}(1+2t)\right]_0^4$$

$$= \frac{1}{4}\left[8(4) - 2e^{-8} - \frac{1}{4}e^{-8}(1+8) + 2 + \frac{1}{4}(1)\right] = \frac{1}{4}\left(32 - 2e^{-8} - \frac{9}{4}e^{-8} + \frac{9}{4}\right)$$

$$= \frac{1}{4}\left(\frac{137}{4} - \frac{17}{4}e^{-8}\right) \approx 8.56, \text{ or } \$8.56.$$

45. The membership will be $N(5) = N(0) + \int_0^5 9\sqrt{t+1}\ln\sqrt{t+1}\,dt = 50 + 9\int_0^5 \sqrt{t+1}\ln\sqrt{t+1}\,dt$. To evaluate the integral, let $u = t+1$, so $du = dt$. If $t = 0$, then $u = 1$ and if $t = 5$, then $u = 6$, so $9\int_0^5 \sqrt{t+1}\ln\sqrt{t+1}\,dt = 9\int_1^6 \sqrt{u}\ln\sqrt{u}\,du$. Using the results of Exercise 17, we have $\int\sqrt{x}\ln\sqrt{x}\,dx = \frac{2}{9}x\sqrt{x}\left(3\ln\sqrt{x} - 1\right) + C$. Thus, $9\int_1^6 \sqrt{u}\ln\sqrt{u}\,du = \left[2u\sqrt{u}\left(3\ln\sqrt{u} - 1\right)\right]_1^6 = 2(6)\sqrt{6}\left(3\ln\sqrt{6} - 1\right) - 2(-1) \approx 51.606$, and so $N = 50 + 51.606 \approx 101.606$, or 101,606 people.

46. The total distance covered is $D = \int_0^6 8t \ln(t+1)\, dt$. To evaluate $I = \int 8t \ln(t+1)\, dt$, let $u = t+1$, so $du = dt$ and $t = u-1$. Then $I = 8\int (u-1)\ln u\, du = 8\int (u\ln u - \ln u)\, du$. Using the results of Example 2 and Exercise 23, we find $I = 8\left[\frac{1}{4}u^2(2\ln u - 1) - u\ln u + u\right] + C$, so

$$D = \int_2^6 8t\ln(t+1)\, dt = 8\left[\tfrac{1}{4}(t+1)^2[2\ln(t+1) - 1] - (t+1)\ln(t+1) + (t+1)\right]_0^6$$

$$= 140\ln 7 - 48 \approx 224.43 \text{ (ft)}.$$

47. The average annual fraud over the period from 2008 through 2012 is $A = \frac{1}{4-0}\int_0^4 [328.9 - 92.07\ln(t+1)]\, dt$. To find $I = \int \ln(t+1)\, dt$, let $u = t+1$, so $du = dt$ and $t = u-1$. Then $I = \int \ln u\, du = u\ln u - u + C$ (See Exercise 23). Thus, $A = \frac{1}{4}[328.9t - 92.07[(t+1)\ln(t+1) - (t+1)]]_0^4 \approx 235.74$ (million GBP).

48. The projected average percentage of the population using smartphones from 2010 through 2015 is $A = \frac{1}{5-0}\int_0^5 [18.952 + 14.088\ln(t+1)]\, dt$. Using the result of Exercise 23, we find $A = \frac{1}{5}[18.952t + 14.088[(t+1)\ln(t+1) - (t+1)]]_0^5 \approx 35.155$.

49. The value of the income stream at the end of 5 years is

$$F = e^{0.05(5)}\int_0^5 (100 + 20t)e^{-0.05t}\, dt = e^{0.25}\left[\int_0^5 100e^{-0.05t}\, dt + 20\int_0^5 te^{-0.05t}\right] dt$$

$$= e^{0.25}\left\{\left[-2000e^{-0.05t}\right]_0^5 + 20\int_0^5 te^{-0.05t}\, dt\right\}.$$

To find $I = \int te^{-0.05t}\, dt$, use parts with $u = t$ and $dv = e^{-0.05t}\, dt$, so $du = dt$ and $v = -\frac{1}{0.05}e^{-0.05t} = -20e^{-0.05t}$. Thus, $I = -20te^{-0.05t} + \int 20e^{-0.05t}\, dt = -20te^{-0.05t} + 20\left(\frac{1}{-0.05}\right)e^{-0.05t} + C = -20te^{-0.05t} - 400e^{-0.05t} + C$. Finally, $F = e^{0.25}\left[-2000e^{-0.05t} + 20\left(-20te^{-0.05t} - 400e^{-0.05t}\right)\right]_0^5 \approx 840.254$, or approximately \$840,254.

50. At the end of the 5-year period, Laura will have $F = e^{0.03(5)}\int_0^5 10{,}000e^{-0.02t}e^{-0.03t}\, dt = e^{0.15}\int_0^5 10{,}000e^{-0.05t}\, dt = \left.\frac{10{,}000e^{0.15}}{-0.05}e^{-0.05t}\right|_0^5 \approx 51{,}399.365$, or approximately \$51,399.

51. $PV = \int_0^5 (30{,}000 + 800t)e^{-0.05t}\, dt = 30{,}000\int_0^5 e^{-0.05t}\, dt + 800\int_0^5 te^{-0.05t}\, dt$. Let $I = \int te^{-0.05t}\, dt$. To evaluate I by parts, let $u = t$ and $dv = e^{-0.05t}\, dt$, so $du = dt$ and $v = -\frac{1}{0.05}e^{-0.05t} = -20e^{-0.05t}$. Then $I = -20te^{-0.05t} + 20\int e^{-0.05t} dt = -20te^{-0.05t} - 400e^{-0.05t} + C$. Thus,

$$PV = \left[-\frac{30{,}000}{0.05}e^{-0.05t} - 800(20)te^{-0.05t} - 800(400)e^{-0.05t}\right]_0^5$$

$$= -600{,}000e^{-0.25} + 600{,}000 - 80{,}000e^{-0.25} - 320{,}000e^{-0.25} + 320{,}000 = 920{,}000 - 1{,}000{,}000e^{-0.25}$$

$$= 141{,}199.22, \text{ or approximately } \$141{,}199.$$

52. The present value of the franchise is $PV = \int_0^T P(t)\,e^{-rt}\,dt = \int_0^{15} (50{,}000 + 3000t)\,e^{-0.04t}\,dt = 50{,}000\int_0^{15} e^{-0.04t}\,dt + 3000\int_0^{15} te^{-0.04t}\,dt$. The first integral is $50{,}000\int_0^{15} e^{-0.04t}\,dt = \frac{50{,}000}{-0.04}e^{-0.04t}\Big|_0^{15} = -1{,}250{,}000\left(e^{-0.6}-1\right) \approx 563{,}985$. The second integral can be evaluated using parts with $u = t$ and $dv = e^{-0.04t}\,dt$, so $du = dt$ and $v = -25e^{-0.04t}$. Thus,

$$\begin{aligned}3000\int_0^{15} te^{-0.04t}\,dt &= 3000\left[-25te^{-0.04t}\Big|_0^{15} + 25\int_0^{15} e^{-0.04t}\,dt\right] = 3000\left(-375e^{-0.6} - 625\,e^{-0.04t}\Big|_0^{15}\right)\\ &= 3000\left(-375e^{-0.6} - 625e^{-0.6} + 625\right) \approx 228{,}565.\end{aligned}$$

Therefore, $PV \approx 563{,}985 + 228{,}565 = 792{,}550$, or \$792,550.

53. $p = 300 - 2(x+1)\ln(x+1)$. If $x = 20$, then we have $p = 300 - 2(21)\ln 21 \approx 172.130$. Therefore, with $\overline{x} = 20$ and $\overline{p} = 172.130$ we find, using Formula (16) from Section 6.7, $CS \approx \int_1^{20}(300 - 2(x+1)\ln(x+1))\,dx - 172.130\cdot 20$. Using the result of Example 2, $\int (x+1)\ln(x+1)\,dx = \frac{1}{4}(x+1)^2(2\ln(x+1)-1) + C$, so we have $CS \approx \left[300x - 2\left(\frac{1}{4}\right)(x+1)^2(2\ln(x+1)-1)\right]_1^{20} - 3442.60 \approx 4877.87 - 299.23 - 3442.60 = 1136.04$, or approximately \$113,604.

54. $p = x\ln\left(0.1x^2+1\right) + 5$. If $x = 7$, then $p = 7\ln\left((0.1)(7)^2+1\right) + 5 \approx 17.425$, so $PS \approx 17.425\cdot 7 - \int_0^7\left[x\ln\left(0.1x^2+1\right)+5\right]dx$. Let $I = \int x\ln\left(0.1x^2+1\right)dx$. To find I, let $u = 0.1x^2 + 1$, so $du = 0.2x\,dx$ and $x\,dx = 5\,du$. Then $I = 5\int \ln u\,du = 5(u\ln u - u)$ [see Exercise 23] $= 5\left[\left(0.1x^2+1\right)\ln\left(0.1x^2+1\right) - \left(0.1x^2+1\right) + 5x\right]$. Thus, $PS \approx 17.425(7) - \left[5\left\{\left(0.1x^2+1\right)\ln\left(0.1x^2+1\right) - \left(0.1x^2+1\right)\right\} + 5x\right]_0^7 \approx 59.114$, or \$59,114.

55. $L = 2\int_0^1\left[x - xe^{2(x-1)}\right]dx = 2\int_0^1\left(x - xe^{2x}e^{-2}\right)dx = 2\left[\frac{1}{2}x^2 - e^{-2}\int_0^1 xe^{2x}\,dx\right]$. Let $I = \int xe^{2x}\,dx$. To find I, use parts with $u = x$ and $dv = e^{2x}\,dx$. Then $du = dx$ and $v = \frac{1}{2}e^{2x}$, so $I = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$, so $L = 2\left[\frac{1}{2}x^2 - e^{-2}\left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right)\right]_0^1 \approx 0.432$.

56. The average concentration from $r = r_1$ to $r = r_2$ is

$$\begin{aligned}A &= \frac{1}{r^2 - r_1}\int_{r_1}^{r_2} c(r)\,dr = \frac{1}{r^2 - r_1}\int_{r_1}^{r_2}\left[\left(\frac{c_1 - c_2}{\ln r_1 - \ln r_2}\right)(\ln r - \ln r_2) + c_2\right]dr\\ &= \frac{1}{r_2 - r_1}\left(\frac{c_1 - c_2}{\ln r_1 - \ln r_2}\right)\left[\int_{r_1}^{r_2}\ln r\,dr - \int_{r_1}^{r_2}\ln r_2\,dr\right] + \frac{1}{r_2 - r_1}\int_{r_1}^{r_2} c_2\,dr.\end{aligned}$$

We integrate $\int \ln r\,dr$ by parts, letting $u = \ln r$ and $dv = dr$, so $du = \frac{dr}{r}$ and $v = r$. Thus, $\int \ln r\,dr = r\ln r - \int r\frac{dr}{r} = r\ln r - r = r(\ln r - 1)$. Then

$$\begin{aligned}A &= \frac{1}{r_2 - r_1}\left(\frac{c_1 - c_2}{\ln r_1 - \ln r_2}\right)\left\{\left[[r(\ln r - 1)]_{r_1}^{r_2} - (\ln r_2)|_{r_1}^{r_2}\right]\right\} + c_2\\ &= \frac{1}{r_2 - r_1}\left(\frac{c_1 - c_2}{\ln r_1 - \ln r_2}\right)\left[r_2(\ln r_2 - 1) - r_1(\ln r_1 - 1) - (r_2 - r_1)\ln r_2\right] + c_2\\ &= \frac{1}{r_2 - r_1}\left(\frac{c_1 - c_2}{\ln r_1 - \ln r_2}\right)\left[r_1(\ln r_2 - \ln r_1) - (r_2 - r_1)\right] + c_2 = (c_2 - c_1)\left(\frac{r_1}{r_2 - r_1} + \frac{1}{\ln r_1 - \ln r_2}\right) + c_2.\end{aligned}$$

57. **a.** $A(0) = 180\left(1 - e^0\right) - 6(0)e^0 = 0$, or 0 lb.

b. $A(180) = 180\left(1 - e^{-180/30}\right) - 6(180)e^{-180/30} \approx 176.877$, or 176.9 lb.

c. The average amount of salt over the first 3 hours is $\overline{A} = \frac{1}{180}\int_0^{180}\left[180\left(1 - e^{-t/30}\right) - 6te^{-t/30}\right]dt = \int_0^{180}\left(1 - e^{-t/30}\right)dt - \frac{1}{30}\int_0^{180} te^{-t/30}\,dt$. Call the first integral I_1 and the second integral I_2. Then

$$I_1 = \int_0^{180}\left(1 - e^{-t/30}\right)dt = \left(t + 30e^{-t/30}\right)\Big|_0^{180} \text{ [substitute } u = -t/30) = \left(180 + 30e^{-6}\right) - \left(0 + 30e^0\right) \approx 150.074.$$

To evaluate I_2, consider the corresponding indefinite integral $J = \int te^{-t/30}\,dt$. Integrating by parts with $u = t$ and $dv = e^{-t/30}dt$, so $du = dt$ and $v = -30e^{-t/30}$, we find $J = -30te^{-t/30} - \int\left(-30e^{-t/30}\right)dt = -30te^{-t/30} - 900e^{-t/30} + C = -30(t + 30)e^{-t/30} + C$.

Therefore, $I_2 = \frac{1}{30}J\Big|_0^{180} = -(t + 30)e^{-t/30}\Big|_0^{180} = -(180 + 30)e^{-6} + 30e^0 \approx 29.479$. Thus, $\overline{A} = I_1 - I_2 \approx 150.074 - 29.479 = 120.595$, or approximately 120.6 lb.

58. Let $u = x$ and $dv = f''(x)\,dx$, so $du = dx$ and $v = f'(x)$. Then $\int xf''(x)\,dx = xf'(x) - \int f'(x)\,dx = xf'(x) - f(x) + C$. Therefore,

$$\int_1^3 xf''(x)\,dx = \left[xf'(x) - f(x)\right]_1^3 = \left[3f'(3) - f(3)\right] - \left[f'(1) - f(1)\right]$$
$$= [3(-1) - (-1)] - (2 - 2) = -2.$$

59. True. This is the integration by parts formula.

60. True. Use the integration by parts formula with $u = e^x$ and $dv = g'(x)\,dx$, so $du = e^x\,dx$ and $v = g(x)$. Then $\int u\,dv = \int e^x g'(x)\,dx = uv - \int v\,du = e^x g(x) - \int e^x g(x)\,dx$.

61. True. Let $U = uv$ and $dV = dw$, so $dU = u\,dv + v\,du$ and $V = w$. Then $\int uv\,dw = \int U\,dV = UV - \int V\,dU = uvw - \int w(u\,dv + v\,du) = uvw - \int uw\,dv - \int vw\,du$.

62. False. $\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du$.

7.2 Integration Using Tables of Integrals

Concept Questions page 515

1. **a.** Formula (19) seems appropriate.

b. Put $a = \sqrt{2}$ and $x = u$. Then using Formula (19), we have

$$\int \frac{\sqrt{2 - x^2}}{x}\,dx = \int \frac{\sqrt{\left(\sqrt{2}\right)^2 - x^2}}{x}\,dx = \sqrt{2 - x^2} - \sqrt{2}\ln\left|\frac{\sqrt{a} + \sqrt{2 - x^2}}{x}\right| + C.$$

2. **a.** Formula (16) seems appropriate.

b. First, we rewrite the indefinite integral as $\int \frac{dx}{\sqrt{2x^2-5}} = \int \frac{dx}{\sqrt{2\left(x^2-\frac{5}{2}\right)}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2-\left(\sqrt{\frac{5}{2}}\right)^2}}$. Then let $u = x$ and $a = \sqrt{\frac{5}{2}} = \frac{\sqrt{10}}{2}$. Formula (16) then gives $\int \frac{dx}{\sqrt{2x^2-5}} = \frac{1}{\sqrt{2}} \ln \left| x + \sqrt{x^2 - \frac{5}{2}} \right| + C$, and so

$$\int_2^3 \frac{dx}{\sqrt{2x^2-5}} = \frac{1}{\sqrt{2}}\left(\ln\left|3+\sqrt{9-\tfrac{5}{2}}\right| - \ln\left|2+\sqrt{4-\tfrac{5}{2}}\right|\right) = \frac{1}{\sqrt{2}}\left(\ln\left|3+\sqrt{\tfrac{13}{2}}\right| - \ln\left|2+\sqrt{\tfrac{3}{2}}\right|\right)$$
$$\approx 0.3839.$$

Exercises page 515

1. First we note that $\int \frac{2x}{2+3x}\,dx = 2\int \frac{x}{2+3x}\,dx$. Next, we use Formula (1) with $a = 2$, $b = 3$, and $u = x$. Then $\int \frac{2x}{2+3x}\,dx = \frac{2}{9}\left(2+3x-2\ln|2+3x|\right) + C$.

2. Use Formula (3) with $a = 1$, $b = 2$, and $u = x$. Then $\int \frac{x}{(1+2x)^2}\,dx = \frac{1}{4}\left(\frac{1}{1+2x} + \ln|1+2x|\right) + C$.

3. $\int \frac{3x^2}{2+4x}\,dx = \frac{3}{2}\int \frac{x^2}{1+2x}\,dx$. Use Formula (2) with $a = 1$, $b = 2$, and $u = x$, obtaining $\int \frac{3x^2}{2+4x}\,dx = \frac{3}{32}\left[(1+2x)^2 - 4(1+2x) + 2\ln|1+2x|\right] + C$.

4. Use Formula (2) with $a = 3$, $b = 1$, and $u = x$ to obtain $\int \frac{x^2}{3+x}\,dx = \frac{1}{2}\left[(3+x)^2 - 12(3+x) + 18\ln|3+x|\right] + C$.

5. $\int x^2\sqrt{9+4x^2}\,dx = \int x^2\sqrt{4\left(\frac{9}{4}+x^2\right)}\,dx = 2\int x^2\sqrt{\left(\frac{3}{2}\right)^2 + x^2}\,dx$. Using Formula (8) with $a = \frac{3}{2}$ and $u = x$, we find that $\int x^2\sqrt{9+4x^2}\,dx = 2\left[\left(\frac{x}{8}\right)\left(\frac{9}{4}+2x^2\right)\sqrt{\frac{9}{4}+x^2} - \frac{81}{128}\ln\left|x+\sqrt{\frac{9}{4}+x^2}\right|\right] + C$.

6. $\int x^2\sqrt{4+x^2}\,dx = \int x^2\sqrt{2^2+x^2}\,dx$. Use Formula (8) with $a = 2$ and $u = x$, obtaining $\int x^2\sqrt{4+x^2}\,dx = \frac{x}{8}\left(4+2x^2\right)\sqrt{4+x^2} - 2\ln\left|x+\sqrt{4+x^2}\right| + C$.

7. Use Formula (6) with $a = 1$, $b = 4$, and $u = x$. Then $\int \frac{dx}{x\sqrt{1+4x}} = \ln\left|\frac{\sqrt{1+4x}-1}{\sqrt{1+4x}+1}\right| + C$.

8. $\int_0^2 \frac{x+1}{\sqrt{2+3x}}\,dx = \int_0^2 \frac{x}{\sqrt{2+3x}}\,dx + \int_0^2 \frac{1}{\sqrt{2+3x}}\,dx$. Use Formula (5) with $a = 2$, $b = 3$, and $u = x$ to evaluate the first integral and substitute $u = 2 + 3x$ in second integral, obtaining

$$\int_0^2 \frac{x+1}{\sqrt{2+3x}}\,dx = \left[\tfrac{2}{27}(3x-4)\sqrt{2+3x}\right]_0^2 + \left[\left(\tfrac{1}{3}\right)2\sqrt{2+3x}\right]_0^2$$
$$= \tfrac{2}{27}\left(2\sqrt{8}+4\sqrt{2}\right) + \tfrac{2}{3}\left(\sqrt{8}-\sqrt{2}\right) = \tfrac{34\sqrt{2}}{27}.$$

9. Use Formula (9) with $a = 3$ and $u = 2x$, so $du = 2\,dx$. Then $\int_0^2 \frac{dx}{\sqrt{9+4x^2}} = \frac{1}{2}\int_0^4 \frac{du}{\sqrt{3^2+u^2}} = \frac{1}{2}\ln\left|u+\sqrt{9+u^2}\right|\Big|_0^4 = \frac{1}{2}(\ln 9 - \ln 3) = \frac{1}{2}\ln 3$. Note that the limits of integration change from $x = 0$ and $x = 2$ to $u = 0$ and $u = 4$ respectively.

10. $\int \frac{dx}{x\sqrt{4+8x^2}} = \int \frac{dx}{2\sqrt{2}x\sqrt{(1/\sqrt{2})^2+x^2}}$. Use Formula (10) with $a = \frac{1}{\sqrt{2}}$ and $u = x$, obtaining

$$\int \frac{dx}{x\sqrt{4+8x^2}} = -\sqrt{2}\ln\left|\frac{\sqrt{(1/\sqrt{2})^2+x^2}+(1/\sqrt{2})}{x}\right| + C.$$

11. Using Formula (22) with $a = 3$ and $u = x$, we see that $\int \frac{dx}{\left(9-x^2\right)^{3/2}} = \frac{x}{9\sqrt{9-x^2}} + C$.

12. $I = \int \frac{dx}{\left(2-x^2\right)^{3/2}}$. Use Formula (22) with $a = \sqrt{2}$ and $u = x$ to obtain $I = \frac{x}{2\sqrt{2-x^2}} + C$.

13. $I = \int x^2\sqrt{x^2-4}\,dx$. Use Formula (14) with $a = 2$ and $u = x$ to obtain $I = \frac{x}{8}\left(2x^2-4\right)\sqrt{x^2-4} - 2\ln\left|x+\sqrt{x^2-4}\right| + C$.

14. $I = \int \frac{dx}{x^2\sqrt{x^2-9}}$. Use Formula (17) with $a = 3$ and $u = x$, obtaining

$$I = \int_4^5 \frac{dx}{x^2\sqrt{x^2-9}} = \frac{\sqrt{x^2-9}}{9x}\Bigg|_4^5 = \frac{1}{9}\left(\frac{\sqrt{16}}{5} - \frac{\sqrt{7}}{4}\right) \approx 0.015.$$

15. Using Formula (19) with $a = 2$ and $u = x$, we have $\int \frac{\sqrt{4-x^2}}{x}\,dx = \sqrt{4-x^2} - 2\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right| + C$.

16. Using Formula (22) with $a = 2$ and $u = x$, we have $\int_0^1 \frac{dx}{\left(4-x^2\right)^{3/2}} = \frac{x}{4\sqrt{4-x^2}}\Big|_0^1 = \frac{1}{4\sqrt{3}} = \frac{\sqrt{3}}{12}$.

17. $I = \int xe^{2x}\,dx$. Use Formula (23) with $a = 2$ and $u = x$ to obtain $I = \frac{1}{4}(2x-1)e^{2x} + C$.

18. Using Formula (25) with $a = -1$, $b = 1$, and $u = x$, we find $\int \frac{dx}{1+e^{-x}} = x + \ln\left(1+e^{-x}\right) + C$.

19. $I = \int \frac{dx}{(x+1)\ln(x+1)}$. Let $u = x+1$, so $du = dx$. Then $\int \frac{dx}{(x+1)\ln(x+1)} = \int \frac{du}{u\ln u}$. Now use Formula (28) with $u = x$ to obtain $\int \frac{du}{u\ln u} = \ln|\ln u| + C$. Therefore, $\int \frac{dx}{(x+1)\ln(x+1)} = \ln|\ln(x+1)| + C$.

20. First we make the substitution $u = x^2+1$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. Then $\int \frac{x\,dx}{\left(x^2+1\right)\ln\left(x^2+1\right)} = \frac{1}{2}\int \frac{du}{u\ln u}$. Using Formula (28), we find $\frac{1}{2}\int \frac{du}{u\ln u} = \frac{1}{2}\ln|\ln u| + C$. Therefore, $\int \frac{x}{\left(x^2+1\right)\ln\left(x^2+1\right)}\,dx = \frac{1}{2}\ln\left(\ln\left(x^2+1\right)\right) + C$.

21. $I = 3\int \frac{e^{2x}}{(1+3e^x)^2}\,dx$. Put $u = e^x$, so $du = e^x\,dx$. Then use Formula (3) with $a = 1$ and $b = 3$. Thus,

$$I = 3\int \frac{u}{(1+3u)^2}du = \frac{1}{3}\left(\frac{1}{1+3u} + \ln|1+3u|\right) + C = \frac{1}{3}\left[\frac{1}{1+3e^x} + \ln(1+3e^x)\right] + C.$$

22. Let $u = e^x$, so $du = e^x\,dx$. Then using Formula (5) with $a = 1$ and $b = 3$, we have

$$\int \frac{e^{2x}}{\sqrt{1+3e^x}}\,dx = \int \frac{e^x e^x}{(1+3e^x)^{1/2}}\,dx = \int \frac{u}{\sqrt{1+3u}}\,du = \frac{2}{27}(3u-2)\sqrt{1+3u} + C$$
$$= \frac{2}{27}(3e^x-2)\sqrt{1+3e^x} + C.$$

23. $\int \frac{3e^x}{1+e^{x/2}}\,dx = 3\int \frac{e^{x/2}}{e^{-x/2}+1}\,dx$. Let $v = e^{x/2}$, so $dv = \frac{1}{2}e^{x/2}\,dx$ and $e^{x/2}\,dx = 2\,dv$. Then $\int \frac{3e^x}{1+e^{x/2}}\,dx = 6\int \frac{dv}{(1/v)+1} = 6\int \frac{v}{v+1}\,dv$. Use Formula (1) with $a = 1$, $b = 1$, and $u = v$, obtaining $6\int \frac{v}{v+1}\,dv = 6(1+v-\ln|1+v|) + C$. Thus, $\int \frac{3e^x}{1+e^{x/2}}\,dx = 6\left[1+e^{x/2} - \ln\left(1+e^{x/2}\right)\right] + C$. This answer may be written in the form $6\left[e^{x/2} - \ln\left(1+e^{x/2}\right)\right] + C_1$, where $C_1 = C + 6$ is an arbitrary constant.

24. $I = \int \frac{dx}{1-2e^{-x}}$. Use Formula (25) with $a = -1$ and $b = -2$, obtaining $I = x + \ln\left(1-2e^{-x}\right) + C$.

25. $I = \int \frac{4\ln x}{x(2+3\ln x)}\,dx$. Let $v = \ln x$ so that $dv = \frac{1}{x}\,dx$. Then $I = 4\int \frac{v}{2+3v}\,dv$. Now use Formula (1) with $a = 2$, $b = 3$, and $u = v$ to obtain $4\int \frac{v}{2+3v}\,dv = \frac{4}{9}(2+3\ln v - 2\ln|2+3\ln v|) + C$. Thus,

$$\int \frac{4\ln x}{x(2+3\ln x)}\,dx = \frac{4}{9}(2+3\ln x - 2\ln|2+3\ln x|) + C.$$

26. Using Formula (29) with $n = 2$ and $u = x$, we find $\int (\ln x)^2\,dx = x(\ln x)^2 - 2\int \ln x\,dx$.Then, using Formula (29) again with $n = 1$ on the second integral on the right, we have

$\int (\ln x)^2\,dx = x(\ln x)^2 - 2\left(x\ln x - \int dx\right) = x(\ln x)^2 - 2x\ln x + 2x + C = x\left[(\ln x)^2 - 2\ln x + 2\right] + C$.

Therefore, $\int_1^e (\ln x)^2\,dx = \left\{x\left[(\ln x)^2 - 2\ln x + 2\right]\right\}\Big|_1^e = e(1-2+2) - 2 = e - 2$.

27. Using Formula (24) with $a = 1$, $n = 2$, and $u = x$, we have

$$\int_0^1 x^2e^x\,dx = \left(x^2e^x\right)\Big|_0^1 - 2\int_0^1 xe^x\,dx = \left[x^2e^x - 2(xe^x - e^x)\right]_0^1 = \left(x^2e^x - 2xe^x + 2e^x\right)\Big|_0^1$$
$$= e - 2e + 2e - 2 = e - 2.$$

28. Using Formula (24) with $a = 2$, $n = 3$, and $u = x$, we have

$$I = \int x^3e^{2x}\,dx = \frac{1}{2}x^3e^{2x} - \frac{3}{2}\int x^2e^{2x}\,dx = \frac{1}{2}x^3e^{2x} - \frac{3}{2}\left(\frac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx\right)$$
$$= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{2}\int xe^{2x}\,dx.$$

Now use Formula (24) with $a = 2$, $n = 2$, and $u = x$ to obtain

$I = \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{2}\left[\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx\right]$. Finally, use Formula (23) with $a = 2$ and $u = x$ to find

$$I = \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C = \frac{1}{8}e^{2x}\left(4x^3 - 6x^2 + 6x - 3\right) + C.$$

29. $I = \int x^2\ln x\,dx$. Use Formula (27) with $n = 2$ and $u = x$ to obtain $I = \int x^2\ln x\,dx = \frac{x^3}{9}(3\ln x - 1) + C$.

30. $I = \int x^3 \ln x \, dx$. Use Formula (27) with $n = 3$ and $u = x$ to obtain $I = \dfrac{x^4}{16}(4 \ln x - 1) + C$.

31. $I = \int (\ln x)^3 \, dx$. Use Formula (29) with $n = 3$ to write $I = x(\ln x)^3 - 3\int (\ln x)^2 \, dx$. Now use Formula (29) again with $n = 2$ to obtain $I = x(\ln x)^3 - 3\left[x(\ln x)^2 - 2\int \ln x \, dx\right]$. Using Formula (29) one more time with $n = 1$ gives $\int (\ln x)^3 \, dx = x(\ln x)^3 - 3x(\ln x)^2 + 6(x \ln x - x) + C = x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x + C$.

32. Repeated use of Formula (29) yields

$$\begin{aligned}\int (\ln x)^4 \, dx &= x(\ln x)^4 - 4\int (\ln x)^3 \, dx = x(\ln x)^4 - 4\left[x(\ln x)^3 - 3\int (\ln x)^2 \, dx\right] \\ &= x(\ln x)^4 - 4x(\ln x)^3 + 12\int (\ln x)^2 \, dx = x(\ln x)^4 - 4x(\ln x)^3 + 12\left[x(\ln x)^2 - 2\int \ln x \, dx\right] \\ &= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24(x \ln x - x) + C \\ &= x\left[(\ln x)^4 - 4(\ln x)^3 + 12(\ln x)^2 - 24 \ln x + 24\right] + C.\end{aligned}$$

33. The number of visitors admitted to the amusement park by noon is found by evaluating the integral $\displaystyle\int_0^3 \frac{60}{(2+t^2)^{3/2}} \, dt = 60\int_0^3 \frac{dt}{(2+t)^{3/2}}$. Using Formula (12) with $a = \sqrt{2}$ and $u = t$, we find

$$60\int_0^3 \frac{dt}{(2+t^2)^{3/2}} = 60\left(\frac{t}{2\sqrt{2+t^2}}\right)\Bigg|_0^3 = 60\left(\frac{3}{2\sqrt{11}-0}\right) = \frac{90}{\sqrt{11}} \approx 27.136, \text{ or approximately } 27{,}136.$$

34. Use Formula (9) with $a = 2$ and $u = t$, obtaining

$N(5) - N(0) = \displaystyle\int_0^5 \frac{3000}{\sqrt{4+t^2}} dt = 3000 \ln\left|t + \sqrt{4+t^2}\right|\Bigg|_0^5 = 3000\left[\ln\left(5 + \sqrt{29}\right) - \ln 2\right] \approx 4941.69$. Therefore, $N(5) = 4942 + 20{,}000 = 24{,}942$.

35. To find the average number of fruit flies over the first 10 days, use Formula (25) with $a = -0.02$, $b = 24$, and $u = t$. Thus,

$$\begin{aligned}\frac{1}{10}\int \frac{1000}{1+24e^{-0.02t}} \, dt &= 100\int_0^{10} \frac{1}{1+24e^{-0.02t}} \, dt = 100\left[t + \tfrac{1}{0.02} \ln\left(1 + 24e^{-0.02t}\right)\right]_0^{10} \\ &= 100(10 + 50 \ln 20.6495 - 50 \ln 25) \approx 44.08, \text{ or approximately 44 fruit flies.}\end{aligned}$$

Over the first 20 days, the average is

$$\begin{aligned}\frac{1}{20}\int_0^{20} \frac{1000}{1+24e^{-0.02t}} \, dt &= 50\int_0^{20} \frac{1}{1+24e^{-0.02t}} \, dt = 50\left[t + \tfrac{1}{0.02} \ln\left(1 + 24e^{-0.02t}\right)\right]_0^{20} \\ &\approx 50(20 + 50 \ln 17.088 - 50 \ln 25) \approx 48.75, \text{ or approximately 49 fruit flies.}\end{aligned}$$

36. Use Formula (25) with $a = -0.2$, $b = 1.5$, and $u = t$. Then

$$\begin{aligned}\frac{1}{5}\int_0^5 \frac{100{,}000}{2\left(1 + 1.5e^{-0.2t}\right)} \, dt &= 10{,}000\int_0^5 \frac{1}{1+1.5e^{-0.2t}} \, dt = 10{,}000\left[t + 5 \ln\left(1 + 1.5e^{-0.2t}\right)\right]_0^5 \\ &= 10{,}000(5 + 5 \ln 1.551819162 - 5 \ln 2.5) \approx 26{,}157, \text{ or about 26,157 people.}\end{aligned}$$

37. To find the average average life expectancy for women from 1907 through 2007, use Formula (26) with $u = t$. Then

$$\begin{aligned}\tfrac{1}{6-1}\textstyle\int_1^6 (49.9 + 17.1 \ln t) \, dt &= \tfrac{1}{5}\left[49.9t + 17.1(t \ln t - t)\right]_1^6 \\ &= \tfrac{1}{5}\{[49.9(6) + 17.1(6 \ln 6 - 6)] - [49.9(1) + 17.1(\ln 1 - 1)]\} \approx 69.6, \text{ or 69.6 years.}\end{aligned}$$

38. Letting $p = 50$ gives $50 = \dfrac{250}{\sqrt{16+x^2}}$, from which we deduce that $\sqrt{16+x^2} = 5$, $16 + x^2 = 25$, and so $x = 3$. Using Formula (9) with $u = 3$, we see that

$$CS = \int_0^3 \frac{250}{\sqrt{16+x^2}}\,dx - 50\,(3) = 250\int_0^3 \frac{1}{\sqrt{16+x^2}}\,dx - 150 = 250\ln\left|x + \sqrt{16+x^2}\right|\Big|_0^3 - 150$$

$$= 250\,(\ln 8 - \ln 4) - 150 \approx 23.286795, \text{ or approximately } \$2329.$$

39. Letting $p = 50$ gives $50 = \dfrac{30x}{5-x}$, so $5 = \dfrac{3x}{5-x}$, $25 - 5x = 3x$, and $x = \frac{25}{8} = 3.125$. Using Formula (1) with $a = 5$, $b = -1$, and $u = x$, we have

$$PS = (50)\,(3.125) - \int_0^{3.125} \frac{30x}{5-x}\,dx = 156.25 - 30\int_0^{3.125} \frac{x}{5-x}\,dx$$

$$= 156.25 - 30\,(5 - x - 5\ln|5-x|)|_0^{3.125} = 156.25 - 30\,[(5 - 3.125 - 5\ln 1.875) - 5 + 5\ln 5]$$

$$\approx 102.875, \text{ or approximately } \$10{,}288.$$

40. If $x = 2$, then $p = \dfrac{160}{\sqrt{4+2^2}} = \dfrac{160}{2\sqrt{2}} = 40\sqrt{2}$, so $CS = \displaystyle\int_0^2 \frac{160}{\sqrt{4+x^2}}\,dx - 80\sqrt{2}$. Using Formula 9 with $a = 2$ and $u = x$, we find $CS = \left[160\ln\left|x + \sqrt{4+x^2}\right|\right]_0^2 - 80\sqrt{2} \approx 27.883$, or \$27,883.

41. If $p = 74$, then we have $74 = 50 + x\sqrt{1+x}$, so $x\sqrt{1+x} = 24$, $x^2\,(1+x) = 576$, and $x^3 + x^2 = 576$. If $x = 8$, then $8^3 + 8^2 = 512 + 64 = 576$, so $x = 8$ is a possible solution, and does in fact satisfy the original equation. Since $x^3 + x^2 - 576 = (x-8)\left(x^2 + 9x + 72\right)$, we see that $x = 8$ is the only real root. Thus, $PS = \int_0^8 \left[74 - \left(50 + x\sqrt{1+x}\right)\right]dx = \int_0^8 \left(24 - x\sqrt{1+x}\right)dx$. Using Formula 4 with $a = 1$, $b = -1$, and $u = x$, we find $PS = \left[24x - \frac{2}{15}\,(3x-2)\,(1+x)^{3/2}\right]_0^8 \approx 112.533$, or approximately \$112,533.

42. Using Formula (18) from Section 6.7 and Formula (23) from the table of integrals with $a = 0.1$ and $u = t$, we obtain

$$I = e^{(0.1)(5)}\int_0^5 20{,}000te^{-0.1t}\,dt = 20{,}000e^{0.5}\int_0^5 te^{-0.1t}\,dt = 20{,}000e^{0.5}\left[\frac{1}{(0.1)^2}\,(-0.1t - 1)\,e^{-0.1t}\right]_0^5$$

$$\approx 297{,}443, \text{ or approximately } \$297{,}443.$$

43. $I = \int_0^{10} \left(250{,}000 + 2000t^2\right)e^{-0.1t}\,dt = -2{,}500{,}000e^{-0.1t}\big|_0^{10} + 2000\int_0^{10} t^2e^{-0.1t}\,dt$. Using Formula (24) with $a = -0.1$, $n = 2$, and $u = t$ to evaluate the second integral, we have

$I = 2{,}500{,}000\left(1 - e^{-1}\right) + 2000\left[\left(-10t^2e^{-0.1t}\right)\big|_0^{10} + \frac{2}{0.1}\int_0^{10} te^{-0.1t}\,dt\right]$. Next, using Formula (23) to evaluate the last integral, we obtain

$$I \approx 1{,}580{,}301.397 + 2000\left\{-1000e^{-1} + 20\left[\tfrac{1}{0.01}\,(-0.1t - 1)\,e^{-0.1t}\right]_0^{10}\right\}$$

$$\approx 1{,}580{,}301.397 + 2000\left[-1000e^{-1} - 2000\left(-2e^{-1} + 1\right)\right]$$

$$\approx 1,580,301.397 + 2000\,(160.60) = \$1{,}901{,}507, \text{ or approximately } \$1{,}901{,}507.$$

44. At the end of the 10-year period, Joanne's account will be worth $F = e^{0.04(10)} \int_0^{10} t^2 e^{-0.02t} e^{-0.04t}\,dt = e^{0.4} \int_0^{10} t^2 e^{-0.06t}\,dt$. Using Formula 24 with $a = -0.06$, $n = 2$, and $u = t$, we find $F = e^{0.4} \left\{ \left[\frac{1}{-0.06} t^2 e^{-0.06t} \right]_0^{10} - \frac{2}{-0.06} \int_0^{10} t e^{-0.06t}\,dt \right\}$. Using Formula 23 with $a = -0.06$, we have $F = e^{0.4} \left[\frac{1}{-0.06} t^2 e^{-0.06t} + \frac{2}{0.06} \left[\frac{1}{(-0.06)^2} (-0.06t - 1) e^{-0.06t} \right] \right]_0^{10} \approx 319.296$, or approximately \$319,296.

45. Using Formula (22) from Section 6.7 and Formula (4) from the table of integrals with $a = 1$, $b = 8$, and $u = x$, we obtain

$L = 2 \int_0^1 \left(x - \frac{1}{3} x \sqrt{1 + 8x} \right) dx = x^2 \Big|_0^1 - \frac{2}{3} \int_0^1 x \sqrt{1 + 8x}\,dx = 1 - \left[\frac{2}{3} \left(\frac{2}{15 \cdot 64} \right) (24x - 2)(1 + 8x)^{3/2} \right]_0^1$
$= 1 - \frac{1}{720} [(22)(27) - (-2)] \approx 0.1722.$

46. The present value of John's investment over the first five years is $PV = \int_0^5 \left(10t + 4e^{0.02t} \right) e^{-0.04t}\,dt = 10 \int_0^5 t e^{-0.04t}\,dt + 4 \int_0^5 e^{-0.02t}\,dt$. Using Formula 23, we have $PV = \left[10 \left(-\frac{1}{0.04} \right)^2 (-0.04t - 1) e^{-0.04t} - \frac{4}{0.02} e^{-0.02t} \right]_0^5 \approx 128.552$, or approximately \$128,552.

47. The Gini Index for country A is $I_A = 2 \int_0^1 \left(x - x^2 e^{x-1} \right) dx = 2 \int_0^1 x\,dx - 2e^{-1} \int_0^1 x^2 e^x\,dx$.
To evaluate the second integral, we use Formula 24 followed by Formula 23, obtaining
$I_A = 1 - 2e^{-1} \left[x^2 e^x - 2 \int x e^x\,dx \right]_0^1 = 1 - 2e^{-1} \left[x^2 e^x - 2(x - 1) e^x \right]_0^1 \approx 0.472.$
For the second country, $I_B = 2 \int_0^1 \left(x - \frac{1}{2} x^2 \sqrt{3 + x^2} \right) dx = 2 \int_0^1 x\,dx - \int_0^1 x^2 \sqrt{3 + x^2}\,dx$.
To evaluate the second integral, we use Formula 8 with $a = \sqrt{3}$ and $u = x$, giving
$I_B = x^2 \Big|_0^1 - \left[\frac{1}{8} x \left(3 + 2x^2 \right) \sqrt{3 + x^2} - \frac{9}{8} \ln \left| x + \sqrt{3 + x^2} \right| \right]_0^1 = 1 - \frac{1}{8} (5)(2) + \frac{9}{8} \ln(1 + 2) \approx 0.368.$
From these results, we see that country B has a more equitable income distribution.

7.3 Numerical Integration

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1. In the trapezoidal rule, each region beneath (or above) the graph of f is approximated by the area of a trapezoid whose base consists of two consecutive points in the partition. Therefore, n can be odd or even. In Simpson's Rule, the area of each subregion is approximated by part of a parabola passing through those points. Therefore, there are two subintervals involved in the approximations, and so n must be even.

2. In the trapezoidal rule, we are in effect approximating the function $f(x)$ on the interval $[x_0, x_1]$ by a linear function through the two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. If f is a linear function then there is no error, because the approximation is exact. When we use Simpson's Rule, we are in effect approximating the function $f(x)$ on the interval $[x_0, x_2]$ that passes through $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$ by a quadratic function whose graph (a parabola) contains these three points. If the graph of $f(x)$ is a parabola, then the approximation is exact.

3. If we use the trapezoidal rule and f is a linear function, then $f''(x) = 0$, so $M = 0$, and consequently the maximum error is 0. If we use Simpson's Rule, then $f^{(4)}(x) = 0$, so $M = 0$, and once again the maximum error is 0.

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1. $\Delta x = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3}$, so $x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1, x_4 = \frac{4}{3}, x_5 = \frac{5}{3}, x_6 = 2$.

Trapezoidal Rule:

$$\int_0^2 x^2\,dx \approx \frac{1}{6}\left[0 + 2\left(\frac{1}{3}\right)^2 + 2\left(\frac{2}{3}\right)^2 + 2\,(1)^2 + 2\left(\frac{4}{3}\right)^2 + 2\left(\frac{5}{3}\right)^2 + 2^2\right]$$

$$\approx \frac{1}{6}\,(0.22222 + 0.88889 + 2 + 3.55556 + 5.55556 + 4) \approx 2.7037.$$

Simpson's Rule:

$$\int x^2\,dx \approx \frac{1}{9}\left[0 + 4\left(\frac{1}{3}\right)^2 + 2\left(\frac{2}{3}\right)^2 + 4\,(1)^2 + 2\left(\frac{4}{3}\right)^2 + 4\left(\frac{5}{3}\right)^2 + 2^2\right]$$

$$\approx \frac{1}{9}\,(0.44444 + 0.88889 + 4 + 3.55556 + 11.11111 + 4) \approx 2.6667.$$

Exact value: $\int_0^2 x^2\,dx = \frac{1}{3}x^3\Big|_0^2 = \frac{8}{3}$.

2. $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$, so $x_0 = 1, x_1 = \frac{3}{2}, x_2 = 2, x_3 = \frac{5}{2}, x_4 = 3$.

Trapezoidal Rule:

$$\int_1^3 \left(x^2 - 1\right) dx \approx \frac{1/2}{2}\left\{\left(1^2 - 1\right) + 2\left[\left(\frac{3}{2}\right)^2 - 1\right] + 2\left[(2)^2 - 1\right] + 2\left[\left(\frac{5}{2}\right)^2 - 1\right] + \left(3^2 - 1\right)\right\} = 6.75.$$

Simpson's Rule:

$$\int_1^3 \left(x^2 - 1\right) dx \approx \frac{\frac{1}{2}}{3}\left\{\left[(1)^2 - 1\right] + 4\left[\left(\frac{3}{2}\right)^2 - 1\right] + 2\left[2^2 - 1\right] + 4\left[\left(\frac{5}{2}\right)^2 - 1\right] + \left[3^2 - 1\right]\right\} = 6.67.$$

Exact value: $\int_1^3 \left(x^2 - 1\right) dx = \left(\frac{1}{3}x^3 - x\right)\Big|_1^3 = (9 - 3) - \left(\frac{1}{3} - 1\right) = \frac{20}{3}$.

3. $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$, so $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, x_4 = 1$.

Trapezoidal Rule:

$$\int_0^1 x^3\,dx \approx \frac{1}{8}\left[0 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{1}{2}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 1^3\right] \approx \frac{1}{8}\,(0 + 0.03125 + 0.25 + 0.84375 + 1) \approx 0.265625.$$

Simpson's Rule:

$$\int_0^1 x^3\,dx \approx \frac{1}{12}\left[0 + 4\left(\frac{1}{4}\right)^3 + 2\left(\frac{1}{2}\right)^3 + 4\left(\frac{3}{4}\right)^3 + 1\right] \approx \frac{1}{12}\,(0 + 0.0625 + 0.25 + 1.6875 + 1) \approx 0.25.$$

Exact value: $\int_0^1 x^3\,dx = \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$.

4. $\Delta x = \frac{b-a}{n} = \frac{2-1}{6} = \frac{1}{6}$, so $x_0 = 0, x_1 = \frac{7}{6}, x_2 = \frac{4}{3}, x_3 = \frac{3}{2}, x_4 = \frac{5}{3}, x_5 = \frac{11}{6}, x_6 = 2$.

Trapezoidal Rule:

$$\int_1^2 x^3\,dx \approx \frac{1}{12}\left[1^3 + 2\left(\frac{7}{6}\right)^3 + 2\left(\frac{4}{3}\right)^3 + 2\left(\frac{3}{2}\right)^3 + 2\left(\frac{5}{3}\right)^3 + 2\left(\frac{11}{6}\right)^3 + 2^3\right]$$

$$\approx \frac{1}{12}\,(1 + 3.17593 + 4.74074 + 6.75 + 9.25926 + 12.32407 + 8) \approx 3.7708.$$

Simpson's Rule:

$$\int_1^2 x^3\,dx \approx \frac{1}{18}\left[1^3 + 4\left(\frac{7}{6}\right)^3 + 2\left(\frac{4}{3}\right)^3 + 4\left(\frac{3}{2}\right)^3 + 2\left(\frac{5}{3}\right)^3 + 4\left(\frac{11}{6}\right)^3 + 2^3\right]$$

$$\approx \frac{1}{18}\,(1 + 6.35185 + 4.74074 + 13.5 + 9.25926 + 24.64815 + 8) \approx 3.7500.$$

Exact value: $\int_1^2 x^3\,dx = \frac{1}{4}x^4\Big|_1^2 = \frac{1}{4}\,(16 - 1) = \frac{15}{4}$.

5. Here $a = 1, b = 2$, and $n = 4$, so $\Delta x = \frac{2-1}{4} = \frac{1}{4} = 0.25$ and $x_0 = 1, x_1 = 1.25, x_2 = 1.5, x_3 = 1.75, x_4 = 2$.

Trapezoidal Rule: $\int_1^2 \frac{1}{x}\,dx \approx \frac{0.25}{2}\left[1 + 2\left(\frac{1}{1.25}\right) + 2\left(\frac{1}{1.5}\right) + 2\left(\frac{1}{1.75}\right) + \frac{1}{2}\right] \approx 0.697$.

Simpson's Rule: $\int_1^2 \frac{1}{x}\,dx \approx \frac{0.25}{3}\left[1 + 4\left(\frac{1}{1.25}\right) + 2\left(\frac{1}{1.5}\right) + 4\left(\frac{1}{1.75}\right) + \frac{1}{2}\right] \approx 0.6933$.

Exact value: $\int_1^2 \frac{1}{x}\,dx = \ln x|_1^2 = \ln 2 - \ln 1 = \ln 2 \approx 0.6931$.

6. $\Delta x = \frac{b-a}{n} = \frac{2-1}{8} = \frac{1}{8}$, so $x_0 = 0, x_1 = \frac{9}{8}, x_2 = \frac{10}{8}, x_3 = \frac{11}{8}, \ldots, x_8 = \frac{16}{8}$.

Trapezoidal Rule: $\int_1^2 \frac{1}{x}\,dx \approx \frac{1/8}{2}\left[1 + 2\left(\frac{8}{9}\right) + 2\left(\frac{8}{10}\right) + 2\left(\frac{8}{11}\right) + \cdots + \left(\frac{8}{16}\right)\right] \approx 0.69412$.

Simpson's Rule: $\int_1^2 \frac{1}{x}\,dx \approx \frac{1/8}{3}\left[1 + 4\left(\frac{8}{9}\right) + 2\left(\frac{8}{10}\right) + 4\left(\frac{8}{11}\right) + \cdots + 4\left(\frac{8}{15}\right) + \left(\frac{8}{16}\right)\right] \approx 0.69315$.

Exact value: $\int_1^2 \frac{1}{x}\,dx = \ln x\Big|_1^2 = \ln 2 \approx 0.69315$.

7. $\Delta x = \frac{1}{4}, x_0 = 1, x_1 = \frac{5}{4}, x_2 = \frac{3}{2}, x_3 = \frac{7}{4}, x_4 = 2$.

Trapezoidal Rule: $\int_1^2 \frac{1}{x^2}\,dx \approx \frac{1}{8}\left[1 + 2\left(\frac{4}{5}\right)^2 + 2\left(\frac{2}{3}\right)^2 + 2\left(\frac{4}{7}\right)^2 + \left(\frac{1}{2}\right)^2\right] \approx 0.5090$.

Simpson's Rule: $\int_1^2 \frac{1}{x^2}\,dx \approx \frac{1}{12}\left[1 + 4\left(\frac{4}{5}\right)^2 + 2\left(\frac{2}{3}\right)^2 + 4\left(\frac{4}{7}\right)^2 + \left(\frac{1}{2}\right)^2\right] \approx 0.5004$.

Exact value: $\int_1^2 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$.

8. $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$, so $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{2}{4}, x_3 = \frac{3}{4}, x_4 = \frac{4}{4}$.

Trapezoidal Rule: $\displaystyle\int_0^1 \frac{1}{1+x}\,dx \approx \frac{\frac{1}{4}}{2}\left[0 + 2\left(\frac{1}{1+\frac{1}{4}}\right) + 2\left(\frac{1}{1+\frac{1}{2}}\right) + 2\left(\frac{1}{1+\frac{3}{4}}\right) + \left(\frac{1}{1+1}\right)\right] \approx 0.57202$.

Simpson's Rule: $\displaystyle\int_0^1 \frac{1}{1+x}\,dx \approx \frac{\frac{1}{4}}{3}\left[0 + 4\left(\frac{1}{1+\frac{1}{4}}\right) + 2\left(\frac{1}{1+\frac{1}{2}}\right) + 4\left(\frac{1}{1+\frac{3}{4}}\right) + \left(\frac{1}{1+1}\right)\right] \approx 0.60992$.

Exact value: $\displaystyle\int_0^1 \frac{1}{1+x}\,dx = \ln(1+x)|_0^1 = \ln 2 \approx 0.69315$.

9. $\Delta x = \frac{b-a}{n} = \frac{4-0}{8} = \frac{1}{2}$, so $x_0 = 0, x_1 = \frac{1}{2}, x_2 = \frac{2}{2}, x_3 = \frac{3}{2}, \ldots, x_8 = \frac{8}{2}$.

Trapezoidal Rule: $\int_0^4 \sqrt{x}\,dx \approx \frac{1/2}{2}\left(0 + 2\sqrt{0.5} + 2\sqrt{1} + 2\sqrt{1.5} + \cdots + 2\sqrt{3.5} + \sqrt{4}\right) \approx 5.26504$.

Simpson's Rule: $\int_0^4 \sqrt{x}\,dx \approx \frac{1/2}{3}\left(0 + 4\sqrt{0.5} + 2\sqrt{1} + 4\sqrt{1.5} + \cdots + 4\sqrt{3.5} + \sqrt{4}\right) \approx 5.30463$.

Exact value: $\int_0^4 \sqrt{x}\,dx \approx \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{2}{3}(8) = \frac{16}{3}$.

10. $\Delta x = \frac{2}{6} = \frac{1}{3}$, so $x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1, x_4 = \frac{4}{3}, x_5 = \frac{5}{3}, x_6 = 2$.

Trapezoidal Rule:

$$\int_0^2 x\left(2x^2+1\right)^{1/2}\,dx \approx \frac{1}{6}\left\{0 + 2\left(\tfrac{1}{3}\right)\left[2\left(\tfrac{1}{3}\right)^2+1\right]^{1/2} + 2\left(\tfrac{2}{3}\right)\left[2\left(\tfrac{2}{3}\right)^2+1\right]^{1/2} + 2\,(1)\left[2\,(1)^2+1\right]^{1/2}\right.$$
$$\left. + 2\left(\tfrac{4}{3}\right)\left[2\left(\tfrac{4}{3}\right)^2+1\right]^{1/2} + 2\left(\tfrac{5}{3}\right)\left[2\left(\tfrac{5}{3}\right)^2+1\right]^{1/2} + 2\left[2\,(2)^2+1\right]^{1/2}\right\} \approx 4.3767.$$

Simpson's Rule:

$$\int_0^2 x\left(2x^2+1\right)^{1/2}\,dx \approx \frac{1}{9}\left\{0 + 4\left(\tfrac{1}{3}\right)\left[2\left(\tfrac{1}{3}\right)^2+1\right]^{1/2} + 2\left(\tfrac{2}{3}\right)\left[2\left(\tfrac{2}{3}\right)^2+1\right]^{1/2} + 4\,(1)\left[2\,(1)^2+1\right]^{1/2}\right.$$
$$\left. + 2\left(\tfrac{4}{3}\right)\left[2\left(\tfrac{4}{3}\right)^2+1\right]^{1/2} + 4\left(\tfrac{5}{3}\right)\left[2\left(\tfrac{5}{3}\right)^2+1\right]^{1/2} + 2\left[2\,(2)^2+1\right]^{1/2}\right\} \approx 4.3329.$$

Exact value: $\int_0^2 x\left(2x^2+1\right)^{1/2}\,dx = \left(\frac{1}{4}\right)\left(\frac{2}{3}\right)\left(2x^2+1\right)^{3/2}\Big|_0^2 = \frac{1}{6}\left(9^{3/2}-1\right) = \frac{13}{3}$.

11. $\Delta x = \frac{1-0}{6} = \frac{1}{6}$, so $x_0 = 0, x_1 = \frac{1}{6}, x_2 = \frac{2}{6}, \ldots, x_6 = \frac{6}{6}$.

Trapezoidal Rule: $\int_0^1 e^{-x}\,dx \approx \frac{1/6}{2}\left(1 + 2e^{-1/6} + 2e^{-2/6} + \cdots + 2e^{-5/6} + e^{-1}\right) \approx 0.633583$.

Simpson's Rule: $\int_0^1 e^{-x}\,dx \approx \frac{1/6}{3}\left(1 + 4e^{-1/6} + 2e^{-2/6} + \cdots + 4e^{-5/6} + e^{-1}\right) \approx 0.632123$.

Exact value: $\int_0^1 e^{-x}\,dx = -e^{-x}\big|_0^1 = -e^{-1} + 1 \approx 0.632121$.

12. $\Delta x = \frac{1-0}{6} = \frac{1}{6}$, so $x_0 = 0, x_1 = \frac{1}{6}, x_2 = \frac{2}{6}, \ldots, x_6 = \frac{6}{6}$.

Trapezoidal Rule: $\int_0^1 xe^{-x^2}\,dx \approx \frac{1/6}{2}\left[0 + 2\cdot\frac{1}{6}e^{-(1/6)^2} + 2\cdot\frac{2}{6}e^{-(2/6)^2} + \cdots + 2\cdot\frac{5}{6}e^{-(5/6)^2} + e^{-1}\right] = 0.3129$.

Simpson's Rule: $\int_0^1 xe^{-x^2}\,dx \approx \frac{1/6}{3}\left[0 + 4\cdot\frac{1}{6}e^{-(1/6)^2} + 2\cdot\frac{2}{6}e^{-(2/6)^2} + \cdots + 4\cdot\frac{5}{6}e^{-(5/6)^2} + e^{-1}\right] = 0.3161$.

Exact value: $\int_0^1 xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}\Big|_0^1 = -\frac{1}{2}\left(e^{-1}-1\right) \approx 0.316060$.

13. $\Delta x = \frac{1}{4}$, so $x_0 = 0, x_1 = \frac{5}{4}, x_2 = \frac{3}{2}, x_3 = \frac{7}{4}, x_4 = 2$.

Trapezoidal Rule: $\int_1^2 \ln x\,dx \approx \frac{1}{8}\left(\ln 1 + 2\ln\frac{5}{4} + 2\ln\frac{3}{2} + 2\ln\frac{7}{4} + \ln 2\right) \approx 0.38370$.

Simpson's Rule: $\int_1^2 \ln x\,dx \approx \frac{1}{12}\left(\ln 1 + 4\ln\frac{5}{4} + 2\ln\frac{3}{2} + 4\ln\frac{7}{4} + \ln 2\right) \approx 0.38626$.

Exact value: $\int_1^2 \ln x\,dx \approx x\,(\ln x - 1)|_1^2 = 2\,(\ln 2 - 1) + 1 = 2\ln 2 - 1 \approx 0.3863$.

14. $\Delta x = \frac{1-0}{8} = \frac{1}{8}$, so $x_0 = 0, x_1 = \frac{1}{8}, x_2 = \frac{2}{8}, \ldots, x_8 = \frac{8}{8}$.
Trapezoidal Rule:

$$\begin{aligned}\int_0^1 x\ln\left(x^2+1\right)dx &\approx \tfrac{1/8}{2}\left[0+2\cdot\tfrac{1}{8}\ln\left(\left(\tfrac{1}{8}\right)^2+1\right)+2\cdot\tfrac{2}{8}\ln\left(\left(\tfrac{2}{8}\right)^2+1\right)+\cdots+2\cdot\tfrac{7}{8}\ln\left(\left(\tfrac{7}{8}\right)^2+1\right)+\ln 2\right]\\ &\approx 0.1954.\end{aligned}$$

Simpson's Rule:

$$\begin{aligned}\int_0^1 x\ln\left(x^2+1\right)dx &\approx \tfrac{1/8}{3}\left[0+4\cdot\tfrac{1}{8}\ln\left(\left(\tfrac{1}{8}\right)^2+1\right)+2\cdot\tfrac{2}{8}\ln\left(\left(\tfrac{2}{8}\right)^2+1\right)+\cdots+4\cdot\tfrac{7}{8}\ln\left(\left(\tfrac{7}{8}\right)^2+1\right)+\ln 2\right]\\ &\approx 0.1931.\end{aligned}$$

Exact value: let $u = x^2 + 1$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. If $x = 0$, then $u = 1$ and if $x = 1$, then $u = 2$, so $\int_0^1 x\ln\left(x^2+1\right)dx = \frac{1}{2}\int_1^2 \ln u\,du = \frac{1}{2}u\left(\ln u - 1\right)\Big|_1^2 = \frac{1}{2}(2)(\ln 2 - 1) - \frac{1}{2}(-1) \approx 0.193147$. (See Exercise 23 in Section 7.1.)

15. $\Delta x = \frac{1-0}{4} = \frac{1}{4}$, so $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{2}{4}, x_3 = \frac{3}{4}, x_4 = \frac{4}{4}$.

Trapezoidal Rule: $\int_0^1 \sqrt{1+x^3}\,dx \approx \frac{1/4}{2}\left[\sqrt{1}+2\sqrt{1+\left(\frac{1}{4}\right)^3}+\cdots+2\sqrt{1+\left(\frac{3}{4}\right)^3}+\sqrt{2}\right] \approx 1.1170.$

Simpson's Rule:

$$\int_0^1 \sqrt{1+x^3}\,dx \approx \tfrac{1/4}{3}\left[\sqrt{1}+4\sqrt{1+\left(\tfrac{1}{4}\right)^3}+2\sqrt{1+\left(\tfrac{2}{4}\right)^3}+\cdots+4\sqrt{1+\left(\tfrac{3}{4}\right)^3}+\sqrt{2}\right] \approx 1.1114.$$

16. $\Delta x = \frac{2}{4} = \frac{1}{2}$, so $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2$.
Trapezoidal Rule:

$$\begin{aligned}\int_0^2 x\left(1+x^3\right)^{1/2}dx &\approx \tfrac{1}{4}\left\{0+2\left(\tfrac{1}{2}\right)\left[1+\left(\tfrac{1}{2}\right)^3\right]^{1/2}+2(1)\left(1+1^3\right)^{1/2}+2\left(\tfrac{3}{2}\right)\left[1+\left(\tfrac{3}{2}\right)^3\right]^{1/2}+2\left(1+2^3\right)^{1/2}\right\}\\ &\approx 4.0410.\end{aligned}$$

Simpson's Rule:

$$\begin{aligned}\int_0^2 x\left(1+x^3\right)^{1/2}dx &\approx \tfrac{1}{6}\left\{0+4\left(\tfrac{1}{2}\right)\left[1+\left(\tfrac{1}{2}\right)^3\right]^{1/2}+2(1)\left(1+1^3\right)^{1/2}+4\left(\tfrac{3}{2}\right)\left[1+\left(\tfrac{3}{2}\right)^3\right]^{1/2}+2\left(1+2^3\right)^{1/2}\right\}\\ &\approx 3.9166.\end{aligned}$$

17. $\Delta x = \frac{2-0}{4} = \frac{1}{2}$, so $x_0 = 0, x_1 = \frac{1}{2}, x_2 = \frac{2}{2}, x_3 = \frac{3}{2}, x_4 = \frac{4}{2}$.
Trapezoidal Rule:

$$\int_0^2 \frac{1}{\sqrt{x^3+1}}\,dx = \frac{1/2}{2}\left[1+\frac{2}{\sqrt{\left(\frac{1}{2}\right)^3+1}}+\frac{2}{\sqrt{(1)^3+1}}+\frac{2}{\sqrt{\left(\frac{3}{2}\right)^3+1}}+\frac{1}{\sqrt{(2)^3+1}}\right] \approx 1.3973.$$

Simpson's Rule:

$$\int_0^2 \frac{1}{\sqrt{x^3+1}}\,dx = \frac{1/2}{3}\left[1+\frac{4}{\sqrt{\left(\frac{1}{2}\right)^3+1}}+\frac{2}{\sqrt{(1)^3+1}}+\frac{4}{\sqrt{\left(\frac{3}{2}\right)^3+1}}+\frac{1}{\sqrt{(2)^3+1}}\right] \approx 1.4052.$$

18. Here $a = 0$, $b = 1$ and $n = 4$, so $\Delta x = \frac{1}{4} = 0.25$ and $x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$.

Trapezoidal Rule:

$$\int_0^1 \left(1 - x^2\right)^{1/2} dx \approx \tfrac{0.25}{2} \left\{1 + 2\left[1 - (0.25)^2\right]^{1/2} + 2\left[1 - (0.5)^2\right]^{1/2} + 2\left[1 - (0.75)^2\right]^{1/2} + 0\right\} \approx 0.7489.$$

Simpson's Rule:

$$\int_0^1 \left(1 - x^2\right)^{1/2} dx \approx \tfrac{0.25}{3} \left\{1 + 4\left[1 - (0.25)^2\right]^{1/2} + 2\left[1 - (0.5)^2\right]^{1/2} + 4\left[1 - (0.75)^2\right]^{1/2} + 0\right\} \approx 0.7709.$$

19. $\Delta x = \frac{2}{4} = \frac{1}{2}$, so $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2$.

Trapezoidal Rule: $\int_0^2 e^{-x^2} dx = \frac{1}{4}\left[e^{-0} + 2e^{-(1/2)^2} + 2e^{-1} + 2e^{-(3/2)^2} + e^{-4}\right] \approx 0.8806$.

Simpson's Rule: $\int_0^2 e^{-x^2} dx = \frac{1}{6}\left[e^{-0} + 4e^{-(1/2)^2} + 2e^{-1} + 4e^{-(3/2)^2} + e^{-4}\right] \approx 0.8818$.

20. $\Delta x = \frac{1-0}{6} = \frac{1}{6}$, so $x_0 = 0, x_1 = \frac{1}{6}, x_2 = \frac{2}{6}, \ldots, x_6 = \frac{6}{6}$.

Trapezoidal Rule: $\int_0^1 e^{x^2} dx = \frac{1/6}{2}\left[1 + 2e^{(1/6)^2} + 2e^{(2/6)^2} + \cdots + 2e^{(5/6)^2} + e^1\right] \approx 1.4752$.

Simpson's Rule: $\int_0^1 e^{x^2} dx = \frac{1/6}{3}\left[1 + 4e^{(1/6)^2} + 2e^{(2/6)^2} + \cdots + 4e^{(5/6)^2} + e^1\right] \approx 1.4629$.

21. $\Delta x = \frac{2-1}{4} = \frac{1}{4}$, so $x_0 = 1, x_1 = \frac{5}{4}, x_2 = \frac{6}{4}, x_3 = \frac{7}{4}, x_4 = \frac{8}{4}$.

Trapezoidal Rule: $\int_1^2 x^{-1/2} e^x \, dx = \dfrac{1/4}{2}\left[e + \dfrac{2e^{5/4}}{\sqrt{5/4}} + \cdots + \dfrac{2e^{7/4}}{\sqrt{7/4}} + \dfrac{e^2}{\sqrt{2}}\right] \approx 3.7757$.

Simpson's Rule: $\int_1^2 x^{-1/2} e^x \, dx = \dfrac{1/4}{3}\left[e + \dfrac{4e^{5/4}}{\sqrt{5/4}} + \cdots + \dfrac{4e^{7/4}}{\sqrt{7/4}} + \dfrac{e^2}{\sqrt{2}}\right] \approx 3.7625$.

22. $\Delta x = \frac{2}{6} = \frac{1}{3}$, so $x_0 = 2, x_1 = \frac{7}{3}, x_2 = \frac{8}{3}, x_3 = 3, x_4 = \frac{10}{3}, x_5 = \frac{11}{3}, x_6 = 4$.

Trapezoidal Rule: $\displaystyle\int_2^4 \frac{dx}{\ln x} \approx \frac{1}{6}\left(\frac{1}{\ln 2} + \frac{2}{\ln \frac{7}{3}} + \frac{2}{\ln \frac{8}{3}} + \frac{2}{\ln 3} + \frac{2}{\ln \frac{10}{3}} + \frac{2}{\ln \frac{11}{3}} + \frac{1}{\ln 4}\right) \approx 1.9308$.

Simpson's Rule: $\displaystyle\int_2^4 \frac{dx}{\ln x} \approx \frac{1}{9}\left[\frac{1}{\ln 2} + \frac{4}{\ln \frac{7}{3}} + \frac{2}{\ln \frac{8}{3}} + \frac{4}{\ln 3} + \frac{2}{\ln \frac{10}{3}} + \frac{4}{\ln \frac{11}{3}} + \frac{1}{\ln 4}\right] \approx 1.9228$.

23. **a.** Here $a = -1$, $b = 2$, $n = 10$, and $f(x) = x^5$. Thus, $f'(x) = 5x^4$ and $f''(x) = 20x^3$. Because $f'''(x) = 60x^2 > 0$ on $(-1, 0) \cup (0, 2)$, we see that $f''(x)$ is increasing on $(-1, 0)$ and $(0, 2)$. Thus, we can take $M = f''(2) = 20\left(2^3\right) = 160$. Using Formula (7), we see that the maximum error incurred is

$$\frac{M(b-a)^3}{12n^2} = \frac{160\,[2 - (-1)]^3}{12\,(100)} = 3.6.$$

b. We compute $f'''(x) = 60x^2$ and $f^{(4)}(x) = 120x$. $f^{(4)}$ is clearly increasing on $(-1, 2)$, so we can take $M = f^{(4)}(2) = 240$. Therefore, using Formula (8), we see that an error bound is

$$\frac{M(b-a)^3}{180n^4} = \frac{240\,(3)^5}{180\left(10^4\right)} \approx 0.0324.$$

24. **a.** Here $a = 0$, $b = 1$, $n = 8$, and $f(x) = e^{-x}$. We find $f'(x) = -e^{-x}$ and $f''(x) = e^{-x}$. Because f'' is positive and decreasing, its maximum occurs at the left endpoint of the interval $[0, 1]$, so $\left|f''(x)\right| \le 1$. Therefore the maximum error is $\dfrac{1\,(2-1)^3}{12\,(8)^2} = \dfrac{1}{768} = 0.0013021$.

b. $f'''(x) = -e^{-x}$ and $f^{(4)}(x) = e^{-x}$, so the maximum error is $\dfrac{1\,(2-1)^5}{180\,(8)^4} = \dfrac{1}{737280} = 0.000001356$.

25. **a.** Here $a = 1$, $b = 3$, $n = 10$, and $f(x) = \dfrac{1}{x}$. We find $f'(x) = -\dfrac{1}{x^2}$ and $f''(x) = \dfrac{2}{x^3}$. Because $f'''(x) = -\dfrac{6}{x^4} < 0$ on $(1, 3)$, we see that f'' is decreasing there. We may take $M = f''(1) = 2$. Using Formula (7), we find the error bound $\dfrac{M(b-a)^3}{12n^2} = \dfrac{2(3-1)^3}{12(100)} \approx 0.013$.

b. $f'''(x) = -\dfrac{6}{x^4}$ and $f^{(4)}(x) = \dfrac{24}{x^5}$. $f^{(4)}(x)$ is decreasing on $(1, 3)$, so we can take $M = f^{(4)}(1) = 24$. Using Formula (8), we find the error bound $\dfrac{24(3-1)^5}{180\left(10^4\right)} \approx 0.00043$.

26. **a.** Here $a = 1$, $b = 3$, $n = 8$, and $f(x) = x^{-2}$. We find $f'(x) = -2x^{-3}$ and $f''(x) = 6x^{-4}$. Because f'' is positive and decreasing on $(1, 3)$, we have $\left|f''(x)\right| \leq 6$, so the maximum error is $\dfrac{6(3-1)^3}{12(8)^2} = \dfrac{48}{768} = 0.0625$.

b. $f'''(x) = -24x^{-5}$ and $f^{(4)}(x) = 120x^{-6}$. Because $f^{(4)}$ is positive and decreasing on $(1, 3)$, we have $\left|f^{(4)}(x)\right| \leq 120$, so the maximum error is $\dfrac{120(3-1)^5}{180(8)^4} = \dfrac{3840}{737{,}280} = 0.00521$.

27. **a.** Here $a = 0$, $b = 2$, $n = 8$, and $f(x) = (1+x)^{-1/2}$. We find $f'(x) = -\frac{1}{2}(1+x)^{-3/2}$ and $f''(x) = \frac{3}{4}(1+x)^{-5/2}$. Because f'' is positive and decreasing on $(0, 2)$, we see that $\left|f''(x)\right| \leq \frac{3}{4}$. So the maximum error is $\dfrac{\frac{3}{4}(2-0)^3}{12(8)^2} = 0.0078125$.

b. $f'''(x) = -\frac{15}{8}(1+x)^{-7/2}$ and $f^{(4)}(x) = \frac{105}{16}(1+x)^{-9/2}$. Because $f^{(4)}$ is positive and decreasing on $(0, 2)$, we find $\left|f^{(4)}(x)\right| \leq \frac{105}{16}$. Therefore, the maximum error is $\dfrac{\frac{105}{16}(2-0)^5}{180(8)^4} = 0.000285$.

28. **a.** Here $a = 1$, $b = 3$, $n = 10$, and $f(x) = \ln x$. We find $f'(x) = \dfrac{1}{x}$ and $f''(x) = -\dfrac{1}{x^2}$. Because $f''$is negative and increasing on $(1, 3)$, we see that $\left|f''(x)\right| \leq |-1| = 1$, so the maximum error is $\dfrac{1(3-1)^3}{12(10)^2} = 0.0067$.

b. $f'''(x) = 2x^{-3}$ and $f^{(4)}(x) = -6x^{-4}$. Because $f^{(4)}$ is negative and increasing on $(1, 3)$, we see that $\left|f^{(4)}(x)\right| \leq 6$, so the maximum error is $\dfrac{6(3-1)^5}{180(10)^4} = 0.000107$.

29. The distance covered is given by

$$d = \int_0^2 V(t)\,dt = \tfrac{1/4}{2}\left[V(0) + 2V\left(\tfrac{1}{4}\right) + \cdots + 2V\left(\tfrac{7}{4}\right) + V(2)\right]$$
$$= \tfrac{1}{8}[19.5 + 2(24.3) + 2(34.2) + 2(40.5) + 2(38.4) + 2(26.2) + 2(18) + 2(16) + 8] \approx 52.84, \text{ or } 52.84 \text{ miles.}$$

30. $$A = \tfrac{1000}{10} \cdot \tfrac{1}{3}[1000 + 4(900) + 2(1000) + 4(1000) + 2(1200) + 4(1400)$$
$$+ 2(1100) + 4(1100) + 2(1000) + 4(1200) + 1400] = \tfrac{100}{3}(33{,}400) \approx 1{,}113{,}333.33.$$

31. $\Delta x = \frac{30-0}{10} = 3$. Trapezoidal Rule:

$$\begin{aligned} A &= \tfrac{1}{30}\textstyle\int_0^{30} f(x)\,dx \\ &\approx \left(\tfrac{3}{2}\right)\left(\tfrac{1}{30}\right)[66 + 2(68 + 72 + 72 + 70 + 64 + 60 + 62 + 62 + 56) + 60] \approx 64.9^\circ\text{F}. \end{aligned}$$

Simpson's Rule:

$$\begin{aligned} A = \tfrac{1}{30}\textstyle\int_0^{30} f(x)\,dx \approx \left(\tfrac{1}{30}\right)\left(\tfrac{3}{3}\right)&[66 + 4(68) + 2(72) + 4(72) + 2(70) + 4(64) \\ &+ 2(60) + 4(62) + 2(62) + 4(56) + 60] \approx 64.73^\circ\text{F}. \end{aligned}$$

32. The required area is

$$\begin{aligned} A \approx \tfrac{206}{3}&[0 + 4(1030) + 2(1349) + 4(1498) + 2(1817) + 4(1910) + 2(1985) \\ &+ 4(2304) + 2(2585) + 4(2323) + 1592] = 3{,}661{,}580, \end{aligned}$$

or approximately 3,661,580 ft^2.

33. $$\begin{aligned} \tfrac{1}{13}\textstyle\int_0^{13} f(t)\,dt = \left(\tfrac{1}{13}\right)\left(\tfrac{1}{2}\right)&\{13.2 + 2[14.8 + 16.6 + 17.2 + 18.7 + 19.3 + 22.6 + 24.2 + 25 \\ &+ 24.6 + 25.6 + 26.4 + 26.6] + 26.6\} \approx 21.65, \text{ or } 21.65 \text{ mpg}. \end{aligned}$$

34. The required rate of flow is

$R =$ (rate of flow) (area of cross-section of the river) $=$ (4) (area of cross-section) $= 4\int_0^{78} y(x)\,dx$.

Approximating the integral using the trapezoidal rule, we have

$$\begin{aligned} R \approx (4)\left(\tfrac{6}{2}\right)&[0.8 + 2(2.6) + 2(5.8) + 2(6.2) + 2(8.2) + 2(10.1) + 2(10.8) \\ &+ 2(9.8) + 2(7.6) + 2(6.4) + 2(5.2) + 2(3.9) + 2(2.4) + 1.4] = 1922.4, \text{ or } 1922.4\ \text{ft}^3/\text{sec}. \end{aligned}$$

35. The total projected cost over the years from 2006 through 2014 is approximately $\int_1^9 C(t)\,dt$. Using the trapezoidal rule with $n = 8$, we have

$$\begin{aligned} \textstyle\int_1^9 C(t)\,dt &\approx \left(\tfrac{9-1}{8}\right)\left(\tfrac{1}{2}\right)[1.00 + 2(1.39) + 2(1.81) + 2(2.31) + 2(2.90) + 2(3.61) + 2(4.48) + 2(5.54) + 6.8] \\ &= 25.94. \end{aligned}$$

Thus, the total projected cost is approximately \$25.94 billion.

36. $\Delta t = \frac{5}{10} = 0.5$, so $t_0 = 0$, $t_1 = 0.5$, $t_2 = 1, \ldots, t_{10} = 5$.

$$\begin{aligned} PSI &= \frac{1}{5}\int_0^5 \left[\frac{136}{1 + 0.25(t-4.5)^2} + 28\right]dt = 27.2\textstyle\int_0^5 \left(\dfrac{1}{1 + 0.25(t-4.5)^2} + 28\right)dt \\ &\approx \frac{(0.5)(27.2)}{2}\left[\frac{1}{1 + 0.25(-4.5)^2} + \frac{2}{1 + 0.25(5-4.5)^2} + \cdots + \frac{1}{1 + 0.25(5-4.5)^2}\right] + 28 \approx 103.9. \end{aligned}$$

37. The average petroleum reserves from 2002 through 2012 were $A = \dfrac{1}{10-0}\displaystyle\int_0^{10} S(t)\,dt = \frac{1}{10}\int_0^{10} \frac{720t^2 + 3480}{t^2 + 6.3}\,dt$.

Using the Trapezoidal Rule with $a = 0$, $b = 10$, and $n = 10$, so that $\Delta t = \frac{10-0}{10} = 1$, we have $t_0 = 0$, $t = 1, \ldots$, $t_{10} = 10$. Thus,

$$\begin{aligned} A &= \tfrac{1}{10}\textstyle\int_0^{10} S(t)\,dt = \left(\tfrac{1}{10}\right)\left(\tfrac{1}{2}\right)[S(0) + 2S(1) + 2S(2) + \cdots + 2S(9) + S(10)] \\ &\approx \tfrac{1}{20}[552.38 + 2(575.34 + 617.48 + 650.98 + 672.65 + 686.26 + \\ &\qquad + 695.04 + 700.90 + 704.98 + 707.90) + 710.07] \approx 664.27, \end{aligned}$$

or approximately 664.27 million barrels.

38. Observe that if $p = 11$, then $x = 7$, so $CS = \int_0^7 D(x)\,dx - 11(7)$. Using the Trapezoidal Rule to approximate the integral, we find

$$\begin{aligned} CS &\approx \tfrac{7-0}{7} \cdot \tfrac{1}{2}\left[D(0) + 2D(1) + 2D(2) + \cdots + 2D(6) + D(7)\right] - 77 \\ &\approx \tfrac{1}{2}\left[80 + 2(61.5 + 44.4 + 32 + 23.5 + 17.8 + 13.8 + 11)\right] - 77 \\ &= 161.5\text{, or approximately \$161,500.} \end{aligned}$$

39. The value of Ivan's account at the end of 5 years is given by the future value of the income stream, that is, by $FV = e^{0.04(5)} \int_0^5 t^{1.5} e^{-0.02t} e^{-0.04t}\,dt = e^{0.2} \int_0^5 t^{1.5} e^{-0.06t}\,dt$. To approximate the integral using the Trapezoidal Rule with $n = 10$, we first find $\Delta t = \frac{5-0}{10} = \frac{1}{2}$. Thus, with $t_0 = 0$, $t_1 = \frac{1}{2}, \ldots, t_{10} = 5$, we have

$$\begin{aligned} \int_0^5 t^{1.5} e^{-0.06t}\,dt &\approx \tfrac{1/2}{2}\Big[0 + 2(0.5)^{1.5} e^{-0.06(0.5)} + 2(1.0)^{1.5} e^{-0.06(1.0)} + \cdots \\ &\qquad \cdots + 2(4.5)^{1.5} e^{-0.06(4.5)} + (5.0)^{1.5} e^{-0.06(5.0)}\Big] \\ &= \tfrac{1}{4}\left[0 + 2(0.34310 + 0.94176 + 1.67900 + 2.50859 + 3.40225\right. \\ &\qquad \left. + 4.34019 + 5.30762 + 6.29302 + 7.28718) + 8.28260\right] \\ &\approx 18.12201. \end{aligned}$$

Thus, $FV \approx e^{0.2}(18.12201) \approx 22.13427$, and at the end of 5 years, Ivan's account will be worth about \$22,134.

40. Solving the equation $25 = \dfrac{50}{0.01x^2 + 1}$, we see that $0.01x^2 + 1 = 2$, $0.01x^2 = 1$, and so $x = 10$. Therefore, $CS = \int_0^{10} \dfrac{50}{0.01x^2 + 1}\,dx - (25)(10)$. We have $\Delta x = \frac{10}{8} = 1.25$, so $x_0 = 0$, $x_1 = 1.25$, $x_2 = 2.50, \ldots, x_8 = 10$.

a. $CS = \dfrac{1.25}{2}\left\{50 + 2\left[\dfrac{50}{0.01(1.25)^2 + 1}\right] + \cdots + \left[\dfrac{50}{0.01(10)^2 + 1}\right]\right\} - 250 \approx 142{,}373.56$, or \$142,373.56.

b. $CS = \dfrac{1.25}{3}\left\{50 + 4\left[\dfrac{50}{0.01(1.25)^2 + 1}\right] + \cdots + \left[\dfrac{50}{0.01(10)^2 + 1}\right]\right\} - 250 \approx 142{,}698.12$, or \$142,698.12.

41. We solve the equation $8 = \sqrt{0.01x^2 + 0.11x + 38}$, finding $64 = 0.01x^2 + 0.11x + 38$, $0.01x^2 + 0.11x - 26 = 0$, $x^2 + 11x - 2600 = 0$, and so $x = \dfrac{-11 \pm \sqrt{121 + 10{,}400}}{2} \approx 45.786$ (we choose the positive root). Therefore, $PS = (8)(45.786) - \int_0^{45.786} \sqrt{0.01x^2 + 0.11x + 38}\,dx$. We have $\Delta x = \dfrac{45.786}{8} = 5.72$, so $x_0 = 0$, $x_1 = 5.72$, $x_2 = 11.44, \ldots, x_8 = 45.79$.

a. $$PS = 366.288 - \tfrac{5.72}{2}\Big[\sqrt{38} + 2\sqrt{0.01(5.72)^2 + 0.11(5.72) + 38} + \cdots + \sqrt{0.01(45.79)^2 + 0.11(45.79) + 38}\Big] \approx 51{,}558\text{, or \$51,558.}$$

b. $$PS = 366.288 - \tfrac{5.72}{3}\Big[\sqrt{38} + 4\sqrt{0.01(5.72)^2 + 0.11(5.72) + 38} + \cdots + \sqrt{0.01(45.79)^2 + 0.11(45.79) + 38}\Big] \approx 51{,}708\text{, or \$51,708.}$$

42. The Gini Index is given by $L = 2\int_0^1 [x - f(x)]\,dx = 2\int_0^1 x\,dx - 2\int_0^1 f(x)\,dx = 1 - 2\int_0^1 f(x)\,dx$. But $\int_0^1 f(x)\,dx \approx \frac{0.1}{2}\left[0.00 + 2(0.01 + 0.02 + 0.05 + 0.09 + 0.15 + 0.21 + 0.31 + 0.40 + 0.56) + 1\right] = 0.23$, so $I \approx 1 - 2(0.23) \approx 0.54$.

43. The Gini Index is given by $L = 2\int_0^1 \left[x - f(x)\right] dx = 2\int_0^1 x\,dx - 2\int_0^1 f(x)\,dx = 1 - 2\int_0^1 f(x)\,dx$. But

$$\begin{aligned}\int_0^1 f(x)\,dx \approx \tfrac{0.1}{3}\,[0.01 + 4(0.02) + 2(0.04) + 4(0.08) + 2(0.13)\\ + 4(0.20) + 2(0.27) + 4(0.36) + 2(0.47) + 4(0.60) + 1] = 0.262,\end{aligned}$$

so $I \approx 1 - 2(0.262) \approx 0.48$.

44. The percentage of the nonfarm work force in the country will continue to grow at the rate of $A = 30 + \int_0^1 5e^{1/(t+1)}\,dt$ percent, t decades from now. $\Delta t = \frac{1}{10} = 0.1$, $t_0 = 0$, $t_1 = 0.1$, …, $t_{10} = 1$. Using Simpson's Rule, we have $A = 30 + \frac{1}{3}\left(5e^1 + 4\cdot 5e^{1/1.1} + 2\cdot 5e^{1/1.2} + 4\cdot 5e^{1/1.3} + \cdots + 4\cdot 5e^{1/1.9} + 5e^{1/2}\right) = 40.1004$, or approximately 40.1%.

45. Observe that if $p = 69.7$, then $x = 12$, so $PS = 69.7(12) - \int_0^{12} S(x)\,dx$. To estimate the integral using Simpson's Rule with $n = 6$, we find $\Delta x = \frac{12-0}{6} = 2$. Thus, with $x_0 = 0$, $x_1 = 2$, …, $x_6 = 12$, we have

$$\begin{aligned}\int_0^{12} S(x)\,dx &\approx \tfrac{2}{3}\left[f(0) + 4f(2) + 2f(4) + 4f(6) + 2f(8) + 4f(10) + f(12)\right]\\ &= \tfrac{2}{3}\,[20.0 + 4(21.5) + 2(25.0) + 4(31.2) + 2(40.3) + 4(53.0) + 69.7]\\ &\approx 428.7333.\end{aligned}$$

Thus, the producers' surplus is $PS \approx 69.7(12) - 428.7333 \approx 407.6667$, or approximately \$407,667.

46. $R = \dfrac{60D}{\int_0^T C(t)dt} = \dfrac{480}{\int_0^{24} C(t)\,dt}$. Now

$$\begin{aligned}\int_0^{24} C(t)\,dt \approx \tfrac{24}{12}\cdot\tfrac{1}{3}\,[0 + 4(0) + 2(2.8) + 4(6.1) + 2(9.7) + 4(7.6) + 2(4.8)\\ + 4(3.7) + 2(1.9) + 4(0.8) + 2(0.3) + 4(0.1) + 0] \approx 74.8\end{aligned}$$

and $R = \frac{480}{74.8} \approx 6.42$, or 6.42 liters/min.

47. The required rate of flow is

$$\begin{aligned}R &= (4.2)\,(\text{area of cross-section})\\ &\approx (4.2)\left(\tfrac{6}{3}\right)[0.8 + 4(1.2) + 2(3) + 4(4.1) + 2(5.8) + 4(6.6) + 2(6.8) + 4(7) + 2(7.2) + 4(7.4)\\ &\qquad + 2(7.8) + 4(7.6) + 2(7.4) + 4(7) + 2(6.6) + 4(6) + 2(5.1) + 4(4.3) + 2(3.2) + 4(2.2) + 1.1]\\ &= 2698.92, \text{ or } 2698.92\ \text{ft}^3/\text{sec}.\end{aligned}$$

48. $\Delta x = \frac{40{,}000-30{,}000}{10} = 1000$, so $x_0 = 30{,}000$, $x_1 = 31{,}000$, x_2, …, $x_{10} = 40{,}000$. Now we approximate $P = \frac{100}{2000\sqrt{2\pi}}\int_{30{,}000}^{40{,}000} e^{-0.5[(x-40{,}000)/2000]^2}\,dx$ by $P = \frac{100(1000)}{2000\sqrt{2\pi}}\left[e^{-0.5[(30{,}000-40{,}000)/2000]^2} + 4e^{-0.5[(31{,}000-40{,}000)/2000]^2} + \cdots + 1\right] \approx 0.50$, or 50%.

49. $\Delta x = \frac{21-19}{10} = 0.2$, so $x_0 = 19$, $x_1 = 19.2$, $x_2 = 19.4$, …, $x_{10} = 21$.

$$\begin{aligned}P &= \tfrac{100}{2.6\sqrt{2\pi}}\int_{19}^{21} e^{-0.5[(x-20)/2.6]^2}\,dx\\ &\approx \tfrac{100}{2.6\sqrt{2\pi}}\left(\tfrac{0.2}{3}\right)\left\{e^{-0.5[(19-20)/2.6]^2} + 4e^{-0.5[(19.2-20)/2.6]^2} + \cdots + e^{-0.5[(21-20)/2.6]^2}\right\} \approx 29.94, \text{ or } 30\%.\end{aligned}$$

50. False. The number n can be odd or even.

51. False. The number n must be even for Simpson's Rule.

52. True.

53. True. Using Formula (8), we see that the error incurred in the approximation is zero since, in this situation $f^{(4)}(x) = 0$ for all x in $[a, b]$.

54. Taking the limit and recalling the definition of a Riemann sum, we find

$$\lim_{\Delta t \to 0} \frac{c(t_1) R\Delta t + c(t_2) R\Delta t + \cdots + c(t_n) R\Delta t}{60} = D, \; \frac{R}{60} \lim_{\Delta t \to 0} [c(t_1) \Delta t + c(t_2) \Delta t + \cdots + c(t_n) \Delta t] = D,$$

$$\frac{R}{60} \int_0^T c(t)\, dt = D, \text{ and so } R = \frac{60D}{\int_0^T c(t)\, dt}.$$

7.4 Improper Integrals

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1. a. $\int_a^\infty f(x)\, dx = \lim_{b \to \infty} \int_a^b f(x)\, dx$

b. $\int_{-\infty}^b f(x)\, dx = \lim_{a \to -\infty} \int_a^b f(x)\, dx$

c. $\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^c f(x)\, dx + \int_c^\infty f(x)\, dx$, where c is any real number.

2. A perpetuity is an annuity in which the payments continue indefinitely. The formula for computing the present value of a perpetuity is $PV = \dfrac{mP}{r}$.

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1. $$A = \int_2^\infty \frac{dx}{(x-1)^{3/2}} = \lim_{b \to \infty} \int_2^b (x-1)^{-3/2}\, dx = \lim_{b \to \infty} \left[-\frac{2}{(x-1)^{1/2}} \right]_2^b = \lim_{b \to \infty} \left[-\frac{2}{(b-1)^{1/2}} + 2 \right] = 2.$$

2. $$A = \int_0^\infty \frac{dx}{(x+1)^{4/3}} = \lim_{b \to \infty} \int_0^b (x+1)^{-4/3}\, dx = \lim_{b \to \infty} \left[-\frac{3}{(x+1)^{1/3}} \right]_0^b = \lim_{b \to \infty} \left[-\frac{3}{(b+1)^{1/3}} + 3 \right] = 3.$$

3. $$A = \int_0^\infty e^{-2x}\, dx = \lim_{b \to \infty} \int_0^b e^{-2x}\, dx = \lim_{b \to \infty} \left(-\tfrac{1}{2} e^{-2x} \right) \Big|_0^b = \lim_{b \to \infty} \left(-\tfrac{1}{2} e^{-2b} + \tfrac{1}{2} \right) = \tfrac{1}{2}.$$

4. $$A = -\int_{-\infty}^0 \left(e^{2x} - e^x \right) dx = -\lim_{a \to -\infty} \int_a^0 \left(e^{2x} - e^x \right) dx = -\lim_{a \to -\infty} \left(\tfrac{1}{2} e^{2x} - e^x \right) \Big|_a^0$$
$$= -\lim_{a \to -\infty} \left[\left(\tfrac{1}{2} - 1 \right) - \left(\tfrac{1}{2} e^{2a} - e^a \right) \right] = -\left(-\tfrac{1}{2} \right) = \tfrac{1}{2}.$$

5. The required area is given by $$\int_3^\infty \frac{2}{x^2}\, dx = \lim_{b \to \infty} \int_3^b \frac{2}{x^2}\, dx = \lim_{b \to \infty} \left(-\frac{2}{x} \right) \Big|_3^b = \lim_{b \to \infty} \left(-\frac{2}{b} + \frac{2}{3} \right) = \frac{2}{3}.$$

6. $$A = \int_2^\infty \frac{2}{x^3}\, dx = \lim_{b \to \infty} \int_2^\infty 2x^{-3}\, dx = \lim_{b \to \infty} - \frac{1}{x^2} \Big|_2^b = \lim_{b \to \infty} \left(-\frac{1}{b^2} + \frac{1}{4} \right) = \frac{1}{4}.$$

7. $$A = \int_3^\infty \frac{1}{(x-2)^2}\, dx = \lim_{b \to \infty} \int_3^b (x-2)^{-2}\, dx = \lim_{b \to \infty} \left(-\frac{1}{x-2} \right) \Big|_3^b = \lim_{b \to \infty} \left(-\frac{1}{b-2} + 1 \right) = 1.$$

8. $$A = \int_0^\infty \frac{2}{(x+1)^3}\, dx = \lim_{b \to \infty} \int_0^b 2(x+1)^{-3}\, dx = \lim_{b \to \infty} \left[-\frac{1}{(x+1)^2} \right]_0^b = \lim_{b \to \infty} \left[-\frac{1}{(b+1)^2} + 1 \right] = 1.$$

9. $A = \int_1^\infty \frac{1}{x^{3/2}}\,dx = \lim_{b\to\infty}\int_1^b x^{-3/2}\,dx = \lim_{b\to\infty}\left(-\frac{2}{\sqrt{x}}\right)\Big|_1^b = \lim_{b\to\infty}\left(-\frac{2}{\sqrt{b}}+2\right) = 2.$

10. $A = \int_4^\infty \frac{3}{x^{5/2}}\,dx = \lim_{b\to\infty}\int_4^\infty 3x^{-5/2}\,dx = \lim_{b\to\infty}\left(-2x^{-3/2}\right)\Big|_4^b = \lim_{b\to\infty}\left(-\frac{2}{b^{3/2}}+\frac{1}{4}\right) = \frac{1}{4}.$

11. $A = \int_0^\infty \frac{1}{(x+1)^{5/2}}\,dx = \lim_{b\to\infty}\int_1^b (x+1)^{-5/2}\,dx = \lim_{b\to\infty}\left[-\frac{2}{3}(x+1)^{-3/2}\right]_0^b$

$= \lim_{b\to\infty}\left[-\frac{2}{3(b+1)^{3/2}}+\frac{2}{3}\right] = \frac{2}{3}.$

12. $A = \int_{-\infty}^0 \frac{1}{(1-x)^{3/2}}dx = \lim_{a\to-\infty}\int_a^0 (1-x)^{-3/2}\,dx = \lim_{a\to-\infty}(-1)(-2)(1-x)^{-1/2}\Big|_a^0 = \lim_{a\to-\infty}\left(2-\frac{2}{\sqrt{1-a}}\right)$

$= 2.$

13. $A = \int_{-\infty}^2 e^{2x}\,dx = \lim_{a\to-\infty}\int_a^2 e^{2x}\,dx = \lim_{a\to-\infty}\frac{1}{2}e^{2x}\Big|_a^2 = \lim_{a\to-\infty}\left(\frac{1}{2}e^4-\frac{1}{2}e^{2a}\right) = \frac{1}{2}e^4.$

14. $a = \int_0^\infty xe^{-x^2}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x^2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Big|_0^b = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2}+\frac{1}{2}\right) = \frac{1}{2}.$

15. Using symmetry, the required area is given by $2\int_0^\infty \frac{x}{\left(1+x^2\right)^2}\,dx = 2\lim_{b\to\infty}\int_0^\infty \frac{x}{\left(1+x^2\right)^2}\,dx.$

To evaluate the indefinite integral $\int \frac{x}{\left(1+x^2\right)^2}\,dx$, put $u = 1+x^2$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. Then $\int \frac{x}{\left(1+x^2\right)^2}\,dx = \frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2u}+C = -\frac{1}{2\left(1+x^2\right)}+C$. Therefore,

$2\lim_{b\to\infty}\int_0^b \frac{x}{\left(1+x^2\right)}\,dx = \lim_{b\to\infty}\left[-\frac{1}{1+x^2}\right]_0^b = \lim_{b\to\infty}\left[-\frac{1}{1+b^2}+1\right] = 1.$

16. $A = \int_{-\infty}^\infty \frac{e^x}{(1+e^x)^2}\,dx = \int_{-\infty}^0 \frac{e^x}{(1+e^x)^2}\,dx + \int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx = \lim_{a\to-\infty}\left(-\frac{1}{1+e^x}\right)\Big|_a^0 + \lim_{b\to\infty}\left(-\frac{1}{1+e^x}\right)\Big|_0^b$

$= \lim_{a\to-\infty}\left(-\frac{1}{2}+\frac{1}{1+e^a}\right) + \lim_{b\to\infty}\left(-\frac{1}{1+e^b}+\frac{1}{2}\right) = \frac{1}{2}+\frac{1}{2} = 1.$

17. a. $I(b) = \int_0^b \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^b = \frac{2}{3}b^{3/2}.$

b. $\lim_{b\to\infty} I(b) = \lim_{b\to\infty}\frac{2}{3}b^{3/2} = \infty.$

18. a. $I(b) = \int_1^b x^{-2/3}\,dx = 3x^{1/3}\Big|_1^b = 3\sqrt[3]{b}-3(1) = 3\left(\sqrt[3]{b}-1\right).$

b. $\lim_{b\to\infty} I(b) = \lim_{b\to\infty} 3\left(\sqrt[3]{b}-1\right) = \infty$ and so the improper integral diverges.

19. $\int_1^\infty \frac{3}{x^4}\,dx = \lim_{b\to\infty}\int_1^b 3x^{-4}\,dx = \lim_{b\to\infty}\left(-\frac{1}{x^3}\right)\Big|_1^b = \lim_{b\to\infty}\left(-\frac{1}{b^3}+1\right) = 1.$

20. $\int_1^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty}\int_1^\infty x^{-3}\,dx = \lim_{b\to\infty}\left(-\frac{1}{2x^2}\right)\Big|_1^b = \lim_{b\to\infty}\left(-\frac{1}{2b^2}+\frac{1}{2}\right) = \frac{1}{2}.$

21. $A=\int_4^\infty \frac{2}{x^{3/2}}\,dx=\lim_{b\to\infty}\int_4^b 2x^{-3/2}\,dx=\lim_{b\to\infty}\left(-4x^{-1/2}\right)\Big|_4^b=\lim_{b\to\infty}\left(-\frac{4}{\sqrt{b}}+2\right)=2.$

22. $\int_1^\infty \frac{1}{\sqrt{x}}\,dx=\lim_{b\to\infty}2\sqrt{x}\Big|_1^b=\lim_{b\to\infty}\left(2\sqrt{b}-2\right)=\infty$, so the improper integral diverges.

23. $\int_1^\infty \frac{4}{x}\,dx=\lim_{b\to\infty}\int_1^b \frac{4}{x}\,dx=\lim_{b\to\infty}4\ln x\Big|_1^b=\lim_{b\to\infty}(4\ln b)=\infty.$

24. $\int_2^\infty \frac{3}{x}\,dx=\lim_{b\to\infty}\int_2^b \frac{3}{x}\,dx=\lim_{b\to\infty}3\ln x\Big|_2^b=\lim_{b\to\infty}(3\ln b-3\ln 2)=\infty.$

25. $\int_{-\infty}^0 (x-2)^{-3}\,dx=\lim_{a\to-\infty}\int_a^0 (x-2)^{-3}\,dx=\lim_{a\to-\infty}\left[-\frac{1}{2(x-2)^2}\right]_a^0=-\frac{1}{8}.$

26. $\int_2^\infty \frac{dx}{(x+1)^2}=\lim_{b\to\infty}\int_2^b (x+1)^{-2}\,dx=\lim_{b\to\infty}\left(-\frac{1}{x+1}\right)\Big|_2^b=\lim_{b\to\infty}\left(-\frac{1}{b+1}+\frac{1}{3}\right)=\frac{1}{3}.$

27. $\int_1^\infty \frac{1}{(2x-1)^{3/2}}\,dx=\lim_{b\to\infty}\int_1^b (2x-1)^{-3/2}\,dx=\lim_{b\to\infty}\left[-\frac{1}{(2x-1)^{1/2}}\right]_1^b=\lim_{b\to\infty}\left(-\frac{1}{\sqrt{2b-1}}+1\right)=1.$

28. $\int_{-\infty}^0 (4-x)^{-3/2}\,dx=\lim_{a\to-\infty}\int_a^0 (4-x)^{-3/2}\,dx=\lim_{a\to-\infty}2(4-x)^{-1/2}\Big|_a^0=1.$

29. $\int_0^\infty e^{-x}\,dx=\lim_{b\to\infty}\int_0^b e^{-x}\,dx=\lim_{b\to\infty}\left(-e^{-x}\right)\Big|_0^b=\lim_{b\to\infty}\left(-e^{-b}+1\right)=1.$

30. $\int_0^\infty e^{-x/2}\,dx=\lim_{b\to\infty}\int_0^b e^{-x/2}dx=\lim_{b\to\infty}\left(-2^{-e/2}\right)\Big|_0^b=\lim_{b\to\infty}\left(-2e^{-b/2}+2\right)=2.$

31. $\int_{-\infty}^0 e^{2x}\,dx=\lim_{a\to-\infty}\frac{1}{2}e^{2x}\Big|_a^0=\lim_{a\to-\infty}\left(\frac{1}{2}-\frac{1}{2}e^{2a}\right)=\frac{1}{2}.$

32. $\int_{-\infty}^0 e^{3x}\,dx=\lim_{a\to-\infty}\int_a^0 e^{3x}\,dx=\lim_{a\to-\infty}\frac{1}{3}e^{3x}\Big|_a^0=\frac{1}{3}.$

33. We use the substitution $u=\sqrt{x}$: $\int_1^\infty \frac{e^{\sqrt{x}}}{\sqrt{x}}dx=\lim_{b\to\infty}\int_1^b \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx=\lim_{b\to\infty}\left(-2e^{\sqrt{x}}\right)\Big|_1^b=\lim_{b\to\infty}\left(2e^{\sqrt{b}}-2e\right)=\infty$, and so the improper integral diverges.

34. We use the substitution $u=-\sqrt{x}$:

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx=\lim_{b\to\infty}\int_1^b \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx=\lim_{b\to\infty}-2e^{-\sqrt{x}}\Big|_1^b=\lim_{b\to\infty}\left(-2e^{-\sqrt{b}}+2e^{-1}\right)=\frac{2}{e}.$$

35. Integrating by parts, we have

$$\int_{-\infty}^0 xe^x\,dx=\lim_{a\to-\infty}\int_a^0 xe^x\,dx=\lim_{a\to-\infty}(x-1)e^x\Big|_a^0=\lim_{a\to-\infty}\left[-1+(a-1)e^a\right]=-1.$$

36. $I=\int_0^\infty xe^{-2x}\,dx$. Integrate by parts with $u=x$ and $dv=e^{-2x}$, so $du=dx$ and $v=-\frac{1}{2}e^{-2x}$. Then $I=-\frac{1}{2}xe^{-2x}+\frac{1}{2}\int e^{-2x}\,dx=-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}$. Next, $\lim_{b\to\infty}\int_0^b xe^{-2x}\,dx=\lim_{b\to\infty}\left(-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}\right)\Big|_0^b=\lim_{b\to\infty}\left(-\frac{1}{2}be^{-2b}-\frac{1}{4}e^{-2b}+\frac{1}{4}\right)=\frac{1}{4}.$

37. $\int_{-\infty}^{\infty} x\,dx = \lim\limits_{a\to-\infty} \frac{1}{2}x^2\Big|_a^0 + \lim\limits_{b\to\infty} \frac{1}{2}x^2\Big|_0^b$, both of which diverge, and so the integral diverges.

38. $\int_{-\infty}^{\infty} x^3\,dx = \lim\limits_{a\to-\infty} \int_a^0 x^3\,dx + \lim\limits_{b\to\infty} \int_0^b x^3\,dx$. But both integrals on the right diverge. For example, $\lim\limits_{b\to\infty} \int_0^b x^3\,dx = \lim\limits_{b\to\infty} \frac{1}{4}x^4\Big|_0^b = \lim\limits_{b\to\infty} \frac{1}{4}b^4 = \infty$. Thus, the given integral diverges.

39.
$$\begin{aligned}\int_{-\infty}^{\infty} x^3\left(1+x^4\right)^{-2}dx &= \int_{-\infty}^{0} x^3\left(1+x^4\right)^{-2}dx + \int_0^{\infty} x^3\left(1+x^4\right)^{-2}dx\\ &= \lim_{a\to-\infty}\int_a^0 x^3\left(1+x^4\right)^{-2}dx + \lim_{b\to\infty}\int_0^b x^3\left(1+x^4\right)^{-2}dx\\ &= \lim_{a\to-\infty}\left[-\tfrac{1}{4}\left(1+x^4\right)^{-1}\right]_a^0 + \lim_{b\to\infty}\left[-\tfrac{1}{4}\left(1+x^4\right)^{-1}\right]_0^b\\ &= \lim_{a\to-\infty}\left[-\frac{1}{4}+\frac{1}{4\left(1+a^4\right)}\right] + \lim_{b\to\infty}\left[-\frac{1}{4\left(1+b^4\right)}+\frac{1}{4}\right] = -\frac{1}{4}+\frac{1}{4} = 0.\end{aligned}$$

40.
$$\begin{aligned}\int_{-\infty}^{\infty} x\left(x^2+4\right)^{-3/2}dx &= \lim_{a\to-\infty}\int_a^0 x\left(x^2+4\right)^{-3/2}dx + \lim_{b\to\infty}\int_0^b x\left(x^2+4\right)^{-3/2}dx\\ &= \lim_{a\to-\infty}\left[-\left(x^2+4\right)^{-1/2}\right]_a^0 + \lim_{b\to\infty}\left[-\left(x^2+4\right)^{-1/2}\right]_0^b\\ &= \lim_{a\to-\infty}\left[-(4)^{-1/2}+\left(a^2+4\right)^{-1/2}\right] + \lim_{b\to\infty}\left[-\left(b^2+4\right)^{-1/2}+(4)^{-1/2}\right] = -\tfrac{1}{2}+\tfrac{1}{2} = 0.\end{aligned}$$

41.
$$\begin{aligned}\int_{-\infty}^{\infty} xe^{1-x^2}dx &= \lim_{a\to-\infty}\int_a^0 xe^{1-x^2}\,dx + \lim_{b\to\infty}\int_0^b xe^{1-x^2}\,dx = \lim_{a\to-\infty} -\tfrac{1}{2}e^{1-x^2}\Big|_a^0 + \lim_{b\to\infty} -\tfrac{1}{2}e^{1-x^2}\Big|_0^b\\ &= \lim_{a\to-\infty}\left(-\tfrac{1}{2}e+\tfrac{1}{2}e^{1-a^2}\right) + \lim_{b\to\infty}\left(-\tfrac{1}{2}e^{1-b^2}+\tfrac{1}{2}e\right) = 0.\end{aligned}$$

42.
$$\begin{aligned}\int_{-\infty}^{\infty}\left(x-\tfrac{1}{2}\right)e^{-x^2+x-1}\,dx &= \lim\nolimits_{a\to\infty}\int_a^0\left(x-\tfrac{1}{2}\right)e^{-x^2+x-1}\,dx + \lim_{b\to\infty}\int_0^b\left(x-\tfrac{1}{2}\right)e^{-x^2+x-1}\,dx\\ &= \lim_{a\to-\infty}\left(-\tfrac{1}{2}e^{-x^2+x-1}\right)\Big|_a^0 + \lim_{b\to\infty}\left(-\tfrac{1}{2}e^{-x^2+x-1}\right)\Big|_0^b = -\tfrac{1}{2}e^{-1}+\tfrac{1}{2}e^{-1} = 0.\end{aligned}$$

43. $\displaystyle\int_{-\infty}^{\infty}\frac{e^{-x}}{1+e^{-x}}dx = \lim_{a\to-\infty}\left[-\ln\left(1+e^{-x}\right)\right]_a^0 + \lim_{b\to\infty}\left[-\ln\left(1+e^{-x}\right)\right]_0^b = \infty$, so the integral diverges.

44.
$$\begin{aligned}\int_{-\infty}^{\infty}\frac{xe^{-x^2}}{1+e^{-x^2}}\,dx &= \lim_{a\to-\infty}\int_a^0\frac{xe^{-x^2}}{1+e^{-x^2}}\,dx + \lim_{b\to\infty}\int_0^b\frac{xe^{-x^2}}{1+e^{-x^2}}\,dx\\ &= \lim_{a\to-\infty}\left[-\tfrac{1}{2}\ln\left(1+e^{-x^2}\right)\right]_a^0 + \lim_{b\to\infty}\left[-\tfrac{1}{2}\ln\left(1+e^{-x^2}\right)\right]_0^b = -\tfrac{1}{2}\ln 2+\tfrac{1}{2}\ln 2 = 0.\end{aligned}$$

45. First, we find the indefinite integral $I = \displaystyle\int\frac{dx}{x\ln^3 x}$. Let $u = \ln x$, so $du = \dfrac{1}{x}\,dx$. Then $I = \displaystyle\int\frac{du}{u^3} = -\frac{1}{2u^2}+C = -\frac{1}{2\ln^2 x}+C$, so $\displaystyle\int_e^{\infty}\frac{dx}{x\ln^3 x} = \lim_{b\to\infty}\int_e^b\frac{dx}{x\ln^3 x} = \lim_{b\to\infty}\left(-\frac{1}{2\ln^2 x}\right)\Big|_e^b = \lim_{b\to\infty}\left[-\frac{1}{2(\ln b)^2}+\frac{1}{2}\right] = \frac{1}{2}$, and so the given integral converges.

46. $\displaystyle\int_{e^2}^{\infty}\frac{dx}{x\ln x} = \lim_{b\to\infty}\int_{e^2}^b\frac{dx}{x\ln x} = \lim_{b\to\infty}\left[\tfrac{1}{2}\ln(\ln x)\right]_{e^2}^b = \lim_{b\to\infty}\left[\tfrac{1}{2}\ln(\ln b)-\tfrac{1}{2}\ln 2\right] = \infty$, and so the given integral diverges.

47. We want the present value PV of a perpetuity with $m = 1$, $P = 1500$, and $r = 0.05$. We find $PV = \frac{(1)(1500)}{0.05} = 30{,}000$, or \$30,000.

48. We want the present value PV of a perpetuity with $m = 1$, $P = 50{,}000$, and $r = 0.06$. We find $PV = \frac{(1)(50{,}000)}{0.06} \approx 833{,}333$, or approximately \$833,333.

49. Using Formula (16) with $m = 1$, $P = 30{,}000$ and $r = 0.04$, we see that the present value of Heidi's investment is $PV = \frac{30{,}000}{0.04} = 750{,}000$, or \$750,000.

50. Integrating by parts, we find

$$PV = \int_0^\infty (10{,}000 + 4000t)\, e^{-rt}\, dt = 10{,}000 \int_0^\infty e^{-rt}\, dt + 4000 \int_0^\infty te^{-rt}\, dt$$
$$= \lim_{b\to\infty} \left[-\frac{10{,}000}{r} e^{-rt} - \frac{4000}{r^2} (rt + 1)\, e^{-rt} \right]_0^b = \frac{10{,}000}{r} + \frac{4000}{r^2} = \frac{10{,}000r + 4000}{r^2} \text{ dollars.}$$

51. Integrating by parts, we find

$$PV = \int_0^\infty (20 + t)\, e^{-0.04t}\, dt = \lim_{b\to\infty} \int_0^b 20e^{-0.04t}\, dt + \lim_{b\to\infty} \int_0^b te^{-0.04t}\, dt$$
$$= \lim_{b\to\infty} \left(-200e^{-0.4t}\right)\Big|_0^b + \lim_{b\to\infty} \left[100\,(-0.1t - 1)\, e^{-0.4t}\right]_0^b = 1125, \text{ or } \$1{,}125{,}000.$$

52. Using the formula in Exercise 50, we see that the present value of the investment is $PV = \int_0^\infty P(t)\, e^{-rt}\, dt = \int_0^\infty 2te^{0.02t} e^{-0.05t}\, dt = 2 \int_0^\infty te^{-0.03t}\, dt$. Consider the indefinite integral $I = \int te^{-0.03t}\, dt$. We integrate by parts with $u = t$ and $dv = e^{-0.03t}$, so $du = dt$ and $v = -\frac{1}{0.03} e^{-0.03t}$, and find that $I = -\dfrac{te^{-0.03t}}{0.03} + \dfrac{1}{0.03} \displaystyle\int e^{-0.03t}\, dt = -\dfrac{te^{-0.03t}}{0.03} - \dfrac{e^{-0.03t}}{0.00009} + C$. Thus, we have

$$PV = 2 \int_0^\infty te^{-0.03t}\, dt = \lim_{b\to\infty} 2 \int_0^b te^{-0.03t}\, dt = 2 \lim_{b\to\infty} \left[-\frac{be^{-0.03b}}{0.03} - \frac{e^{-0.03b}}{0.0009} + \frac{1}{0.0009} \right] = \frac{2}{0.0009} \approx 2222.22.$$

(We have used the hint from Exercise 50.) Thus, the present value of the investment is approximately \$22.222 million.

53. True. $\int_a^\infty f(x)\, dx = \int_a^b f(x)\, dx + \int_b^\infty f(x)\, dx$, so if $\int_a^\infty f(x)\, dx$ exists, then $\int_b^\infty f(x)\, dx = \int_a^\infty f(x)\, dx - \int_a^b f(x)\, dx$.

54. False. Take $f(x) = x$. Then

$\lim_{t\to\infty} \int_{-t}^t f(x)\, dx = \lim_{t\to\infty} \int_{-t}^t x\, dx = \lim_{t\to\infty} \left(\frac{1}{2}x^2\right)\Big|_{-t}^t = \lim_{t\to\infty} \left(\frac{1}{2}t^2 - \frac{1}{2}t^2\right) = \lim_{t\to\infty} 0 = 0.$

But $\int_{-\infty}^\infty f(x)\, dx$ does not exist because $\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^0 f(x)\, dx + \int_0^\infty f(x)\, dx$ and $\int_0^\infty f(x)\, dx = \lim_{b\to\infty} \left(\int_0^b x\, dx\right) = \lim_{b\to\infty} \frac{1}{2}b^2 = \infty$.

55. False. Let $f(x) = \begin{cases} e^{-2x} & \text{if } x \le 0 \\ e^{-x} & \text{if } x > 0 \end{cases}$. Then $\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^0 e^{2x}\, dx + \int_0^\infty e^{-x}\, dx = \frac{1}{2} + 1 = \frac{3}{2}$. But $2 \int_0^\infty f(x)\, dx = 2 \int_0^\infty e^{-x}\, dx = 2$.

56. False. Take $f(x) = e^{-x}$ and $a = 0$. Then $\int_0^\infty e^{-x}\, dx$ exists, but $\int_{-\infty}^0 e^{-x}\, dx$ does not exist.

57. a. $CV \approx \int_0^\infty Re^{-it}\,dt = \lim_{b\to\infty}\int_0^b Re^{-it}\,dt = \lim_{b\to\infty}\left(-\frac{R}{i}e^{-it}\right)\Big|_0^b = \lim_{b\to\infty}\left(-\frac{R}{i}e^{-ib}+\frac{R}{i}\right) = \frac{R}{i}.$

b. $CV = \frac{10{,}000}{0.06} \approx 166{,}667$, or \$166,667.

58. $\int_0^\infty e^{-px}\,dx = \lim_{b\to\infty}\int_a^b e^{-px}\,dx = \lim_{b\to\infty}\left(-\frac{1}{p}e^{-px}\right)\Big|_a^b = \lim_{b\to\infty}\left(-\frac{1}{p}e^{-pb}+\frac{1}{p}e^{-pa}\right) = \frac{1}{pe^{pa}}$ if $p > 0$. If $p \le 0$, the integral diverges.

59. $\int_{-\infty}^b e^{px}\,dx = \lim_{a\to-\infty}\int_a^b e^{px}\,dx = \lim_{a\to-\infty}\left(\frac{1}{p}e^{px}\right)\Big|_a^b = \lim_{a\to-\infty}\left(\frac{1}{p}e^{pb}-\frac{1}{p}e^{pa}\right) = -\frac{1}{p}e^{pa}$ if $p > 0$. If $p \le 0$, the integral diverges.

60. If $p < 0$, then $1/x^p$ is unbounded and the improper integral diverges. If $p > 0$, then $p \ne -1$, and so $\int_1^\infty \frac{1}{x^p\,dx} = \lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty}\left(\frac{x^{1-p}}{1-p}\right)\Big|_1^b = \lim_{b\to\infty}\left(\frac{b^{1-p}}{1-p}-\frac{1}{1-p}\right) = \frac{1}{p-1}$ if $p > 1$. If $p < 1$, the integral diverges to infinity. If $p = 1$, then $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty}\ln b = \infty$. So the integral converges if $p > 1$.

7.5 Applications of Calculus to Probability

Concept Questions page 551

1. See the definition on page 542 of the test. For example, $f(x) = \frac{3}{125}x^2$ on the interval $[0, 5]$.

2. **a.** See the definition on page 548 of the text.

   b. $E(x) = 1/k$

Exercises page 551

1. $f(x) = \frac{1}{16}x \ge 0$ on $[2, 6]$. Next $\int_2^6 \frac{1}{16}x\,dx = \frac{1}{32}x^2\Big|_2^6 = \frac{1}{32}(36-4) = 1$, and so f is a probability density function on $[2, 6]$.

2. $f(x) = \frac{2}{9}\left(3x-x^2\right) = \frac{2}{9}x(3-x)$ is nonnegative on $[0, 3]$ because the factors x and and $3-x$ are nonnegative there. Next, we compute $\int_0^3 \frac{2}{9}\left(3x-x^2\right)dx = \frac{2}{9}\left(\frac{3}{2}x^2-\frac{1}{3}x^3\right)\Big|_0^3 = \frac{2}{9}\left(\frac{27}{2}-9\right) = 1$, and so f is a probability density function.

3. $f(x) = \frac{3}{8}x^2$ is nonnegative on $[0, 2]$. Next, we compute $\int_0^2 \frac{3}{8}x^2\,dx = \frac{1}{8}x^3\Big|_0^2 = \frac{1}{8}(8) = 1$, and so f is a probability density function on $[0, 2]$.

4. Because $x-1 \ge 0$ and $5-x \ge 0$ on $[1, 5]$, we see that $f(x) \ge 0$ on $[1, 5]$. Next,

$$\int_1^5 f(x)\,dx = \frac{3}{32}\int_1^5 (x-1)(5-x)\,dx = \frac{3}{32}\int_1^5\left(-x^2+6x-5\right)dx = \frac{3}{32}\left(-\frac{1}{3}x^3+3x^2-5x\right)\Big|_1^5$$
$$= \frac{3}{32}\left[\left(-\frac{125}{3}+75-25\right)-\left(-\frac{1}{3}+3-5\right)\right] = 1,$$

and so f is a probability density function on $[1, 5]$.

5. $\int_0^1 20\left(x^3 - x^4\right) dx = 20\left(\frac{1}{4}x^4 - \frac{1}{5}x^5\right)\Big|_0^1 = 20\left(\frac{1}{4} - \frac{1}{5}\right) = 20\left(\frac{1}{20}\right) = 1$. Furthermore, $f(x) = 20\left(x^3 - x^4\right) = 20x^3(1 - x) \geq 0$ on $[0, 1]$. Therefore, f is a density function on $[0, 1]$, as asserted.

6. $f(x) = \dfrac{8}{7x^2}$ is nonnegative on $[1, 8]$. Next, we compute $\displaystyle\int_1^8 \frac{8}{7x^2}\, dx = -\frac{8}{7x}\bigg|_1^8 = -\frac{8}{7}\left(\frac{1}{8} - 1\right) = 1$, and so f is a probability density function on $[1, 8]$.

7. Clearly $f(x) \geq 0$ on $[1, 4]$. Next, $\int_1^4 f(x)\, dx = \frac{3}{14}\int_1^4 x^{1/2}\, dx = \left(\frac{3}{14}\right)\left(\frac{2}{3}x^{3/2}\right)\Big|_1^4 = \frac{1}{7}(8 - 1) = 1$, and so f is a probability density function on $[1, 4]$.

8. $f(x) = \dfrac{12 - x}{72}$ is nonnegative on $[0, 12]$. Next, we see that $\displaystyle\int_0^{12} \frac{12 - x}{72}\, dx = \int_0^{12}\left(\frac{1}{6} - \frac{x}{72}\right) dx = \left(\frac{1}{6}x - \frac{1}{144}x^2\right)\Big|_0^{12} = 2 - 1 = 1$, and conclude that f is a probability function on $[0, 12]$.

9. First, note that $f(x) \geq 0$ on $[0, \infty)$. Next, let $I = \int x\left(x^2 + 1\right)^{-3/2} dx$. Integrate I using the substitution $u = x^2 + 1$, so $du = 2x\, dx$. Then $I = \frac{1}{2}\int u^{-3/2} du = \frac{1}{2}\left(-2u^{-1/2}\right) + C = -\dfrac{1}{\sqrt{u}} + C = -\dfrac{1}{\sqrt{x^2 + 1}} + C$.

Therefore, $\displaystyle\int_0^\infty \frac{x\, dx}{\left(x^2 + 1\right)^{3/2}} = \lim_{b\to\infty}\left(-\frac{1}{\sqrt{x^2 + 1}}\right)\bigg|_0^b = \lim_{b\to\infty}\left(-\frac{1}{\sqrt{b^2 + 1}} + 1\right) = 1$, completing the proof.

10. First, it is easy to see that $f(x) = xe^{-2x^2} \geq 0$ on $[0, \infty)$. Next, $\int_0^\infty 4xe^{-2x^2}\, dx = \lim_{b\to\infty}\int_0^b 4xe^{-2x^2}\, dx = \lim_{b\to\infty}\left(-e^{-2x^2}\right)\Big|_0^b = \lim_{b\to\infty}\left(-e^{-2b^2} + 1\right) = 1$, and the proof is complete.

11. $\int_1^4 k\, dx = kx|_1^4 = 3k = 1$ implies that $k = \frac{1}{3}$.

12. $\int_0^4 kx\, dx = \frac{1}{2}kx^2\Big|_0^4 = 8k = 1$ implies that $k = \frac{1}{8}$.

13. $\int_0^4 k(4 - x)\, dx = k\int_0^4 (4 - x)\, dx = k\left(4x - \frac{1}{2}x^2\right)\Big|_0^4 = k(16 - 8) = 8k = 1$ implies that $k = \frac{1}{8}$.

14. $\int_0^1 kx^3\, dx = \frac{1}{4}kx^4\Big|_0^1 = \frac{1}{4}k = 1$ implies that $k = 4$.

15. $\int_0^4 kx^{1/2}\, dx = \frac{2}{3}kx^{3/2}\Big|_0^4 = \frac{16}{3}k = 1$ implies that $k = \frac{3}{16}$.

16. $\displaystyle\int_1^5 \frac{k}{x}\, dx = k\ln x|_1^5 = k\ln 5 = 1$ implies that $k = \dfrac{1}{\ln 5}$.

17. $\displaystyle\int_1^\infty \frac{k}{x^3}\, dx = \lim_{b\to\infty}\int_1^b kx^{-3}\, dx = \lim_{b\to\infty}\left(-\frac{k}{2x^2}\right)\bigg|_1^b = \lim_{b\to\infty}\left(-\frac{k}{2b^2} + \frac{k}{2}\right) = \frac{k}{2} = 1$ implies that $k = 2$.

18. $\int_0^\infty ke^{-x/2}\, dx = \lim_{b\to\infty}\int_0^b ke^{-x/2}\, dx = \lim_{b\to\infty} -2ke^{-x/2}\Big|_0^b = \lim_{b\to\infty}\left(-2ke^{-b/2} + 2k\right) = 2k = 1$ implies that $k = \frac{1}{2}$.

19. a. $P(2 \le X \le 4) = \int_2^4 \frac{1}{12}x\,dx = \frac{1}{24}x^2\Big|_2^4 = \frac{1}{24}(16-4) = \frac{1}{2}.$

b. $P(1 \le X \le 4) = \int_1^4 \frac{1}{12}x\,dx = \frac{1}{24}x^2\Big|_1^4 = \frac{1}{24}(16-1) = \frac{5}{8}.$

c. $P(X \ge 2) = \int_2^5 \frac{1}{12}x\,dx = \frac{1}{24}x^2\Big|_2^5 = \frac{1}{24}(25-4) = \frac{7}{8}.$

d. $P(X = 2) = \int_2^2 \frac{1}{12}x\,dx = 0.$

20. a. $P(1 \le X \le 2) = \int_1^2 \frac{1}{9}x^2\,dx = \frac{1}{27}x^3\Big|_1^2 = \frac{1}{27}(8-1) = \frac{7}{27}.$

b. $P(1 < X \le 3) = \int_1^3 \frac{1}{9}x^2\,dx = \frac{1}{27}x^3\Big|_1^3 = \frac{1}{27}(27-1) = \frac{26}{27}.$

c. $P(X \le 2) = \int_0^2 \frac{1}{9}x^2\,dx = \frac{1}{27}x^3\Big|_0^2 = \frac{1}{27}(8-0) = \frac{8}{27}.$

d. $P(X = 1) = \int_1^1 \frac{1}{9}x^2\,dx = 0.$

21. a. $P(-1 \le X \le 1) = \int_{-1}^1 \frac{3}{32}\left(4-x^2\right)dx = \frac{3}{32}\left(4x - \frac{1}{3}x^3\right)\Big|_{-1}^1 = \frac{3}{32}\left[\left(4-\frac{1}{3}\right) - \left(-4+\frac{1}{3}\right)\right] = \frac{11}{16}.$

b. $P(X \le 0) = \int_{-2}^0 \frac{3}{32}\left(4-x^2\right)dx = \frac{3}{32}\left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^0 = \frac{3}{32}\left[0 - \left(-8+\frac{8}{3}\right)\right] = \frac{1}{2}.$

c. $P(X > -1) = \int_{-1}^2 \frac{3}{32}\left(4-x^2\right)dx = \frac{3}{32}\left(4x - \frac{1}{3}x^3\right)\Big|_{-1}^2 = \frac{3}{32}\left[\left(8-\frac{8}{3}\right) - \left(-4+\frac{1}{3}\right)\right] = \frac{27}{32}.$

d. $P(X = 0) = \int_0^0 \frac{3}{32}\left(4-x^2\right)dx = 0.$

22. a. $P(1 < X < 3) = \int_1^3 \frac{3}{16}x^{1/2}\,dx = \frac{1}{8}x^{3/2}\Big|_1^3 = \frac{1}{8}\left(3\sqrt{3}-1\right) = 0.5245.$

b. $P(X \le 2) = \int_0^2 \frac{3}{16}x^{1/2}\,dx = \frac{1}{8}x^{3/2}\Big|_0^2 = \frac{\sqrt{2}}{4} \approx 0.3536.$

c. $P(X = 2) = \int_2^2 \frac{3}{16}x^{1/2}\,dx = 0.$

d. $P(X \ge 1) = \int_1^4 \frac{3}{16}x^{1/2}\,dx = \frac{1}{8}x^{3/2}\Big|_1^4 = \frac{1}{8}(8-1) = \frac{7}{8}.$

23. a. $P(X \ge 4) = \int_4^9 \frac{1}{4}x^{-1/2}\,dx = \frac{1}{2}x^{1/2}\Big|_4^9 = \frac{1}{2}(3-2) = \frac{1}{2}.$

b. $P(1 \le X < 8) = \int_1^8 \frac{1}{4}x^{-1/2}\,dx = \frac{1}{2}x^{1/2}\Big|_1^8 = \frac{1}{2}\left(2\sqrt{2}-1\right) \approx 0.9142.$

c. $P(X = 3) = \int_3^3 \frac{1}{4}x^{-1/2}\,dx = 0.$

d. $P(X \le 4) = \int_1^4 \frac{1}{4}x^{-1/2}\,dx = \frac{1}{2}x^{1/2}\Big|_1^4 = \frac{1}{2}(2-1) = \frac{1}{2}.$

24. a. $P(X \le 4) = \int_0^4 \frac{1}{2}e^{-x/2}\,dx = -e^{-x/2}\big|_0^4 = -e^{-2}+1 \approx 0.8647.$

b. $P(1 < X < 2) = \int_1^2 \frac{1}{2}e^{-x/2}\,dx = -e^{-x/2}\big|_1^2 = -e^{-1}+e^{-1/2} \approx 0.2387.$

c. $P(X = 50) = 0.$

d. $P(X \ge 2) = \int_2^\infty \frac{1}{2}e^{-x/2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{2}e^{-x/2}dx = \lim_{b\to\infty} \left(-e^{-x/2}\right)\Big|_2^b = \lim_{b\to\infty} \left(-e^{-b/2}+e^{-1}\right) = e^{-1} \approx 0.3679.$

25. a. $P(0 \le X \le 4) = \int_0^4 4xe^{-2x^2}\,dx = -e^{-2x^2}\Big|_0^4 = -e^{-32}+1 \approx 1.$

b. $P\left(X \geq 1\right) = \int_1^\infty 4xe^{-2x^2}dx = \lim\limits_{b\to\infty} \int_1^b 4xe^{-2x^2}dx = \lim\limits_{b\to\infty} \left(-e^{-2x^2}\right)\Big|_1^b = \lim\limits_{b\to\infty} \left(-e^{-2b^2} + e^{-2}\right) = e^{-2}$

$\approx 0.1353.$

26. a. $P\left(0 \leq X \leq 3\right) = \int_0^3 \frac{1}{9}xe^{-x/3}\,dx$. Let $I = \int xe^{-x/3}dx$ and integrate by parts with $u = x, dv = \frac{1}{9}e^{-x/3}dx$, $du = dx$, and $v = -\frac{1}{3}e^{-x/3}$. Then $I = -\frac{1}{3}xe^{-x/3} + \int \frac{1}{3}e^{-x/3}\,dx = -\frac{1}{3}xe^{-x/3} - e^{-x/3} + C = -\frac{1}{3}\left(x+3\right)e^{-x/3} + C$. Therefore, $P\left(0 \leq X \leq 3\right) = -\frac{1}{3}\left(x+3\right)e^{-x/3}\Big|_0^3 = -2e^{-1} + \frac{1}{3}\left(3\right) \approx 0.2642.$

b. $P\left(X \geq 1\right) = \int_1^\infty \frac{1}{9}xe^{-x/3} = \lim\limits_{b\to\infty} \int_1^b \frac{1}{9}xe^{-x/3}\,dx = \lim\limits_{b\to\infty} \left[-\frac{1}{3}\left(b+3\right)e^{-b/3} + \frac{1}{3}\left(4\right)e^{-1/3}\right] = \frac{4}{3}e^{-1/3}$ because $\lim\limits_{b\to\infty} e^{-b/3} = 0$ and $\lim\limits_{b\to\infty} be^{-b/3} = 0$. Thus, $P\left(x \geq 1\right) = \frac{4}{3}e^{-1/3} \approx 0.9554.$

27. $\mu = \int_3^6 \frac{1}{3}x\,dx = \frac{1}{6}x^2\Big|_3^6 = \frac{1}{6}\left(36-9\right) = \frac{9}{2}.$

28. $\mu = \int_2^6 \frac{1}{4}x\,dx = \frac{1}{8}x^2\Big|_2^6 = \frac{1}{8}\left(36-4\right) = 4.$

29. $\mu = \int_0^5 \frac{3}{125}x^3\,dx = \frac{3}{500}x^4\Big|_0^5 = \frac{15}{4}.$

30. $\mu = \int_0^2 \frac{3}{8}x^3\,dx = \frac{3}{32}x^4\Big|_0^2 = \frac{3}{2}.$

31. $\mu = \int_1^5 \frac{3}{32}x\left(x-1\right)\left(5-x\right)dx = \frac{3}{32}\int_1^5 \left(-x^3 + 6x^2 - 5x\right)dx = \frac{3}{32}\left(-\frac{1}{4}x^4 + 2x^3 - \frac{5}{2}x^2\right)\Big|_1^5$
$= \frac{3}{32}\left[\left(-\frac{625}{4} + 250 - \frac{125}{2}\right) - \left(-\frac{1}{4} + 2 - \frac{5}{2}\right)\right] = 3.$

32. $\mu = \int_0^1 20\left(x^4 - x^5\right)\,dx = 20\left(\frac{1}{5}x^5 - \frac{1}{6}x^6\right)\Big|_0^1 = 20\left(\frac{1}{5} - \frac{1}{6}\right) = \frac{2}{3}.$

33. $\mu = \displaystyle\int_1^8 \frac{8}{7x}\,dx = \frac{8}{7}\ln x\Big|_1^8 = \frac{8}{7}\ln 8 \approx 2.3765.$

34. $\mu = \displaystyle\int_1^4 \frac{4}{3x}\,dx = \frac{4}{3}\ln x\Big|_1^4 = \frac{4}{3}\ln 4 \approx 1.84839.$

35. $\mu = \int_1^4 \frac{3}{14}x^{3/2}dx = \frac{3}{35}x^{5/2}\Big|_1^4 = \frac{3}{35}\left(32-1\right) = \frac{93}{35}.$

36. $\mu = \int_0^1 \frac{5}{2}x^{5/2}dx = \frac{5}{7}x^{7/2}\Big|_0^1 = \frac{5}{7}.$

37. $\mu = \displaystyle\int_1^\infty \frac{3}{x^3}\,dx = \lim\limits_{b\to\infty} \int_1^b 3x^{-3}\,dx = \lim\limits_{b\to\infty} \left(-\frac{3}{2x^2}\right)\Big|_1^b = \lim\limits_{b\to\infty} \left(-\frac{3}{2b^2} + \frac{3}{2}\right) = \frac{3}{2}.$

38. $\mu = \int_1^\infty 3.5x^{-3.5}\,dx = \lim\limits_{b\to\infty} \int_1^b 3.5x^{-3.5}\,dx = \lim\limits_{b\to\infty} \left(-\frac{3.5}{2.5}x^{-2.5}\right)\Big|_1^b = \lim\limits_{b\to\infty} \left(-\frac{7}{5b^{2.5}} + \frac{7}{5}\right) = \frac{7}{5}.$

39. $\mu = \int_0^\infty \frac{1}{4} x e^{-x/4}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{4} x e^{-x/4}\,dx = \lim_{b\to\infty} 4\left(-\frac{1}{4} - 1\right) e^{-x/4}\Big|_0^b = \lim_{b\to\infty} \left[4\left(-\frac{1}{4}b - 1\right) e^{-b/4} + 4\right] = 4.$

40. $\mu = \int_0^\infty \frac{1}{9} x^2 e^{-x/3}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{9} x^2 e^{-x/3}\,dx = \lim_{b\to\infty} \left[-\frac{1}{3}x^2 e^{-x/3} + 6\left(-\frac{1}{3}x - 1\right) e^{-x/3}\right]_0^b$

$= \lim_{b\to\infty} \left[-\frac{1}{3}b^2 e^{-b/3} + 6\left(-\frac{1}{3}b - 1\right) e^{-b/3} + 6\right] = 6.$

41. **a.** Here $k = \frac{1}{15}$, so $f(x) = \frac{1}{15} e^{(-1/15)x}$.

b. The probability is $\int_{10}^{12} \frac{1}{15} e^{(-1/15)x}\,dx = -e^{(-1/15)x}\big|_{10}^{12} = -e^{-12/15} + e^{-10/15} \approx 0.06.$

c. The probability is

$\int_{15}^\infty \frac{1}{15} e^{(-1/15)x}\,dx = \lim_{b\to\infty} \int_{15}^b \frac{1}{15} e^{(-1/15)x}\,dx = \lim_{b\to\infty} \left[-e^{(-1/15)x}\right]_{15}^b = \lim_{b\to\infty} \left[-e^{(-1/15)b} + e^{-1}\right] \approx 0.37.$

42. **a.** $P(x \le 100) = \frac{1}{100} \int_0^{100} e^{-x/100}\,dx = -e^{-x/100}\big|_0^{100} = -e^{-1} + 1 \approx 0.63.$

b. $P(x \ge 120) = \frac{1}{100} \int_{120}^\infty e^{-x/100}\,dx = \lim_{b\to\infty} \left(-e^{-b/100} + e^{-1.2}\right) = e^{-1.2} \approx 0.30.$

c. $P(60 \le x \le 140) = \frac{1}{100} \int_{60}^{140} e^{-x/100}\,dx = -e^{-x/100}\big|_{60}^{140} = -e^{-1.4} + e^{-0.6} \approx 0.30.$

43. **a.** $P(X \le 30) = \int_0^{30} 0.02 e^{-0.02x}\,dx = -e^{-0.02x}\big|_0^{30} = -e^{-0.6} + 1 \approx 0.4512.$

b. $P(40 \le X \le 60) = \int_{40}^{60} 0.02 e^{-0.02x}\,dx = -e^{-0.02x}\big|_{40}^{60} = -e^{-1.2} + e^{-0.8} \approx 0.1481.$

c. $P(X \ge 70) = \int_{70}^\infty 0.02 e^{-0.02x}\,dx = \lim_{b\to\infty} \int_{70}^b 0.02 e^{-0.02x}\,dx = \lim_{b\to\infty} \left(-e^{-0.02x}\right)\big|_{70}^b = \lim_{b\to\infty} \left(-e^{-0.02b} + e^{-1.4}\right)$

$= e^{-1.4} \approx 0.2466.$

44. The required probability is given by

$P\left(0 \le x \le \frac{1}{2}\right) = \int_0^{1/2} 12x^2(1 - x)\,dx = 12 \int_0^{1/2} \left(x^2 - x^3\right) dx = 12\left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^{1/2}$

$= \left[12x^3\left(\frac{1}{3} - \frac{1}{4}x\right)\right]_0^{1/2} = 12\left(\frac{1}{2}\right)^3 \left[\frac{1}{3} - \frac{1}{4}\left(\frac{1}{2}\right)\right] - 0 = \frac{5}{16}.$

45. **a.** The probability density function is $0.001e^{-0.001x}$. The required probability is

$P(600 \le x \le 800) = 0.001 \int_{600}^{800} e^{-0.001x}\,dx = -e^{-0.001x}\big|_{600}^{800} = -e^{-0.8} + e^{-0.6} \approx 0.099.$

b. The probability is

$P(x \ge 1200) = 0.001 \int_{1200}^\infty e^{-0.001x}\,dx = 0.001 \lim_{b\to\infty} \int_{1200}^b e^{-0.001x}\,dx = \lim_{b\to\infty} \left(-e^{-0.001x}\right)\big|_{1200}^b$

$= \lim_{b\to\infty} -e^{-0.001b} + e^{-1.2} \approx 0.30.$

46. The probability density function is $f(x) = \frac{1}{8} e^{-x/8}$. The required probability is

$P(x \ge 8) = \frac{1}{8} \int_8^\infty e^{-x/8}\,dx = \lim_{b\to\infty} \left(-e^{-x/8}\right)\big|_8^b = \lim_{b\to\infty} \left(-e^{-b/8} + e^{-1}\right) = e^{-1} \approx 0.37.$

47. Here $f(x) = \frac{1}{30} e^{-x/30}$. Thus, $P(x \ge 120) = \frac{1}{30} \int_{120}^\infty e^{-x/30}\,dx = \lim_{b\to\infty} \left(-e^{-x/30}\right)\big|_{120}^b = e^{-120/30} \approx 0.02.$

48. The probability density function is $f(x) = 0.00001e^{-0.00001x}$. The required probability is

$P(x \le 20{,}000) = 0.00001 \int_0^{20{,}000} e^{-0.00001x}\,dx = -e^{-0.00001x}\big|_0^{20{,}000} = -e^{-0.2} + 1 \approx 0.18.$

49. Let X be a random variable that denotes the waiting time. Then X is uniformly distributed over the interval $[0, 15]$. So the associated probability density function is $f(x) = \frac{1}{15}$, $0 \leq X \leq 15$.

a. The desired probability is $P(X \geq 5) = P(5 \leq X \leq 15) = \int_5^{15} \frac{1}{15}\,dx = \frac{1}{15}x\Big|_5^{15} = \frac{10}{15} = \frac{2}{3}$.

b. The desired probability is $P(5 \leq X \leq 8) = \int_5^8 \frac{1}{15}\,dx = \frac{1}{15}x\Big|_5^8 = \frac{1}{5}$.

50. Let X denote the time (in minutes) that Joan has to wait for the next show. Then X is uniformly distributed over $[0, 8]$, and the uniform density function associated with this problem is $f(x) = \frac{1}{8}$, $0 \leq x \leq 8$. Joan will have to wait at most 5 minutes if and only if Joan arrives between 7:10 P.M. and 7:15 P.M., so the required probability is $P(3 \leq X \leq 8) = \int_3^8 \frac{1}{8}\,dx = \frac{1}{8}x\Big|_5^8 = \frac{1}{8}(8-5) = \frac{3}{8}$.

51. Let X denote the number of minutes past 8:00 P.M. that Joan arrives for the show. Since X is uniformly distributed over the interval $[0, 30]$, the uniform density function associated with this problem is $f(x) = \frac{1}{30}$, $0 \leq x \leq 30$.

a. In this case, Joan must arrive between 8:10 P.M. and 8:15 P.M. or between 8:25 P.M. and 8:30 P.M. Therefore, the required probability is $P(10 < X < 15) + P(25 < X < 30) = \int_{10}^{15} \frac{1}{30}\,dx + \int_{20}^{30} \frac{1}{30}\,dx = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

b. In this case, Joan must arrive between 8:00 P.M. and 8:05 P.M. or between 8:15 P.M. and 8:20 P.M. Therefore, the required probability is $P(0 < X < 5) + P(15 < X < 20) = \int_0^5 \frac{1}{30}\,dx + \int_{15}^{20} \frac{1}{30}\,dx = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

52. $\mu = \int_0^5 t \cdot \frac{2}{25}t\,dt = \frac{2}{25}\int_0^5 t^2\,dt = \frac{2}{75}t^3\Big|_0^5 = \frac{2}{75}(125) = \frac{10}{3}$, so a shopper is expected to spend 3 minutes 20 seconds in the magazine section.

53. $\mu = \displaystyle\int_1^3 t \cdot \frac{9}{4t^3}\,dt = \frac{9}{4}\int_1^3 t^{-2}\,dt = -\frac{9}{4t}\Big|_1^3 = -\frac{9}{4}\left(\frac{1}{3} - 1\right) = \frac{3}{2}$, so the expected reaction time is 1.5 seconds.

54. $\mu = \int_0^5 x \cdot \frac{6}{125}x(5-x)\,dx = \frac{6}{125}\int_0^5 \left(5x^2 - x^3\right)dx = \frac{6}{125}\left(\frac{5}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^5 = \frac{6}{125}\left(\frac{625}{3} - \frac{625}{4}\right) = 2.5$, so the expected demand is 2500 lb/wk.

55. $\mu = \int_0^3 x \cdot \frac{2}{9}x(3-x)\,dx = \frac{2}{9}\int_0^3 \left(3x^2 - x^3\right)dx = \frac{2}{9}\left(x^3 - \frac{1}{4}x^4\right)\Big|_0^3 = \frac{2}{9}\left(27 - \frac{81}{4}\right) = 1.5$, so the expected amount of snowfall is 1.5 ft.

56. $\mu = \int_2^3 x \cdot 4(x-2)^3\,dx = 4\int_2^3 \left(x^4 - 6x^3 + 12x^2 - 8x\right)dx = 4\left(\frac{1}{5}x^5 - \frac{3}{2}x^4 + 4x^3 - 4x^2\right)\Big|_2^3$

$= 4\left[\left(\frac{243}{5} - \frac{243}{2} + 108 - 36\right) - \left(\frac{32}{5} - 24 + 32 - 16\right)\right] = \frac{14}{5}$,

so the station can expect to sell 2800 gallons of gas on each Monday.

57. $\mu = \int_0^\infty t \cdot 9\left(9+t^2\right)^{-3/2}dt = \lim\limits_{b\to\infty}\int_0^b 9t\left(9+t^2\right)^{-3/2}dt = \lim\limits_{b\to\infty}\left[(9)\left(\frac{1}{2}\right)(-2)\left(9+t^2\right)^{-1/2}\right]_0^b$

$= \lim\limits_{b\to\infty}\left[-\dfrac{9}{\sqrt{9+b^2}} + 3\right] = 3$, so the plasma TVs are expected to last 3 years.

58. a. The probability that the product will fail in the time interval $[0, x]$ is $F(x) = \int_0^x f(x)\,dx$, so the probability that it will not fail in the time interval $[0, x]$ is $R(x) = 1 - F(x) = 1 - \int_0^x f(x)\,dx$.

b. $R(x) = 1 - \int_0^x ke^{-kx}\,dx = 1 - \left[k\left(-\frac{1}{k}\right)e^{-kx}\right]_0^x = 1 - \left(-e^{-kx} + 1\right) = e^{-kx}$.

c. Here $k = 0.05$ and $x = 10{,}000$, so $R(10) = e^{-(0.05)(10)} \approx 0.607$. Thus, the probability that the component will survive at least 10,000 hours is approximately 0.607.

59. We need to find $E(X) = \int_0^\infty x\, f(x)\, dx = \frac{1}{10}\int_0^\infty xe^{-x/10}\, dx = \frac{1}{10}\lim_{b\to\infty}\int_0^b xe^{-x/10}\, dx$. Using integration by parts or Formula (23) from Section 7.2, we find that

$$\int xe^{-x/10}\, dx = \frac{1}{\left(-\frac{1}{10}\right)^2}\left(-\frac{x}{10}-1\right)e^{-x/10} + C = -100\left(\frac{x+10}{10}\right)e^{-x/10} + C = -10\,(x+10)\,e^{-x/10}.$$

Using this result, we have

$E(x) = \frac{1}{10}(-10)\lim_{b\to\infty}\left[(x+10)\,e^{-x/10}\right]_0^b = -\lim_{b\to\infty}\left(be^{-b/10} + 10e^{-b/10} - 10\right) = (-1)(-10) = 10$, or 10 minutes.

60. $$E(X) = \int_a^b xf(x)\, dx = \int_a^b \frac{x}{b-a}\, dx = \frac{1}{2(b-a)}x^2\bigg|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2}$$

61. Refer to Exercise 60. The required average waiting time is $E(X) = \frac{1}{2}(a-b) = \frac{1}{2}(0+15) = \frac{15}{2}$, or $7\frac{1}{2}$ minutes.

62. a. Take $a = 0$ (6 a.m.) and $b = 60$ (7 a.m.) Then the required probability is
$P(30 \le X \le 45) = \frac{1}{60-0}\int_{30}^{45} dx = \frac{45-30}{60} = \frac{1}{4}$.

b. The expected delivery time is $E(X) = \frac{1}{2}(b+a) = \frac{1}{2}(60+0) = 30$; that is, 6:30 A.M.

63. Because f is a probability density function on $[0,1]$, we have
$\int_0^1 \left(ax^2+bx\right)dx = \left(\frac{1}{3}ax^3+\frac{1}{2}bx^2\right)\Big|_0^1 = \frac{1}{3}a+\frac{1}{2}b = 1$ and
$\int_0^1 \left(ax^3+bx^2\right)dx = \left(\frac{1}{4}ax^4+\frac{1}{3}bx^3\right)\Big|_0^1 = \frac{1}{4}a+\frac{1}{3}b = 0.6$. Solving the system of equations, we find $a = -2.4$ and $b = 3.6$.

64. Because f is a probability density function on $[1,e]$, we have
$\int_1^2 f(x)\,dx = \int_1^2 \left(ax+\frac{b}{x}\right)dx = \left(\frac{1}{2}ax^2+b\ln x\right)\Big|_1^2 = 1$, so $3a+(2\ln 2)\,b = 2$. Next, we want
$\int_1^2 xf(x)\,dx = \int_1^2 \left(ax^2+b\right)dx = \left(\frac{1}{3}ax^3+bx\right)\Big|_1^2 = \left(\frac{8}{3}a+2b\right)-\left(\frac{1}{3}a+b\right) = 2$, so $7a+3b = 6$. Solving the system $3a+(2\ln 2)\,b = 2$, $7a+3b = 6$, we find $a = \dfrac{6(1-2\ln 2)}{9-14\ln 2} \approx 3.3$ and $b = \dfrac{4}{9-14\ln 2} \approx -5.7$.

65. We require that $\int_0^\infty e^{-ax}(bx+c)\,dx = 1$. Consider $I = \int e^{-ax}(bx+c)\,dx$. Let $u = bx+c$ and $dv = e^{-ax}\,dx$, so $du = b\,dx$ and $v = -\frac{1}{a}e^{-ax}$. Thus, $I = -\dfrac{1}{a}(bx+c)\,e^{-ax} + \dfrac{b}{a}\displaystyle\int e^{-ax}\,dx = -\dfrac{1}{a}(bx+c)\,e^{-ax} - \dfrac{b}{a^2}e^{-ax} + C$.
Thus,

$$\int_0^\infty e^{-ax}(bx+c)\,dx = \lim_{L\to\infty}\int_0^L e^{-ax}(bx+c)\,dx = \lim_{L\to\infty}\left\{\left[-\frac{1}{a}(bL+C)\,e^{-aL}+\frac{b}{a^2}e^{-aL}\right]-\left(-\frac{c}{a}-\frac{b}{a^2}\right)\right\}$$
$$= \frac{b}{a^2}+\frac{c}{a} = \frac{b+ac}{a^2}.$$

Because $f(x) = e^{-ax}(bx+c)$ must be nonnegative, we see that $a > 0$, $b \ge 0$, $c \ge 0$, and $\dfrac{b+ac}{a^2} = 1$; that is, $b+ac = a^2$.

66. False. Since $f(1) = \frac{3}{2}-3 = -\frac{3}{2} < 0$, we see that f is not nonnegative on $[1,3]$.

67. True. Observe that $P(x < a) = \int_{-\infty}^{a} f(x)\,dx$ and $P(x > b) = \int_{b}^{\infty} f(x)\,dx$, so $P(x < a) + P(a < x < b) + P(x > b) = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$. Therefore, $P(x < a) + P(x > b) = 1 - \int_{a}^{b} f(x)\,dx$.

68. False. f must be nonnegative on $[a, b]$ as well.

69. False. The expected value of x is $\int_{a}^{b} xf(x)\,dx$.

70. False. Let $f(x) = 1$ for all x in the interval [0, 1]. Then f is a probability density function on [0, 1], but f is not a probability density function on $\left[\frac{1}{2}, \frac{3}{4}\right]$ because $\int_{1/2}^{3/4} 1\,dx = \frac{1}{4} \neq 1$.

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1. product, $uv - \int v\,du$, u, easy to integrate

2. $x^2 + 1$, $2x\,dx$, 27

3. $\frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$, $\frac{M(b-a)^3}{12n^2}$

4. $\frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\right]$, even, $\frac{M(b-a)^5}{180n^4}$

5. $\lim_{a\to-\infty} \int_{a}^{b} f(x)\,dx$, $\lim_{b\to\infty} \int_{a}^{b} f(x)\,dx$, $\int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$

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1. Let $u = 2x$ and $dv = e^{-x}\,dx$, so $du = 2\,dx$ and $v = -e^{-x}$. Then $\int 2xe^{-x}\,dx = uv - \int v\,du = -2xe^{-x} + 2\int e^{-x}\,dx = -2xe^{-x} - 2e^{-x} + C = -2(1+x)e^{-x} + C$.

2. Let $u = x$ and $dv = e^{4x}\,dx$, so $du = dx$ and $v = \frac{1}{4}e^{4x}$. Then $\int xe^{4x}\,dx = \frac{1}{4}xe^{4x} - \frac{1}{4}\int e^{4x}\,dx = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C = \frac{1}{16}(4x-1)e^{4x} + C$.

3. Let $u = \ln 5x$ and $dv = dx$, so $du = \frac{1}{x}\,dx$ and $v = x$. Then $\int \ln 5x\,dx = x\ln 5x\,dx - \int dx = x\ln 5x - x + C = x(\ln 5x - 1) + C$.

4. Let $u = \ln 2x$ and $dv = dx$, so $du = \frac{1}{x}\,dx$ and $v = x$. Then $\int_{1}^{4} \ln 2x\,dx = x\ln 2x|_{1}^{4} - \int_{1}^{4} dx = 4\ln 8 - \ln 2 - (4-1) = 4\ln 8 - \ln 2 - 3$.

5. Let $u = x$ and $dv = e^{-2x}\,dx$, so $du = dx$ and $v = -\frac{1}{2}e^{-2x}$. Then $\int_{0}^{1} xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x}\Big|_{0}^{1} + \frac{1}{2}\int_{0}^{1} e^{-2x}\,dx = -\frac{1}{2}e^{-2} - \left(\frac{1}{4}e^{-2x}\right)\Big|_{0}^{1} = -\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4} = \frac{1}{4}\left(1 - 3e^{-2}\right)$.

6. Let $u = x$ and $dv = e^{2x}\,dx$, so $du = dx$ and $v = \frac{1}{2}e^{2x}$. Then $\int_{0}^{2} xe^{2x}\,dx = \frac{1}{2}xe^{2x}\Big|_{0}^{2} - \frac{1}{2}\int_{0}^{2} e^{2x}\,dx = e^4 - \left(\frac{1}{4}e^{2x}\right)\Big|_{0}^{2} = e^4 - \frac{1}{4}e^4 + \frac{1}{4} = \frac{1}{4}\left(1 + 3e^4\right)$.

7. $f(x) = \int f'(x)\,dx = \int \frac{\ln x}{\sqrt{x}}\,dx$. To evaluate the integral, we use parts with $u = \ln x$, $dv = x^{-1/2}\,dx$, $du = \frac{1}{x}\,dx$ and $v = 2x^{1/2}$. Then

$$\int \frac{\ln x}{x^{1/2}}\,dx = 2x^{1/2}\ln x - \int 2x^{-1/2}\,dx = 2x^{1/2}\ln x - 4x^{1/2} + C = 2x^{1/2}(\ln x - 2) + C = 2\sqrt{x}(\ln x - 2) + C.$$

But $f(1) = -2$, giving $2\sqrt{1}(\ln 1 - 2) + C = -2$, so $C = 2$. Therefore, $f(x) = 2\sqrt{x}(\ln x - 2) + 2$.

8. $f'(x) = xe^{-3x}$. Let $u = x$ and $dv = e^{-3x}\,dx$, so $du = dx$ and $v = -\frac{1}{3}e^{-3x}$. Then $f(x) = uv - \int v\,du = -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}\,dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C$. Because $f(0) = 0$, $-\frac{1}{9} + C = 0$, and so $C = \frac{1}{9}$. Therefore, $f(x) = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + \frac{1}{9}$.

9. Using Formula (4) with $a = 3$, $b = 2$, and $u = x$, we have

$$\int \frac{x^2}{(3+2x)^2}\,dx = \frac{1}{8}\left(3 + 2x - \frac{9}{3+2x} - 6\ln|3+2x|\right) + C.$$

10. Using Formula (5) with $a = 3$, $b = 2$, and $u = x$, we have

$$\int \frac{2x}{\sqrt{2x+3}}\,dx = 2\int \frac{x}{\sqrt{2x+3}}\,dx = 2 \cdot \frac{2}{3(4)}(2x-6)\sqrt{2x+3} + C = \frac{2}{3}(x-3)\sqrt{2x+3} + C.$$

11. Using Formula (24) with $a = 4$, $n = 2$, and $u = x$, we have $\int x^2e^{4x}\,dx = \frac{1}{4}x^2e^{4x} - \frac{1}{2}\int xe^{4x}\,dx$. Now we use Formula (23) to obtain $\int x^2e^{4x}\,dx = \frac{1}{4}x^2e^{4x} - \frac{1}{2}\left[\frac{1}{16}(4x-1)e^{4x}\right] + C = \frac{1}{32}\left(8x^2 - 4x + 1\right)e^{4x} + C$.

12. Using Formula (18) with $a = 5$ and $u = x$, we have $\displaystyle\int \frac{dx}{\left(x^2-25\right)^{3/2}} = -\frac{x}{25\sqrt{x^2-25}} + C$.

13. Using Formula (17) with $a = 2$ and $u = x$, we have $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-4}} = \frac{\sqrt{x^2-4}}{4x} + C$.

14. First, we make the substitution $u = 2x$ so $du = 2\,dx$ and $dx = \frac{1}{2}\,du$. Then with $x = \frac{1}{2}u$, we have $\int 8x^3\ln 2x\,dx = \int 8\left(\frac{1}{2}u\right)^3\ln u\left(\frac{1}{2}du\right) = \frac{1}{2}\int u^3\ln u\,du$. Now we use Formula (27) with $n = 3$, obtaining $\int u^3\ln u\,du = \frac{1}{16}u^4(4\ln u - 1) + C$. Therefore, $\int 8x^3\ln 2x\,dx = \frac{1}{2}\cdot\frac{1}{16}(2x)^4(4\ln 2x - 1) + C = \frac{1}{2}x^4(4\ln 2x - 1) + C$.

15. $\displaystyle\int_0^\infty e^{-2x}\,dx = \lim_{b\to\infty}\int_0^b e^{-2x}\,dx = \lim_{b\to\infty}\left(-\tfrac{1}{2}e^{-2x}\right)\Big|_0^b = \lim_{b\to\infty}\left(-\tfrac{1}{2}e^{-2b} + \tfrac{1}{2}\right) = \tfrac{1}{2}$.

16. $\displaystyle\int_{-\infty}^0 e^{3x}\,dx = \lim_{a\to-\infty} e^{3x}\,dx = \lim_{a\to-\infty}\left(\tfrac{1}{3}e^{3x}\right)\Big|_a^0 = \lim_{a\to-\infty}\left(\tfrac{1}{3} - \tfrac{1}{3}e^{3a}\right) = \tfrac{1}{3}$.

17. $\displaystyle\int_3^\infty \frac{2}{x}\,dx = \lim_{b\to\infty}\int_3^b \frac{2}{x}\,dx = \lim_{b\to\infty} 2\ln x\Big|_3^b = \lim_{b\to\infty}(2\ln b - 2\ln 3) = \infty$.

18. $\displaystyle\int_2^\infty \frac{dx}{(x+2)^{3/2}} = \lim_{b\to\infty}\int_2^b (x+2)^{-3/2}\,dx = \lim_{b\to\infty}\left[-2(x+2)^{-1/2}\right]_2^b = \lim_{b\to\infty}\left[-\frac{2}{(b+2)^{1/2}} + 1\right] = 1$.

19. $\displaystyle\int_2^\infty \frac{dx}{(1+2x)^2} = \lim_{b\to\infty}\int_2^b (1+2x)^{-2}\,dx = \lim_{b\to\infty}\left(\tfrac{1}{2}\right)(-1)(1+2x)^{-1}\Big|_2^b = \lim_{b\to\infty}\left[-\frac{1}{2(1+2b)} + \frac{1}{2(5)}\right] = \frac{1}{10}$.

20. $\int_1^\infty 3e^{1-x}\,dx = \lim_{b\to\infty} \int_1^b 3e^{1-x}\,dx = \lim_{b\to\infty} \left(-3e^{1-x}\right)\Big|_1^b = \lim_{b\to\infty} \left(-3e^{1-b}+3\right) = 3.$

21. $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$, so $x_0 = 1, x_1 = \frac{3}{2}, x_2 = 2, x_3 = \frac{5}{2}, x_4 = 3$.

Trapezoidal Rule: $\displaystyle\int_1^3 \frac{dx}{1+\sqrt{x}} \approx \frac{\frac{1}{2}}{2}\left[\frac{1}{2} + \frac{2}{1+\sqrt{1.5}} + \frac{2}{1+\sqrt{2}} + \frac{2}{1+\sqrt{2.5}} + \frac{1}{1+\sqrt{3}}\right] \approx 0.8421.$

Simpson's Rule $\int_1^3 \displaystyle\frac{dx}{1+\sqrt{x}} \approx \frac{\frac{1}{2}}{3}\left[\frac{1}{2} + \frac{4}{1+\sqrt{1.5}} + \frac{2}{1+\sqrt{2}} + \frac{4}{1+\sqrt{2.5}} + \frac{1}{1+\sqrt{3}}\right] \approx 0.8404.$

22. $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$, so $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{2}{4}, x_3 = \frac{3}{4}, x_4 = \frac{4}{4}$.

Trapezoidal Rule: $\int_0^1 e^{x^2}\,dx \approx \frac{1/4}{2}\left[1 + 2e^{(0.25)^2} + 2e^{(0.5)^2} + 2e^{(0.75)^2} + e\right] \approx 1.4907.$

Simpson's Rule: $\int_0^1 e^{x^2}\,dx \approx \frac{1/4}{3}\left[1 + 4e^{(0.25)^2} + 2e^{(0.5)^2} + 4e^{(0.75)^2} + e\right] \approx 1.4637.$

23. $\Delta x = \frac{1-(-1)}{4} = \frac{1}{2}$, so $x_0 = -1, x_1 = -\frac{1}{2}, x_2 = 0, x_3 = \frac{1}{2}, x_4 = 1$.

Trapezoidal Rule: $\int_{-1}^1 \sqrt{1+x^4}\,dx \approx \frac{0.5}{2}\left[\sqrt{2} + 2\sqrt{1+(-0.5)^4} + 2 + 2\sqrt{1+(0.5)^4} + \sqrt{2}\right] \approx 2.2379.$

Simpson's Rule: $\int_{-1}^1 \sqrt{1+x^4}\,dx \approx \frac{0.5}{3}\left[\sqrt{2} + 4\sqrt{1+(-0.5)^4} + 2 + 4\sqrt{1+(0.5)^4} + \sqrt{2}\right] \approx 2.1791.$

24. Here $a = 1, b = 3, n = 4$ and $\Delta x = 0.5$, so $x_0 = 1, x_1 = 1.5, x_2 = 2, x_3 = 2.5, x_4 = 3$.

Trapezoidal Rule:

$$\int_1^3 \frac{e^x}{x}\,dx \approx \frac{0.5}{2}\left(e^1 + \frac{2e^{1.5}}{1.5} + \frac{2e^2}{2} + \frac{2e^{2.5}}{2.5} + \frac{e^3}{3}\right)$$

$$\approx 0.25\,(2.7182818 + 5.9755854 + 7.389056 + 9.7459952 + 6.695179) \approx 8.1310.$$

Simpson's Rule:

$$\int_1^3 \frac{e^x}{x}\,dx \approx \frac{0.5}{3}\left(e^1 + \frac{4e^{1.5}}{1.5} + \frac{2e^2}{2} + \frac{4e^{2.5}}{2.5} + \frac{e^3}{3}\right)$$

$$\approx 0.1666667\,(2.7182818 + 11.951171 + 7.389056 + 19.49199 + 6.695179) \approx 8.0409.$$

25. **a.** Here $a = 0, b = 1$, and $f(x) = \dfrac{1}{x+1}$. We have $f'(x) = -\dfrac{1}{(x+1)^2}$ and $f''(x) = \dfrac{2}{(x+1)^3}$. Because f'' is positive and decreasing on $(0, 1)$, it attains its maximum value of 2 at $x = 0$, so we take $M = 2$. Using Formula (7) from Section 7.3, we see that the maximum error incurred is $\dfrac{M(b-a)^3}{12n^2} = \dfrac{2(1^3)}{12(8^2)} = \dfrac{1}{384} \approx 0.002604.$

b. We compute $f'''(x) = -\dfrac{6}{(x+1)^4}$ and $f^{(4)}(x) = \dfrac{24}{(x+1)^5}$. Because $f^{(4)}(x)$ is positive and decreasing on $(0, 1)$, we take $M = 24$. The maximum error is $\dfrac{24(1^5)}{180(8^4)} = \dfrac{1}{30720} \approx 0.000033.$

26. $\frac{3}{128}\int_0^4 \left(16-x^2\right)dx = \frac{3}{128}\left(16x - \frac{1}{3}x^3\right)\Big|_0^4 = \frac{3}{128}\left(64 - \frac{64}{3}\right) = \frac{3}{128}\left(\frac{192-64}{3}\right) = 1.$ Also, $f(x) \geq 0$ on $[0, 4]$.

27. $f(x) \geq 0$ on $[0, 3]$. Next,

$$\tfrac{1}{9}\int_0^3 x\sqrt{9-x^2}\,dx = \tfrac{1}{9}\int_0^3 x\left(9-x^2\right)^{1/2}\,dx = \left(\tfrac{1}{9}\right)\left(-\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)\left(9-x^2\right)^{3/2}\Big|_0^3 = -\tfrac{1}{27}\left(9-x^2\right)^{3/2}\Big|_0^3$$
$$= 0 + \tfrac{1}{27}(9)^{3/2} = 1.$$

28. a. $$\int_0^2 kx\sqrt{4-x^2}\,dx = k\int_0^2 x\left(4-x^2\right)^{1/2}\,dx = k\left(-\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)\left(4-x^2\right)^{3/2}\Big|_{x=0}^{x=2} = \left(-\tfrac{k}{3}\right)\left(0-4^{3/2}\right)$$
$$= \tfrac{k}{3}(8) = 1, \text{ so } k = \tfrac{3}{8}.$$

b. $$\int_1^2 \tfrac{3}{8}x\sqrt{4-x^2}\,dx = \tfrac{3}{8}\left(-\tfrac{1}{3}\right)\left(4-x^2\right)^{3/2}\Big|_1^2 = -\tfrac{1}{8}\left(0-3^{3/2}\right) = 0.6495.$$

29. a. $$\int_1^4 \frac{k}{\sqrt{x}}\,dx = k\int_1^4 x^{-1/2}\,dx = 2k\sqrt{x}\Big|_{x=1}^{x=4} = 2k(2-1) = 2k = 1, \text{ so } k = \tfrac{1}{2}.$$

b. $$\int_2^3 \frac{\frac{1}{2}}{\sqrt{x}}\,dx = \frac{1}{2}\int_2^3 x^{-1/2}\,dx = 2\left(\tfrac{1}{2}\right)\sqrt{x}\Big|_2^3 = \sqrt{3}-\sqrt{2} = 0.3178.$$

30. a. $$\int_0^3 kx^2(3-x)\,dx = k\int_0^3\left(3x^2-x^3\right)dx = k\left(x^3-\tfrac{1}{4}x^4\right)\Big|_{x=0}^{x=3} = k\left(27-\tfrac{81}{4}\right) = \tfrac{27}{4}k = 1, \text{ so } k = \tfrac{4}{27}.$$

b. $$\int_1^2 \tfrac{4}{27}x^2(3-x)\,dx = \tfrac{4}{27}\int_1^2\left(3x^2-x^3\right)dx = \tfrac{4}{27}\left(x^3-\tfrac{1}{4}x^4\right)\Big|_1^2 = \tfrac{4}{27}\left[(8-4)-\left(1-\tfrac{1}{4}\right)\right] = \tfrac{13}{27}$$
$$\approx 0.4815.$$

31. Since f is a probability density function on $[0, 1]$, we have $\int_0^1 f(x)\,dx = 1$, so $\int_0^1\left(a+bx^2\right)dx = \left[ax+\frac{1}{3}bx^3\right]_0^1 = a+\frac{1}{3}b = 1$. Next $E(X) = 0.6$ gives $E(x) = \int_0^1 xf(x)\,dx = \int_0^1 x\left(a+bx^2\right)dx = \int_0^1\left(ax+bx^3\right)dx = \left[\frac{1}{2}ax^2+\frac{1}{4}bx^4\right]_0^1 = \frac{1}{2}a+\frac{1}{4}b = 0.6 = \frac{3}{5}$. So we have a system of equations in a and b: $a+\frac{1}{3}b = 1$ and $\frac{1}{2}a+\frac{1}{4}b = \frac{3}{5}$. The first equation gives $a = 1-\frac{1}{3}b$. Substituting into the second equation gives $\frac{1}{2}\left(1-\frac{1}{3}b\right)+\frac{1}{4}b = \frac{3}{5}$, so $\frac{1}{2}-\frac{1}{6}b+\frac{1}{4}b = \frac{3}{5}$, $30-10b+15b = 36$, $5b = 6$, and $b = \frac{6}{5}$. Using the first equation in the original system, we find that $a = 1-\frac{1}{3}b = 1-\frac{1}{3}\left(\frac{6}{5}\right) = \frac{3}{5}$. Therefore, $a = \frac{3}{5}$ and $b = \frac{6}{5}$.

32. a. The probability that a woman entering the maternity wing stays for more than 6 days is given by

$$\int_6^\infty \tfrac{1}{4}e^{-0.25t}\,dt = \lim_{b\to\infty}\int_6^b \tfrac{1}{4}e^{-0.25t}\,dt = \lim_{b\to\infty}\left(-e^{-0.25t}\right)\Big|_6^b = \lim_{b\to\infty}\left(-e^{-0.25b}+e^{-1.5}\right) \approx 0.22.$$

b. The probability that a woman entering the maternity wing at the hospital stays for less than 2 days is given by

$$\int_0^2 \tfrac{1}{4}e^{-0.25t}\,dt = -e^{-0.25t}\Big|_0^2 = -e^{-0.5}+1 \approx 0.39.$$

c. $E = \dfrac{1}{k} = \dfrac{1}{1/4} = 4$, or 4 days.

33. $$P(0 \leq X \leq 2) = \int_0^2 \tfrac{1}{16}\left(6x-x^2\right)dx = \tfrac{1}{16}\left(3x^2-\tfrac{1}{3}x^3\right)\Big|_0^2 = \tfrac{1}{16}\left(12-\tfrac{8}{3}\right) = \tfrac{7}{12} \approx 0.5833.$$

34. a. $$\int_0^4 \tfrac{1}{5}e^{-x/5}\,dx = 1-e^{-4/5} = 0.5507.$$

b. $$\int_6^\infty \tfrac{1}{5}e^{-x/5}\,dx = \lim_{b\to\infty}\left(-e^{-x/5}\right)\Big|_6^\infty = 0.3012.$$

c. $\int_2^4 \frac{1}{5}e^{-x/5}\,dx = 0.2210$.

35. The producer's surplus is given by $PS = \overline{p}\,\overline{x} - \int_0^{\overline{x}} s(x)\,dx$, where $\overline{x}$ is found by solving the equation $2\sqrt{25+x^2} = 26$. Thus, $\sqrt{25+x^2} = 13$, $25+x^2 = 169$, and $x = \pm 12$. Therefore, $\overline{x} = 12$, and so $PS = (26)(12) - 2\int_0^{12}\left(25+x^2\right)^{1/2}dx$. Using Formula (7) from Section 7.2 with $a = 5$, we obtain

$$PS = (26)(12) - 2\int_0^{12}\left(25+x^2\right)^{1/2}dx = 312 - 2\left[\tfrac{1}{2}x\left(25+x^2\right)^{1/2} + \tfrac{25}{2}\ln\left|x + \left(25+x^2\right)^{1/2}\right|\right]_0^{12}$$

$$= 312 - 2\left[6(13) + \tfrac{25}{2}\ln(12+13) - \tfrac{25}{2}\ln 5\right] \approx 115.76405, \text{ or } \$1{,}157{,}641.$$

36. Integrate by parts with $u = t$ and $dv = e^{-0.05t}$, so $du = dt$ and $v = -20e^{-0.05t}$. Thus,

$$S(t) = -20te^{-0.05t} + \int 20e^{-0.05t}dt = -20te^{-0.05t} - 400e^{-0.05t} + C = -20te^{-0.05t} - 400e^{-0.05t} + C$$

$$= -20e^{-0.05t}(t+20) + C.$$

The initial condition implies that $S(0) = 0$, giving $-20(20) + C = 0$, and so $C = 400$. Therefore, $S(t) = -20e^{-0.05t}(t+20) + 400$. By the end of the first year, the number of units sold is given by $S(12) = -20e^{-0.6}(32) + 400 = 48.761$, or 48,761 games.

37. If $p = 30$, we have $2\sqrt{325-x^2} = 30$, $\sqrt{325-x^2} = 15$, $325 - x^2 = 225$, $x^2 = 100$, and so $x = \pm 10$. The equilibrium point is thus $(10, 30)$, and $CS = \int_0^{10} 2\sqrt{325-x^2}\,dx - (30)(10)$. To approximate the integral using Simpson's Rule with $n = 10$, we have $\Delta x = \frac{10-0}{10} = 1$, so $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, …, $x_{10} = 10$. Thus,

$$2\int_0^{10}\sqrt{325-x^2}\,dx \approx \tfrac{2}{3}\left(\sqrt{325} + 4\sqrt{325-1} + 2\sqrt{325-4} + \cdots + 4\sqrt{325-81} + \sqrt{325-100}\right)$$

$$\approx 341.1, \text{ so } CS \approx 341.1 - 300 \approx 41.1, \text{ or } \$41{,}100.$$

38. Trapezoidal Rule: $A \approx \frac{100}{2}(0 + 480 + 520 + 600 + 680 + 680 + 800 + 680 + 600 + 440 + 0) = 274{,}000$, or 274,000 ft^2.

Simpson's Rule: $A \approx \frac{100}{3}(0 + 960 + 520 + 1200 + 680 + 1360 + 800 + 1360 + 600 + 880 + 0) = 278{,}667$, or 278,667 ft^2.

39. Think of the upper curve as the graph of f and the lower curve as the graph of g. Then the required area is given by $A = \int_0^{150}\left[f(x) - g(x)\right]dx = \int_0^{150} h(x)\,dx$,where $h = f - g$. Using Simpson's Rule,

$$A \approx \tfrac{15}{3}\left[h(0) + 4h(1) + 2f(2) + 4f(3) + 2f(4) + \cdots + 4f(9) + f(10)\right]$$

$$= 5\left[0 + 4(25) + 2(40) + 4(70) + 2(80) + 4(90) + 2(65) + 4(50) + 2(60) + 4(35) + 0\right] = 7850,$$

or 7850 ft^2.

40. We want the present value of a perpetuity with $m = 1$, $P = 10{,}000$, and $r = 0.09$. We find

$$PV = \frac{(1)(10{,}000)}{0.09} \approx 111{,}111, \text{ or approximately } \$111{,}111.$$

CHAPTER 7 Before Moving On... page 558

1. Let $u = \ln x$ and $dv = x^2\,dx$, so $du = \dfrac{1}{x}\,dx$ and $v = \frac{1}{3}x^3$. Then
$\int x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2\,dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C = \frac{1}{9}x^3\,(3\ln x - 1) + C.$

2. $I = \displaystyle\int \frac{dx}{x^2\sqrt{8+2x^2}}$. Let $u = \sqrt{2}x$, so $du = \sqrt{2}\,dx$ and $dx = \frac{\sqrt{2}}{2}\,du$. Then
$I = \dfrac{\sqrt{2}}{2}\displaystyle\int \frac{du}{\frac{1}{2}u^2\sqrt{8+u^2}} = \sqrt{2}\int \frac{du}{u^2\sqrt{\left(2\sqrt{2}\right)^2+u^2}}$. Using Formula (11) from Section 7.2 with $a = 2\sqrt{2}$ and
$x = u$, $I = \sqrt{2}\displaystyle\int \frac{du}{u^2\sqrt{\left(2\sqrt{2}\right)^2+u^2}} = \sqrt{2}\left(-\frac{\sqrt{8+u^2}}{8u}\right) + C = -\frac{\sqrt{8+2x^2}}{8x} + C.$

3. $n = 5$, so $\Delta x = \frac{4-2}{5} = 0.4$ and $x_0 = 2$, $x_1 = 2.4$, $x_2 = 2.8$, $x_3 = 3.2$, $x_4 = 3.6$, $x_5 = 4$. Thus,
$\int_2^4 \sqrt{x^2+1}\,dx \approx \frac{0.4}{2}\left[f\,(2) + 2f\,(2.4) + 2f\,(2.8) + 2f\,(3.2) + 2f\,(3.2) + 2f\,(3.6) + f\,(4)\right]$
$= 0.2\,[2.23607 + 2\,(2.6) + 2\,(2.97321) + 2\,(3.35261) + 2\,(3.73631) + 4.12311] \approx 6.3367.$

4. $n = 6$, so $\Delta x = \frac{3-1}{6} = \frac{1}{3}$ and $x_0 = 1$, $x_1 = \frac{4}{3}$, $x_2 = \frac{5}{3}$, $x_3 = 2$, $x_4 = \frac{7}{3}$, $x_5 = \frac{8}{3}$, $x_6 = 3$.
$\int_1^3 e^{0.2x}\,dx \approx \frac{1/3}{3}\left[f\,(1) + 4f\left(\frac{4}{3}\right) + 2f\left(\frac{5}{3}\right) + 4f\,(2) + 2f\left(\frac{7}{3}\right) + 4f\left(\frac{8}{3}\right) + f\,(3)\right]$
$\approx \frac{1}{9}\,[1.2214 + 4\,(1.30561) + 2\,(1.39561) + 4\,(1.49182) + 2\,(1.59467) + 4\,(1.7046) + 1.82212] \approx 3.0036.$

5. $\int_1^\infty e^{-2x}\,dx = \lim\limits_{b\to\infty} \int_1^b e^{-2x}\,dx = \lim\limits_{b\to\infty}\left(-\frac{1}{2}\,e^{-2x}\Big|_1^b\right) = \lim\limits_{b\to\infty}\left(-\frac{1}{2}e^{-2b} + \frac{1}{2}e^{-2}\right) = \frac{1}{2}e^{-2} = \dfrac{1}{2e^2}.$

6. a. $f\,(x) \geq 0$ on $[0, 8]$ and $\int_0^8 \frac{5}{96}x^{2/3}\,dx = \frac{5}{96}\left(\frac{3}{5}x^{5/3}\right)\Big|_0^8 = \frac{5}{96}\left(\frac{3}{5}\right)\left(8^{5/3}\right) = 1$. Therefore, f is a probability density function on $[0, 8]$.

b. $P\,(1 \leq X \leq 8) = \int_1^8 \frac{5}{96}x^{2/3}\,dx = \frac{5}{96}\left(\frac{3}{5}x^{5/3}\right)\Big|_1^8 = \frac{1}{32}\,(32 - 1) = \frac{31}{32}.$

CHAPTER 7 Explore & Discuss

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1. Let $u = x^n$ and $dv = e^{ax}\,dx$, so $du = nx^{n-1}\,dx$ and $v = \dfrac{1}{a}e^{ax}$. Then
$\int x^n e^{ax}\,dx = x^n \cdot \frac{1}{a}e^{ax} - \frac{1}{a}\int e^{ax} nx^{n-1}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx.$

2. Let $n = 3$ and $a = 1$. Then $\int x^3 e^x\,dx = x^3 e^x - 3\int x^2 e^x\,dx$. Using the results of Example 4, we have
$\int x^3 e^x\,dx = x^3 e^x - 3e^x\left(x^2 - 2x + 2\right) + C = e^x\left(x^3 - 3x^2 + 6x - 6\right) + C.$

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Differentiate the antiderivative (the expression on the right of the formula) and show that it is equal to the integrand. For example, we can verify Formula (16) as follows:

$$\frac{d}{du}\left(\ln\left|u+\sqrt{u^2-a^2}\right|+C\right) = \frac{\frac{d}{du}\left[u+\left(u^2-a^2\right)^{1/2}\right]}{u+\left(u^2-a^2\right)^{1/2}} = \frac{1+\frac{1}{2}\left(u^2-a^2\right)^{-1/2}(2u)}{u+\left(u^2-a^2\right)^{1/2}}$$

$$= \frac{\left(u^2-a^2\right)^{-1/2}\left[\left(u^2-a^2\right)^{1/2}+u\right]}{u+\left(u^2-a^2\right)^{1/2}} = \frac{1}{\sqrt{u^2-a^2}}.$$

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Let $v = a + bu$, so $dv = b\,du$ and $du = \frac{1}{b}\,dv$. From the first equation, we find $u = \frac{1}{b}(v-a)$. Substituting, we have

$$\int \frac{u\,du}{a+bu} = \int \frac{\frac{1}{b}(v-a)}{v}\cdot\frac{1}{b}\,dv = \frac{1}{b^2}\int\left(1-\frac{a}{v}\right)dv = \frac{1}{b^2}\left(v-a\ln|v|\right) = \frac{1}{b^2}\left(a+bu-a\ln|a+bu|\right)+C.$$

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Here $a = 0$, $b = 2$, and $n = 10$, so $\Delta x = \frac{b-a}{n} = \frac{2}{10} = 0.2$ and $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4, \ldots, x_{10} = 2$. The Trapezoidal Rule yields

$$\int_0^2 f(x)\,dx = \frac{0.2}{2}\left[1+2\sqrt{1+(0.2)^2}+2\sqrt{1+(0.4)^2}+\cdots+2\sqrt{1+(1)^2} + 2\cdot\frac{2}{\sqrt{1+(1.2)^2}}+2\cdot\frac{2}{\sqrt{1+(1.4)^2}}+\cdots+\frac{2}{\sqrt{1+(2)^2}}\right]$$

$$\approx 0.1\,(1+2.0396+2.1541+2.3324+2.5612+2.8284+2.5607+2.3250+2.1200+1.9426+0.8944)$$

$$\approx 2.276.$$

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Here $a = 0$, $b = 2$, and $n = 10$, so $\Delta x = \frac{b-a}{n} = \frac{2}{10} = 0.2$ and $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, ..., $x_{10} = 2$. Simpson's Rule yields $\int f(x)\,dx = \frac{0.2}{3}\left[1+4\sqrt{1+(0.2)^2}+2\sqrt{1+(0.4)^2}+\cdots+4\sqrt{1+(1)^2}\right.$

$$\left.+2\cdot\frac{2}{\sqrt{1+(1.2)^2}}+4\cdot\frac{2}{\sqrt{1+(1.4)^2}}+\cdots+\frac{2}{\sqrt{1+(2)^2}}\right]$$

$$\approx \frac{0.2}{3}\,(1+4.0792+2.1541+4.6648+2.5612+5.6569+2.5607+4.6499+2.1200+3.8851+0.8944)$$

$$\approx 2.282.$$

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1. To find the maximum error in using the Trapezoidal Rule to approximate $\int_0^2 f(x)\,dx$ with $n = 10$, we find M_1 such that $\left|f''(x)\right| < M_1$, where $f(x) = \sqrt{1+x^2}$ on $[0, 1]$, and M_2 such that $\left|f''(x)\right| < M_2$, where $f(x) = \dfrac{2}{\sqrt{1+x^2}}$ on $[1, 2]$. Take M to be the larger of M_1 and M_2, then use Formula (7) from Section 7.3.
2. This is similar to Part 1, but we use $f^{(4)}(x)$ in lieu of $f''(x)$ and we use Formula (8) from Section 7.3.

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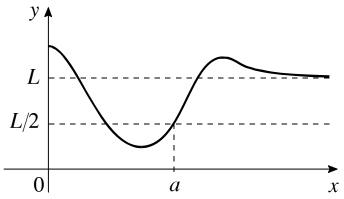

1. Because $L > 0$, $\frac{1}{2}L > 0$. Next, because $\lim_{x\to\infty} f(x) = L$, there is some number $a \geq 0$ such that $f(x) > \frac{1}{2}L$ whenever $x \geq a$. Therefore,

$$\begin{aligned}\int_0^\infty f(x)\,dx &= \int_0^a f(x)\,dx + \int_a^\infty f(x)\,dx\\ &\geq \int_0^a f(x)\,dx + \int_a^\infty \tfrac{1}{2}L\,dx\\ &= \int_0^a f(x)\,dx + \lim_{b\to\infty}\int_a^b \tfrac{1}{2}L\,dx = \int_0^a f(x)\,dx + \lim_{b\to\infty}\tfrac{1}{2}L(b-a) = \infty.\end{aligned}$$

Thus, we see that $\int_0^\infty f(x)\,dx$ is divergent.

2. In this case, $\int_0^\infty f(x)\,dx$ may converge or diverge. For example, if $f(x) = \dfrac{1}{x+1}$ on $[0, \infty)$, then clearly $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{1}{x+1} = 0$, but you can verify that $\displaystyle\int_0^\infty \frac{1}{x+1}dx = \infty$. Next, if $f(x) = \dfrac{1}{(x+1)^2}$ on $[0, \infty)$, then $\lim_{x\to\infty} \dfrac{1}{(x+1)^2} = 0$, but $\displaystyle\int_0^\infty \frac{1}{(x+1)^2}\,dx = 1$, as you can verify.

CHAPTER 7 Exploring with Technology

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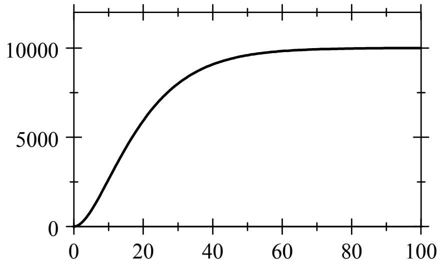
1.



2. $$\begin{aligned}\lim_{t\to\infty} T(t) &= \lim_{t\to\infty}\left[-1000e^{-0.1t}(t+10) + 10{,}000\right]\\ &= -\lim_{t\to\infty}\frac{1000(t+10)}{e^{0.1t}} + 10{,}000\\ &= 10{,}000.\end{aligned}$$

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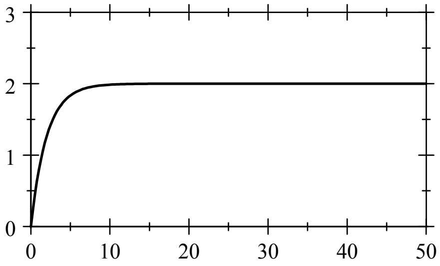
1, 2.



8

CALCULUS OF SEVERAL VARIABLES

8.1

Functions of Several Variables

Concept Questions

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1. A function of two variables is a rule that assigns to each point (x, y) in a subset of the plane a unique number $f(x, y)$. For example, $f(x, y) = x^2 + 2y^2$ has the whole xy-plane as its domain.

2. By the uniqueness property in the definition of a function, we must have $f(a, b) = f(c, d)$.

3. a. The graph of $f(x, y)$ is the set $S = \{(x, y, z) \mid z = f(x, y), (x, y) \in D\}$, where D is the domain of f.

b. The level curve of f is the projection onto the xy-plane of the trace of $f(x, y)$ in the plane $z = k$, where k is a constant in the range of f.

Exercises

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1. $f(x, y) = 2x + 3y - 4$, so $f(0, 0) = 2(0) + 3(0) - 4 = -4$, $f(1, 0) = 2(1) + 3(0) - 4 = -2$, $f(0, 1) = 2(0) + 3(1) - 4 = -1$, $f(1, 2) = 2(1) + 3(2) - 4 = 4$, and $f(2, -1) = 2(2) + 3(-1) - 4 = -3$.

2. $g(x, y) = 2x^2 - y^2$, so $g(1, 2) = 2 - 4 = -2$, $g(2, 1) = 8 - 1 = 7$, $g(1, 1) = 2 - 1 = 1$, $g(-1, 1) = 2 - 1 = 1$, and $g(2, -1) = 8 - 1 = 7$.

3. $f(x, y) = x^2 + 2xy - x + 3$, so $f(1, 2) = 1^2 + 2(1)(2) - 1 + 3 = 7$, $f(2, 1) = 2^2 + 2(2)(1) - 2 + 3 = 9$, $f(-1, 2) = (-1)^2 + 2(-1)(2) - (-1) + 3 = 1$, and $f(2, -1) = 2^2 + 2(2)(-1) - 2 + 3 = 1$.

4. $h(x, y) = \dfrac{x + y}{x - y}$, so $h(0, 1) = \dfrac{0 + 1}{0 - 1} = -1$, $h(-1, 1) = \dfrac{-1 + 1}{-1 - 1} = 0$, $h(2, 1) = \dfrac{2 + 1}{2 - 1} = 3$, and $h(\pi, -\pi) = \dfrac{\pi - \pi}{\pi - (-\pi)} = 0$.

5. $g(s, t) = 3s\sqrt{t} + t\sqrt{s} + 2$, so $g(1, 2) = 3(1)\sqrt{2} + 2\sqrt{1} + 2 = 4 + 3\sqrt{2}$, $g(2, 1) = 3(2)\sqrt{1} + \sqrt{2} + 2 = 8 + \sqrt{2}$, $g(0, 4) = 0 + 0 + 2 = 2$, and $g(4, 9) = 3(4)\sqrt{9} + 9\sqrt{4} + 2 = 56$.

6. $f(x, y) = xye^{x^2+y^2}$, so $f(0, 0) = 0$, $f(0, 1) = 0$, $f(1, 1) = e^2$, and $f(-1, -1) = e^{1+1} = e^2$.

7. $h(s, t) = s \ln t - t \ln s$, so $h(1, e) = \ln e - e \ln 1 = \ln e = 1$, so $h(e, 1) = e \ln 1 - \ln e = -1$, and $h(e, e) = e \ln e - e \ln e = 0$.

8. $f(u, v) = \left(u^2 + v^2\right) e^{uv^2}$, so $f(0, 1) = e^0 = 1$, so $f(-1, -1) = (1 + 1) e^{-1} = 2e^{-1}$, $f(a, b) = \left(a^2 + b^2\right) e^{ab^2}$, and $f(b, a) = \left(a^2 + b^2\right) e^{a^2 b}$.

9. $g(r, s, t) = re^{s/t}$, so $g(1, 1, 1) = e$, $g(1, 0, 1) = 1$, and $g(-1, -1, -1) = -e^{-1/(-1)} = -e$.

10. $g(u,v,w) = \dfrac{ue^{vw}+ve^{uw}+we^{uv}}{u^2+v^2+w^2}$, so $g(1,2,3) = \dfrac{e^6+2e^3+3e^2}{1+4+9} = \dfrac{e^2\left(3+2e+e^4\right)}{14}$ and $g(3,2,1) = \dfrac{3e^2+2e^3+e^6}{9+4+1} = \dfrac{e^2\left(3+2e+e^4\right)}{14}$.

11. $f(x,y) = 2x+3y$. The domain of f is the set of all ordered pairs (x,y), where x and y are real numbers.

12. $g(x,y,z) = x^2+y^2+z^2$. The domain of g is the set of all ordered triples (x,y,z), where x, y, and z are real numbers.

13. $h(u,v) = \dfrac{uv}{u-v}$. The domain is all real values of u and v except those satisfying the equation $u=v$.

14. $f(s,t) = \sqrt{s^2+t^2}$. Because $s^2+t^2 \geq 0$ for all values of s and t, the domain of f is the set of all ordered pairs (s,t).

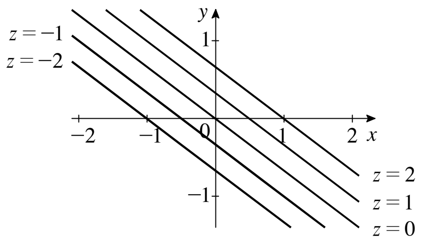
15. $g(r,s) = \sqrt{rs}$. The domain of g is the set of all ordered pairs (r,s) satisfying $rs \geq 0$, that is the set of all ordered pairs whose members have the same sign (allowing zeros).

16. $f(x,y) = e^{-xy}$. The domain of f is the set of all ordered pairs (x,y) where x and y are real numbers.

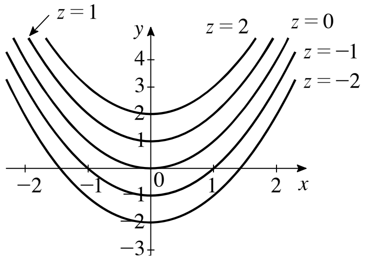
17. $h(x,y) = \ln(x+y-5)$. The domain of h is the set of all ordered pairs (x,y) such that $x+y>5$.

18. $h(u,v) = \sqrt{4-u^2-v^2}$. The domain of h is all real values of u and v satisfying $u^2+v^2 \leq 4$.

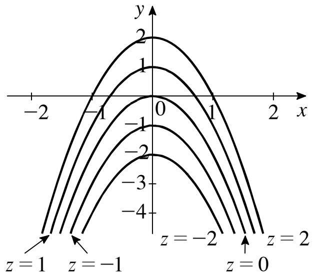
19. The graph shows level curves of $z = f(x,y) = 2x+3y$ for $z = -2, -1, 0, 1$, and 2.



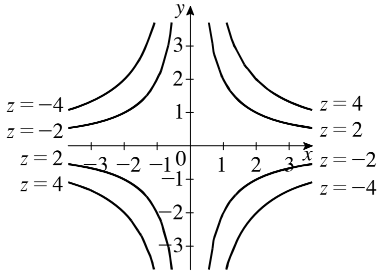
20. The graph shows level curves of $z = f(x,y) = -x^2+y$ for $z = -2, -1, 0, 1$, and 2.



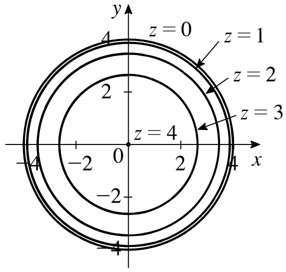
21. The graph shows level curves of $z = f(x,y) = 2x^2+y$ for $z = -2, -1, 0, 1$, and 2.



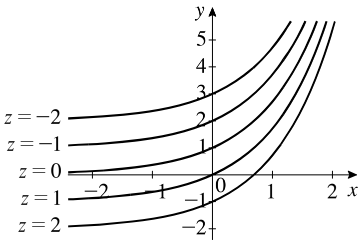
22. The graph shows level curves of $z = f(x,y) = xy$ for $z = -4, -2, 2$, and 4.



23. The graph shows level curves of $z = f(x, y) = \sqrt{16 - x^2 - y^2}$ for $z = 0, 1, 2, 3$, and 4.



24. The graph shows level curves of $z = f(x, y) = e^x - y$ for $z = -2, -1, 0, 1$, and 2.



25. The level curves of f have equations $f(x, y) = \sqrt{x^2 + y^2} = C$. An equation of the curve containing the point $(3, 4)$ satisfies $\sqrt{3^2 + 4^2} = C$, so $C = \sqrt{9 + 16} = 5$. Thus, an equation is $\sqrt{x^2 + y^2} = 5$.

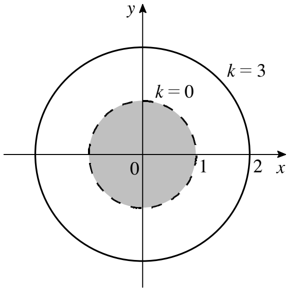
26. The level surfaces of f have equations $f(x, y, z) = 2x^2 + 3y^2 - z = C$. An equation of the level surface containing the point $(-1, 2, -3)$ satisfies $2(-1)^2 + 3\left(2^2\right) - (-3) = C$, so $C = 17$. Thus, an equation is $2x^2 + 3y^2 - z = 17$.

27. (b)

28. (a)

29. No. Suppose the level curves $f(x, y) = c_1$ and $f(x, y) = c_2$ intersect at a point (x_0, y_0) and $c_1 \neq c_2$. Then $f(x_0, y_0) = c_1$ and $f(x_0, y_0) = c_2$ where $c_1 \neq c_2$. Thus, f takes on two distinct values at (x_0, y_0), contradicting the definition of a function.

30. The level set of f for $k = 0$ is the open unit disk with center the origin. The level curve of f for $k = 3$ is the circle with radius 2 and center the origin.



31. $V = f(1.5, 4) = \pi (1.5)^2 (4) = 9\pi$, or 9π ft^3.

32. $f(13.5, 9) = \dfrac{100(13.5)}{9} = 150$.

33. **a.** P and E are real numbers with $E \neq 0$.

b. $R = \frac{205.56}{13.09} \approx 15.704$.

34. **a.** Since the price per share P must be positive and the dividend D must be nonnegative, we see that the required domain is the set of all ordered pairs (D, P) where for which $D \geq 0$ and $P > 0$.

b. $Y = \frac{3}{205.56} \approx 0.01459$, or approximately 1.46%.

35. **a.** $M = \dfrac{80}{(1.8)^2} = 24.69$.

b. We must have $\dfrac{w}{(1.8)^2} < 25$; that is, $w < 25\,(1.8)^2 = 81$. Thus, the maximum weight is 81 kg.

36. $R\,(4, 0.1) = \dfrac{4k}{(0.1)^4} = 40{,}000k$ dynes.

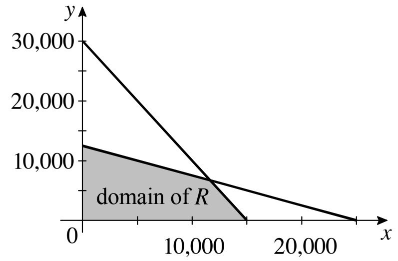
37. **a.** $C\,(x, y) = 200x + 120y + 20{,}000$.

b. The domain of C is the set of all x and y such that $x \geq 0$ and $y \geq 0$.

c. The total cost is $C\,(1000, 200) = 200\,(1000) + 120\,(200) + 20{,}000$, or \$244,000.

38. **a.** $R\,(x, y) = px + qy = (60{,}000 - 4x - 2y)\,x + (50{,}000 - 2x - 4y)\,y = -4x^2 - 4y^2 - 4xy + 60{,}000x + 50{,}000y$.

b. The domain of R is the set of all x and y satisfying the system of inequalities $60000 - 4x - 2y \geq 0$, $50000 - 2x - 4y \geq 0$, $x \geq 0$, $y \geq 0$, or equivalently, $4x + 2y \leq 60000$, $2x + 4y \leq 50000$, $x \geq 0$, $y \geq 0$.



c. Substituting $x = 3000$ and $y = 2000$ into each of the inequalities in part (b) gives $4\,(3000) + 2\,(2000) = 16000 \leq 60000$, $2\,(3000) + 4\,(2000) = 14000 \leq 50000$, $3000 \geq 0$, $2000 \geq 0$. All of the inequalities are satisfied. You can also see this by observing that the point (3000, 2000) lies in the domain of R sketched in part (b).

d. $R\,(3000, 2000) = -4\,(3000)^2 - 4\,(2000)^2 - 4\,(3000)\,(2000) + 60000\,(3000) + 50000\,(2000) = 204{,}000{,}000$, or \$204 million.

39. **a.** $R\,(x, y) = xp + yq = x\left(200 - \frac{1}{5}x - \frac{1}{10}y\right) + y\left(160 - \frac{1}{10}x - \frac{1}{4}y\right) = -\frac{1}{5}x^2 - \frac{1}{4}y^2 - \frac{1}{5}xy + 200x + 160y$.

b. The domain of R is the set of all points (x, y) satisfying $200 - \frac{1}{5}x - \frac{1}{10}y \geq 0$, $160 - \frac{1}{10}x - \frac{1}{4}y \geq 0$, $x \geq 0$, and $y \geq 0$.

40. $R\,(x, y) = xp + yq = x\left(200 - \frac{1}{5}x - \frac{1}{10}y\right) + y\left(160 - \frac{1}{10}x - \frac{1}{4}y\right) = -\frac{1}{5}x^2 - \frac{1}{4}y^2 - \frac{1}{5}xy + 200x + 160y$. Thus, $R\,(100, 60) = -\frac{1}{5}\,(10{,}000) - \frac{1}{4}\,(3600) - \frac{1}{5}\,(6000) + 200\,(100) + 160\,(60) = 25{,}500$. This says that the revenue from sales of 100 of the finished and 60 of the unfinished furniture pieces per week is \$25,500.
$R\,(60, 100) = -\frac{1}{5}\,(3600) - \frac{1}{4}\,(10{,}000) - \frac{1}{5}\,(6000) + 200\,(60) + 160\,(100) = 23{,}580$. This says that the revenue from sales of 60 of the finished and 100 of the unfinished furniture pieces per week is \$23,580.

41. **a.**
$$\begin{aligned} R\,(x, y) &= xp + yq = 20x - 0.005x^2 - 0.001xy + 15y - 0.001xy - 0.003y^2 \\ &= -0.005x^2 - 0.003y^2 - 0.002xy + 20x + 15y. \end{aligned}$$

b. Because p and q must both be nonnegative, the domain of R is the set of all ordered pairs (x, y) for which $20 - 0.005x - 0.001y \geq 0$, $15 - 0.001x - 0.003y \geq 0$, $x \geq 0$, and $y \geq 0$.

42. $R\,(300, 200) = -0.005\,(90{,}000) - 0.003\,(40{,}000) - 0.002\,(60{,}000) + 20\,(300) + 15\,(200) = 8310$, or \$8310.
$R\,(200, 300) = -0.005\,(40{,}000) - 0.003\,(90{,}000) - 0.002\,(60{,}000) + 20\,(200) + 15\,(300) = 7910$, or \$7910.

43. a. The domain of V is the set of all ordered pairs (P, T) where P and T are positive real numbers.

b. $V = \dfrac{30.9\,(273)}{760} \approx 11.10$ liters.

44. a. The domain of S is the set of all ordered pairs (W, H) such that W and H are nonnegative real numbers.

b. $S = 0.007184\,(70)^{0.425}\,(178)^{0.725} \approx 1.87\text{ m}^2$.

45. a. The domain of W is the set of all ordered pairs (L, G) for which $L > 0$ and $G > 0$.

b. The approximate weight of Sue's catch is $W = \dfrac{(20)(12)^2}{800} = 3.6$, or 3.6 lb.

46. Suppose that the trout caught by Jane has length L inches and girth G inches. Then the trout caught by Ashley has length $1.2L$ inches and girth $0.9G$ inches. The ratio of the weight of Ashley's catch to that of Jane's catch is $\dfrac{W(1.2L, 0.9G)}{W(L, G)} = \dfrac{\dfrac{(1.2L)(0.9G)^2}{800}}{\dfrac{LG^2}{800}} = (1.2)(0.9)^2 = 0.972$, or $W(1.2L, 0.9G) = 0.972W(L, G)$. This says that Ashley's catch is estimated to weigh 2.8% less than Jane's catch.

47. The output is $f(32, 243) = 100\,(32)^{3/5}\,(243)^{2/5} = 100\,(8)\,(9) = 7200$, or \$7200 billion.

48. a. $f(px, py) = a\,(px)^b\,(py)^{1-b} = ap^bx^bp^{1-b}y^{1-b} = apx^by^{1-b} = pf(x, y)$.

b. Increasing the amount of money expended for labor by $r\%$ gives $x + 0.01rx = (1.01r)\,x$ as the new amount spent on labor. Similarly, the new amount spent on capital is $(1 + 0.01r)\,y$. Using the result of part (a) with $p = 1.01r$, we see that the resulting output is $f((1 + 0.01r)\,x, (1 + 0.01r)\,y) = (1.01r)\,f(x, y) = f(x, y) + (0.01r)\,f(x, y)$. That is, the output is increased by $(0.01r)\,f(x, y)$, or $r\%$.

49. The number of suspicious fires is $N(100, 20) = \dfrac{100\left[1000 + 0.03\left(100^2\right)(20)\right]^{1/2}}{[5 + 0.2\,(20)]^2} = 103.29$, or about 103.

50. $A = f(10000, 0.06, 3) = 10{,}000e^{(0.06)(3)} = 10{,}000e^{0.18}$, or approximately \$11,972.17.

51. a. If $r = 4\%$, then $P = f(300000, 0.04, 30) = \dfrac{300{,}000\,(0.04)}{12\left[1 - \left(1 + \frac{0.04}{12}\right)^{-360}\right]} \approx 1432.25$, or \$1432.25. If $r = 6\%$, then $P = f(300000, 0.06, 30) = \dfrac{300{,}000\,(0.06)}{12\left[1 - \left(1 + \frac{0.06}{12}\right)^{-360}\right]} \approx 1798.65$, or \$1798.65.

b. $P = f(300000, 0.06, 20) = \dfrac{300{,}000\,(0.06)}{12\left[1 - \left(1 + \frac{0.06}{12}\right)^{-240}\right]} \approx 2149.29$, or \$2149.29.

52. If $i = 60$, then $B = f(280000, 0.06, 30, 60) = 280{,}000\left[\frac{\left(1+\frac{0.06}{12}\right)^{60}-1}{\left(1+\frac{0.06}{12}\right)^{360}-1}\right] \approx 19{,}447.80$, so they owe $280{,}000 - 19{,}447.80$, or \$260,552.20.

If $i = 240$, then $B = f(280000, 0.06, 30, 240) = 280{,}000\left[\frac{\left(1+\frac{0.06}{12}\right)^{240}-1}{\left(1+\frac{0.06}{12}\right)^{360}-1}\right] = 128{,}789.96$, so they owe $280{,}000 - 128{,}789.96$, or \$151,210.04.

53. $f(20, 40, 5) = \sqrt{\frac{2(20)(40)}{5}} = \sqrt{320} \approx 17.9$, or approximately 18 bicycles.

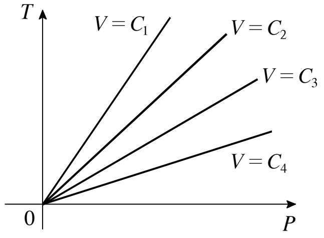
54. The required power output is $P = f(30, 16) = 0.772(30)^2(16)^3 \approx 2{,}845{,}900.8$, or approximately 2,845,901 watts.

55. For yacht A, we have $f(20.95, 277.3, 17.56) = \frac{20.95 + 1.25(277.3)^{1/2} - 9.80(17.56)^{1/3}}{0.388} \approx 41.993$. Because this is less than 42, yacht A satisfies the formula.

For yacht B, we have $f(21.87, 311.78, 22.48) = \frac{21.87 + 1.25(311.78)^{1/2} - 9.80(22.48)^{1/3}}{0.388} \approx 41.967$. Because this is less than 42, yacht B satisfies the formula as well.

56. $f(M, 600, 10) = \frac{\pi^2(360{,}000)M(10)}{900} \approx 39{,}478.42M$, or $\frac{39{,}478.42}{980} \approx 40.28$ times gravity.

57. The level curves of V have equation $\frac{kT}{P} = C$, where C is a positive constant. The level curves are the family of straight lines $T = \frac{C}{k}P$ lying in the first quadrant, because k, T, and P are positive. Every point on the level curve $V = C$ gives the same volume C.



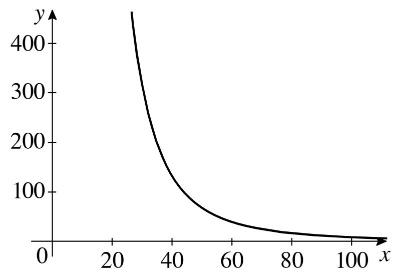
58. **a.** $F(81, 16) = 100(81)^{3/4}(16)^{1/4} = 5400$.

b. $100x^{3/4}y^{1/4} = 5400$ so $y^{1/4} = 54x^{-3/4}$ and $y = \left(54x^{-3/4}\right)^4 = \frac{8{,}503{,}056}{x^3}$.

c.

x	50	60	70	80	90
y	68.02	39.37	24.79	16.61	11.66

d.



59. False. Let $h(x, y) = xy$. Then there is no pair of functions f and g such that $h(x, y) = f(x) + g(y)$.

60. False. Let $f(x, y) = xy$. Then $f(ax, ay) = (ax)(ay) = a^2xy \neq axy = af(x, y)$.

61. False. Because $x^2 - y^2 = (x + y)(x - y)$, we see that $x^2 - y^2 = 0$ if $y = \pm x$. Therefore, the domain of f is $\{(x, y) \mid y \neq \pm x\}$.

62. True. If $c > 0$, then $z = f(x, y) = c$ and the point (x, y, c) on the graph of f is c units above the xy-plane. Similarly, if $c < 0$, then $z = f(x, y) = c$ and the point (x, y, c) on the graph of f lies $|c|$ units below the xy-plane.

63. False. Take $f(x, y) = \sqrt{x^2 + y^2}$, $P_1(-1, 1)$, and $P_2(1, 1)$. Then $f(x_1, y_1) = f(-1, 1) = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ and $f(x_2, y_2) = f(1, 1) = \sqrt{1^2 + 1^2} = \sqrt{2}$. So $f(x_1, y_1) = f(x_2, y_2)$, but $P(x_1, y_1) \neq P(x_2, y_2)$.

64. False. Take $f(x, y) = x^2 + y^2$. Then $f(-1, 1) = 2$ and $f(1, -1) = 2$.

8.2 Partial Derivatives

Concept Questions page 582

1. a. $\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial x}(x, y)\Big|_{(a,b)} = \left[\lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}\right]_{(a,b)}$.

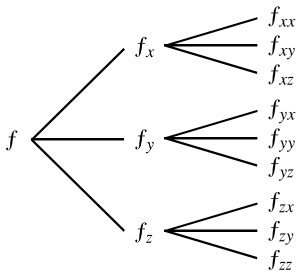
b. See pages 572–576 of the text.

2. a. Two commodities are substitute commodities if a decrease in the demand for one corresponds to an increase in the demand for the other. Examples are coffee and tea. Two commodities are complementary commodities if a decrease in the demand for one corresponds to a decrease in the demand for the other as well. Examples are (non-digital) cameras and film.

b. If the demand equation relating the quantities x and y to the unit prices are $x = f(p, q)$ and $y = g(p, q)$, then the commodities are substitute commodities if $\frac{\partial f}{\partial q} > 0$ and $\frac{\partial g}{\partial p} > 0$ and complementary commodities if $\frac{\partial f}{\partial q} < 0$ and $\frac{\partial g}{\partial p} < 0$.

3. f_{xx}, f_{yy}, f_{xy}, and f_{yx}.

4. There are nine partial derivatives of order two, as shown: f_{xx}, f_{xy}, f_{xz}, f_{yx}, f_{yy}, f_{yz}, f_{zx}, f_{zy}, and f_{zz}.



Exercises page 582

1. a. $f(x, y) = x^2 + 2y^2$, so $f_x(2, 1) = 4$ and $f_y(2, 1) = 4$.

b. $f_x(2, 1) = 4$ says that the slope of the tangent line to the curve of intersection of the surface $z = x^2 + 2y^2$ and the plane $y = 1$ at the point $(2, 1, 6)$ is 4. $f_y(2, 1) = 4$ says that the slope of the tangent line to the curve of intersection of the surface $z = x^2 + 2y^2$ and the plane $x = 2$ at the point $(2, 1, 6)$ is 4.

c. $f_x(2, 1) = 4$ says that the rate of change of $f(x, y)$ with respect to x with y held fixed with a value of 1 is 4 units per unit change in x. $f_y(2, 1) = 4$ says that the rate of change of $f(x, y)$ with respect to y with x held fixed with a value of 2 is 4 units per unit change in y.

2. **a.** $f(x,y) = 9 - x^2 + xy - 2y^2$, so $f_x(1,2) = 0$ and $f_y(1,2) = -7$.

b. $f_x(1,2) = 0$ says that the slope of the tangent line to the curve of intersection of the surface $z = x^2 + 2y^2$ and the plane $y = 1$ at the point $(1,2,2)$ is 0. $f_y(1,2) = -7$ says that the slope of the tangent line to the curve of intersection of the surface $z = x^2 + 2y^2$ and the plane $x = 1$ at the point $(1,2,2)$ is -7.

c. $f_x(1,2) = 0$ says that the rate of change of $f(x,y)$ with respect to x with y held fixed with a value of 2 is 0 units per unit change in x. $f_y(1,2) = -7$ says that the rate of change of $f(x,y)$ with respect to y with x held fixed with a value of 1 is -7 units per unit change in y.

3. $f(x,y) = 2x + 3y + 5$, so $f_x = 2$ and $f_y = 3$.

4. $f(x,y) = 2xy$, so $f_x = 2y$ and $f_y = 2x$.

5. $g(x,y) = 2x^2 + 4y + 1$, so $g_x = 4x$ and $g_y = 4$.

6. $f(x,y) = 1 + x^2 + y^2$, so $f_x = 2x$ and $f_y = 2y$.

7. $f(x,y) = \frac{2y}{x^2}$, so $f_x = -\frac{4y}{x^3}$ and $f_y = \frac{2}{x^2}$.

8. $f(x,y) = \frac{x}{1+y}$, so $f_x = \frac{1}{1+y}$ and $f_y = -\frac{x}{(1+y)^2}$.

9. $g(u,v) = \frac{u-v}{u+v}$, so $\frac{\partial g}{\partial u} = \frac{(u+v)(1) - (u-v)(1)}{(u+v)^2} = \frac{2v}{(u+v)^2}$ and $\frac{\partial g}{\partial v} = \frac{(u+v)(-1) - (u-v)(1)}{(u+v)^2} = -\frac{2u}{(u+v)^2}$.

10. $f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$, so $f_x = \frac{\left(x^2+y^2\right)(2x) - \left(x^2-y^2\right)(2x)}{\left(x^2+y^2\right)^2} = \frac{4xy^2}{\left(x^2+y^2\right)^2}$ and $f_y = \frac{\left(x^2+y^2\right)(-2y) - \left(x^2-y^2\right)(2y)}{\left(x^2+y^2\right)^2} = -\frac{4x^2y}{\left(x^2+y^2\right)^2}$.

11. $f(s,t) = \left(s^2 - st + t^2\right)^3$, so $f_s = 3\left(s^2 - st + t^2\right)^2(2s - t)$ and $f_t = 3\left(s^2 - st + t^2\right)^2(2t - s)$.

12. $g(s,t) = s^2t + st^{-3}$, so $g_s = 2st + t^{-3}$ and $g_t = s^2 - 3st^{-4}$.

13. $f(x,y) = \left(x^2+y^2\right)^{2/3}$, so $f_x = \frac{2}{3}\left(x^2+y^2\right)^{-1/3}(2x) = \frac{4}{3}x\left(x^2+y^2\right)^{-1/3}$ and $f_y = \frac{4}{3}y\left(x^2+y^2\right)^{-1/3}$.

14. $f(x,y) = x\left(1+y^2\right)^{1/2}$, so $f_x = \sqrt{1+y^2}$ and $f_y = x\left(\frac{1}{2}\right)\left(1+y^2\right)^{-1/2}(2y) = \frac{xy}{\sqrt{1+y^2}}$.

15. $f(x,y) = e^{xy+1}$, so $f_x = ye^{xy+1}$ and $f_y = xe^{xy+1}$.

16. $f(x,y) = \left(e^x + e^y\right)^5$, so $f_x = 5e^x\left(e^x + e^y\right)^4$ and $f_y = 5e^y\left(e^x + e^y\right)^4$.

17. $f(x,y) = x\ln y + y\ln x$, so $f_x = \ln y + \frac{y}{x}$ and $f_y = \frac{x}{y} + \ln x$.

18. $f(x,y) = x^2e^{y^2}$, so $\frac{\partial f}{\partial x} = 2xe^{y^2}$ and $\frac{\partial f}{\partial y} = x^2e^{y^2}(2y) = 2x^2ye^{y^2}$.

19. $g(u,v) = e^u \ln v$, so $g_u = e^u \ln v$ and $g_v = \frac{e^u}{v}$.

20. $f(x) = \frac{e^{xy}}{x+y}$, so $f_x = \frac{(x+y)e^{xy}(y) - e^{xy}(1)}{(x+y)^2} = \frac{\left(xy + y^2 - 1\right)e^{xy}}{(x+y)^2}$ and $f_y = \frac{\left(x^2 + xy - 1\right)e^{xy}}{(x+y)^2}$.

21. $f(x,y,z) = xyz + xy^2 + yz^2 + zx^2$, so $f_x = yz + y^2 + 2xz$ and $f_y = xz + 2xy + z^2$, $f_z = xy + 2yz + x^2$.

22. $g(u,v,w) = \dfrac{2uvw}{u^2+v^2+w^2}$, so $g_u = \dfrac{(u^2+v^2+w^2)(2vw) - 2uvw(2u)}{(u^2+v^2+w^2)^2} = \dfrac{2vw(v^2+w^2-u^2)}{(u^2+v^2+w^2)^2}$ and

$g_v = \dfrac{2uw(u^2+w^2-v^2)}{(u^2+v^2+w^2)^2}$, $g_w = \dfrac{2uv(u^2+v^2-w^2)}{(u^2+v^2+w^2)^2}$.

23. $h(r,s,t) = e^{rst}$, so $h_r = ste^{rst}$, $h_s = rte^{rst}$, and $h_t = rse^{rst}$.

24. $f(x,y,z) = xe^{y/z}$, so $\dfrac{\partial f}{\partial x} = e^{y/z}$, $\dfrac{\partial f}{\partial y} = xe^{y/z}\left(\dfrac{1}{z}\right) = \dfrac{x}{z}e^{yz}$, and $\dfrac{\partial f}{\partial z} = xe^{y/z}\left(-\dfrac{y}{z^2}\right) = -\dfrac{xy}{z^2}e^{y/z}$.

25. $f(x,y) = x^2y + xy^2$, so $f_x(1,2) = \left.(2xy+y^2)\right|_{(1,2)} = 8$ and $f_y(1,2) = \left.(x^2+2xy)\right|_{(1,2)} = 5$.

26. $f(x,y) = x^2 + xy + y^2 + 2x - y$, so $f_x(-1,2) = (2x+y+2)|_{(-1,2)} = 2(-1) + 2 + 2 = 2$ and $f_y(-1,2) = (x+2y-1)|_{(-1,2)} = -1 + 2(2) - 1 = 2$.

27. $f(x,y) = x\sqrt{y} + y^2 = xy^{1/2} + y^2$, so $f_x(2,1) = \sqrt{y}\big|_{(2,1)} = 1$ and $f_y(2,1) = \left.\left(\dfrac{x}{2\sqrt{y}} + 2y\right)\right|_{(2,1)} = 3$.

28. $g(x,y) = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$, so $g_x(3,4) = \left.\dfrac{x}{\sqrt{x^2+y^2}}\right|_{(3,4)} = \dfrac{3}{5}$ and $g_y(3,4) = \left.\dfrac{y}{\sqrt{x^2+y^2}}\right|_{(3,4)} = \dfrac{4}{5}$.

29. $f(x,y) = \dfrac{x}{y}$, so $f_x(1,2) = \left.\dfrac{1}{y}\right|_{(1,2)} = \dfrac{1}{2}$ and $f_y(1,2) = -\left.\dfrac{x}{y^2}\right|_{(1,2)} = -\dfrac{1}{4}$.

30. $f(x,y) = \dfrac{x+y}{x-y}$, so $f_x(1,-2) = \left.\dfrac{(x-y)(1)-(x+y)(1)}{(x-y)^2}\right|_{(1,-2)} = -\left.\dfrac{2y}{(x-y)^2}\right|_{(1,-2)} = \dfrac{4}{9}$ and

$f_y(1,-2) = \left.\dfrac{(x-y)(1)-(x+y)(-1)}{(x-y)^2}\right|_{(1,-2)} = \left.\dfrac{2x}{(x-y)^2}\right|_{(1,-2)} = \dfrac{2}{9}$.

31. $f(x,y) = e^{xy}$, so $f_x(1,1) = ye^{xy}|_{(1,1)} = e$ and $f_y(1,1) = xe^{xy}|_{(1,1)} = e$.

32. $f(x,y) = e^x \ln y$, so $f_x(0,e) = e^x \ln y|_{(0,e)} = \ln e = 1$ and $f_y(0,e) = \left.\dfrac{e^x}{y}\right|_{(0,e)} = \dfrac{1}{e}$.

33. $f(x,y,z) = x^2yz^3$, so $f_x(1,0,2) = 2xyz^3\big|_{(1,0,2)} = 0$, $f_y(1,0,2) = x^2z^3\big|_{(1,0,2)} = 8$, and $f_z(1,0,2) = 3x^2yz^2\big|_{(1,0,2)} = 0$.

34. $f(x,y,z) = x^2y^2 + z^2$, so $f_x(1,1,2) = 2xy^2\big|_{(1,1,2)} = 2$, $f_y(1,1,2) = 2x^2y\big|_{(1,1,2)} = 2$, and $f_z(1,1,2) = 2z|_{(1,1,2)} = 4$.

35. $f(x,y) = x^2y + xy^3$, so $f_x = 2xy + y^3$ and $f_y = x^2 + 3xy^2$. Therefore, $f_{xx} = 2y$, $f_{xy} = 2x + 3y^2 = f_{yx}$, and $f_{yy} = 6xy$.

36. $f(x,y) = x^3 + x^2y + x + 4$, so $f_x = 3x^2 + 2xy + 1$ and $f_y = x^2$. Therefore, $f_{xx} = 6x + 2y$, $f_{xy} = 2x = f_{yx}$, and $f_{yy} = 0$.

37. $f(x,y) = x^2 - 2xy + 2y^2 + x - 2y$, so $f_x = 2x - 2y + 1$ and $f_y = -2x + 4y - 2$. Therefore, $f_{xx} = 2$, $f_{xy} = -2 = f_{yx}$, and $f_{yy} = 4$.

38. $f(x,y) = x^3 + x^2y^2 + y^3 + x + y$, so $f_x = 3x^2 + 2xy^2 + 1$ and $f_y = 2x^2y + 3y^2 + 1$. Therefore, $f_{xx} = 6x + 2y^2$, $f_{yy} = 2x^2 + 6y$, and $f_{xy} = 4xy = f_{yx}$.

39. $f(x,y) = \left(x^2 + y^2\right)^{1/2}$, so $f_x = \frac{1}{2}\left(x^2 + y^2\right)^{-1/2}(2x) = x\left(x^2 + y^2\right)^{-1/2}$ and $f_y = y\left(x^2 + y^2\right)^{-1/2}$. Therefore,

$$f_{xx} = \left(x^2 + y^2\right)^{-1/2} + x\left(-\tfrac{1}{2}\right)\left(x^2 + y^2\right)^{-3/2}(2x) = \left(x^2 + y^2\right)^{-1/2} - x^2\left(x^2 + y^2\right)^{-3/2}$$

$$= \left(x^2 + y^2\right)^{-3/2}\left(x^2 + y^2 - x^2\right) = \frac{y^2}{\left(x^2 + y^2\right)^{3/2}},$$

$$f_{xy} = x\left(-\tfrac{1}{2}\right)\left(x^2 + y^2\right)^{-3/2}(2y) = -\frac{xy}{\left(x^2 + y^2\right)^{3/2}} = f_{yx}, \text{ and}$$

$$f_{yy} = \left(x^2 + y^2\right)^{-1/2} + y\left(-\tfrac{1}{2}\right)\left(x^2 + y^2\right)^{-3/2}(2y) = \left(x^2 + y^2\right)^{-1/2} - y^2\left(x^2 + y^2\right)^{-3/2}$$

$$= \left(x^2 + y^2\right)^{-3/2}\left(x^2 + y^2 - y^2\right) = \frac{x^2}{\left(x^2 + y^2\right)^{3/2}}.$$

40. $f(x,y) = xy^{1/2} + yx^{1/2}$, so $f_x = y^{1/2} + \frac{1}{2}yx^{-1/2}$ and $f_y = \frac{1}{2}xy^{-1/2} + x^{1/2}$. Therefore, $f_{xx} = -\frac{1}{4}yx^{-3/2}$, $f_{xy} = \frac{1}{2}y^{-1/2} + \frac{1}{2}x^{-1/2} = f_{yx}$, and $f_{yy} = -\frac{1}{4}xy^{-3/2}$.

41. $f(x,y) = e^{-x/y}$, so $f_x = -\dfrac{1}{y}e^{-x/y}$ and $f_y = \dfrac{x}{y^2}e^{-x/y}$. Therefore, $f_{xx} = \dfrac{1}{y^2}e^{-x/y}$,

$$f_{xy} = -\frac{x}{y^3}e^{-x/y} + \frac{1}{y^2}e^{-x/y} = \left(\frac{-x+y}{y^3}\right)e^{-x/y} = f_{yx}, \text{ and } f_{yy} = -\frac{2x}{y^3}e^{-x/y} + \frac{x^2}{y^4}e^{-x/y} = \frac{x}{y^3}\left(\frac{x}{y} - 2\right)e^{-x/y}.$$

42. $f(x,y) = \ln\left(1 + x^2y^2\right)$, so $f_x = \dfrac{2xy^2}{1 + x^2y^2}$ and $f_y = \dfrac{2x^2y}{1 + x^2y}$.

Therefore, $f_{xx} = \dfrac{\left(1 + x^2y^2\right)\left(2y^2\right) - 2xy^2\left(2xy^2\right)}{\left(1 + x^2y^2\right)^2} = \dfrac{2y^2\left(1 - x^2y^2\right)}{\left(1 + x^2y^2\right)^2}$,

$$f_{yy} = \frac{\left(1 + x^2y^2\right)\left(2x^2\right) - \left(2x^2y\right)\left(2x^2y\right)}{\left(1 + x^2y^2\right)^2} = \frac{2x^2\left(1 - x^2y^2\right)}{\left(1 + x^2y^2\right)^2}, \text{ and}$$

$$f_{xy} = \frac{\left(1 + x^2y^2\right)(4xy) - 2xy^2\left(2x^2y\right)}{\left(1 + x^2y^2\right)^2} = \frac{4xy}{\left(1 + x^2y^2\right)^2} = f_{yx}.$$

43. a. $f(x,y) = 20x^{3/4}y^{1/4}$, so $f_x(256,16) = 15\left(\frac{y}{x}\right)^{1/4}\Big|_{(256,16)} = 15\left(\frac{16}{256}\right)^{1/4} = 15\left(\frac{2}{4}\right) = 7.5$ and

$f_y(256,16) = 5\left(\frac{x}{y}\right)^{3/4}\Big|_{(256,16)} = 5\left(\frac{256}{16}\right)^{3/4} = 5(80) = 40.$

b. Yes.

44. a. $f(x,y) = 40x^{4/5}y^{1/5}$, so $f_x = 32x^{-1/5}y^{1/5}$ and $f_x(32,243) = 32(32)^{-1/5}(243)^{1/5} = 48$, or 48 units per unit change in labor. $f_y = 8x^{4/5}y^{-4/5}$, so $f_y(32,243) = \frac{8(16)}{81} = \frac{128}{81}$, or $\frac{128}{81}$ units per unit change in capital

b. No.

45. $P(x, y) = x^2 + 5x + 2xy + 3y^2 + 2y$, so $P_x(x, y) = 2x + 5 + 2y$ and $P_y(x, y) = 6y + 2 + 2x$. Thus, $P_x(400, 300) = 2(400) + 5 + 2(300) = 1405$ and $P_y(400, 300) = 6(300) + 2 + 2(400) = 2602$.

Thus, with labor at 400 work-hours per day and capital expenditure at \$300/day:

- an increase of 1 work-hour per day with capital expenditure held fixed results in a production increase of approximately 1405 candles per day;
- an increase of \$1 per day in capital expenditure with labor held fixed results in a production increase of approximately 2602 candles per day.

46. $P(x, y) = 0.2x^2 + 2x + 3xy + 0.4y^2 + 3y$, so $P_x(x, y) = 0.4x + 2 + 3y$ and $P_y(x, y) = 0.8y + 3 + 3x$. Thus, $P_x(10, 5) = 0.4(10) + 2 + 3(5) = 21$ and $P_y(10, 5) = 0.8(5) + 3 + 3(10) = 37$.

Thus, with labor at 10,000 work-hours per month and capital investment at \$5000 per month:

- an increase of 1000 work-hours per month with capital investment held fixed results in a production increase of approximately 21,000 mugs per month.
- an increase of \$1000 per month in capital expenditure with labor held fixed results in a production increase of approximately 37,000 mugs per month.

47. $p(x, y) = 200 - 10\left(x - \frac{1}{2}\right)^2 - 15(y - 1)^2$, so $\frac{\partial p}{\partial x}(0, 1) = -20\left(x - \frac{1}{2}\right)\Big|_{(0,1)} = 10$. At the location (0, 1) in the figure, the price of land is increasing by \$10 per square foot per mile to the east. $\frac{\partial p}{\partial y}(0, 1) = -30(y - 1)|_{(0,1)} = 0$, so at the point (0, 1) in the figure, the price of land is unchanging with respect to north-south change.

48. $R(x, y) = -0.2x^2 - 0.25y^2 - 0.2xy + 200x + 160y$, so $\frac{\partial R}{\partial x}(300, 250) = (-0.4x - 0.2y + 200)|_{(300,250)} = -0.4(300) - 0.2(250) + 200 = 30$. This says that at sales levels of 300 finished and 250 unfinished units, revenue is increasing by \$30 per week per unit increase in finished pieces. $\frac{\partial R}{\partial y}(300, 250) = -0.5y - 0.2x + 160|_{(300,250)} = -0.5(250) - 0.2(300) + 160 = -25$, and this says that at the same sales levels, revenue is decreasing by \$25 per week per unit increase in unfinished pieces.

49. $P(x, y) = -0.02x^2 - 15y^2 + xy + 39x + 25y - 20{,}000$, so $\frac{\partial P}{\partial x}(4000, 150) = (-0.04x + y + 39)|_{(4000,150)} = -0.04(4000) + 150 + 39 = 29$. This says that when inventory is \$4,000,000 and floor space is 150,000 ft^2, monthly profit increases by \$29 per thousand dollars increase in inventory. $\frac{\partial P}{\partial y}(4000, 150) = (-30y + x + 25)|_{(4000,150)} = -475$, and this says that with the same inventory and floor space as above, monthly profit decreases by \$475 per thousand-square-foot increase in floor space.

Similarly, $\frac{\partial P}{\partial x}(5000, 150) = (-0.04x + y + 39)|_{(5000,150)} = -0.04(5000) + 150 + 39 = -11$ and $\frac{\partial P}{\partial y}(5000, 150) = (-30y + x + 25)|_{(5000,150)} = -30(150) + 5000 + 25 = 525$.

50. **a.** $D(x, y) = 500e^{-0.02(x-120)}\left(1 - 0.7e^{-0.0001y}\right)$, so

$D_x(x, y) = 500(-0.02)e^{-0.02(x-120)}\left(1 - 0.7e^{-0.0001y}\right) = -10e^{-0.02(x-120)}\left(1 - 0.7e^{-0.0001y}\right) < 0$ for all x and y, as specified. Thus, D is a decreasing function of x with y held fast.

Also, $D_y(x, y) = 500e^{-0.02(x-120)}\left[-0.7(-0.0001)e^{-0.0001y}\right] = 0.035e^{-0.02(x-120)}e^{-0.0001y} > 0$ for all x and y, as specified. So D is an increasing function of y with x held fast. Thus:

- if advertising spending is held fast, then the number of rooms demanded drops if the daily room rate increases.

- if the daily room rate is held fast, then the number of rooms demanded increases if advertising spending increases.

b. If the daily room rate is cut by \$1 from the current \$200 per night, then demand for rooms will change by $-D_x(200, 25000) = 10e^{-0.02(200-120)}\left[1 - 0.7e^{-0.0001(25000)}\right] \approx 1.90$; that is, demand will increase by approximately 2 rooms.

51. $N(x, y) = \dfrac{120\sqrt{1000 + 0.03x^2y}}{(5 + 0.2y)^2}$, so

$\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\dfrac{120\left(1000 + 0.03x^2y\right)^{1/2}}{(5 + 0.2y)^2} = \dfrac{120\left(\frac{1}{2}\right)\left(1000 + 0.03x^2y\right)^{-1/2}(0.06xy)}{(5 + 0.2y)^2}$. Thus,

$\dfrac{\partial N}{\partial x}(100, 20) = \left.\dfrac{3.6xy}{(5 + 0.2y)^2\sqrt{1000 + 0.03x^2y}}\right|_{(100,20)} \approx 1.06$. This means that with the level of reinvestment held constant at 20 cents per dollar deposited, the number of suspicious fires will grow at the rate of approximately 1 fire per increase of 1 person per census tract when the number of people per census tract is 100. Next,

$$\frac{\partial N}{\partial y}(100, 20) = 120\frac{\partial}{\partial y}\left.\left[\left(1000 + 0.03x^2y\right)^{1/2}(5 + 0.2y)^{-2}\right]\right|_{(100,20)}$$

$$= 120\left.\left[\tfrac{1}{2}\left(1000 + 0.03x^2y\right)^{-1/2}\left(0.03x^2\right)(5 + 0.2y)^{-2} + \left(1000 + 0.03x^2y\right)^{1/2}(-2)(5 + 0.2y)^{-3}(0.2)\right]\right|_{(100,20)}$$

$$= \left.\frac{9x^2 - 1.08x^2y - 48{,}000}{(5 + 0.2y)^3\sqrt{1000 + 0.03x^2y}}\right|_{(100,20)} \approx -2.85$$

which tells us that if the number of people per census tract is constant at 100 per tract, the number of suspicious fires decreases at a rate of approximately 2.9 per increase of 1 cent per dollar deposited for reinvestment when the level of reinvestment is 20 cents per dollar deposited.

52. $S = 0.007184W^{0.425}H^{0.725}$, so

$\dfrac{\partial S}{\partial W}(70, 180) = 0.007184(0.425)\left.W^{-0.575}H^{0.725}\right|_{(70,180)} = 0.0030532(70)^{-0.575}(180)^{0.725} \approx 0.01145$. This says at a weight of 70 kg and a height of 180 cm, surface area increases at a rate of 0.01145 square meters per 1 kg weight increase. $\dfrac{\partial S}{\partial H}(70, 180) = 0.007184(0.725)\left.W^{0.425}H^{-0.275}\right|_{(70,180)} = 0.0052084(70)^{0.425}(180)^{-0.275} = 0.00760$, and this says that at the same weight and height, surface area is increasing at a rate of 0.0076 square meters per 1 cm increase in height.

53. a. The rate of change of T with respect to t with s held constant is given by $\dfrac{\partial T}{\partial t}(t, s) = \lim\limits_{h\to 0}\dfrac{T(t + h, s) - T(t, s)}{h}$.

So if h is small, then $\dfrac{\partial T}{\partial t}(t, s) \approx \dfrac{T(t + h, s) - T(t, s)}{h}$. In particular, with $t = 34$, $s = 25$, and $h = 2$, we see that the required rate of change is $\dfrac{\partial T}{\partial t}(34, 25) \approx \dfrac{T(36, 25) - T(34, 25)}{2} = \dfrac{24 - 21.4}{2} = 1.3$ °F/°F.

b. The required rate of change is given by

$\dfrac{\partial T}{\partial s}(34, 25) \approx \dfrac{T(34, 30) - T(34, 25)}{5} = \dfrac{20.3 - 21.4}{5} = -0.22$ °F/mi/h.

54. a. $T = f(t, s) = 35.74 + 0.6215t - 35.75s^{0.16} + 0.4275ts^{0.16}$, so

$f(32, 20) = 35.74 + 0.6215(32) - 35.75\left(20^{0.16}\right) + 0.4275(32)\left(20^{0.16}\right) \approx 19.99$, or approximately 20°F.

b. $\dfrac{\partial T}{\partial s} = -35.75\left(0.16S^{-0.84}\right) + 0.4275t\left(0.16S^{-0.84}\right) = 0.16\left(-35.75 + 0.4275t\right)s^{-0.84}$, so $\left.\dfrac{\partial T}{\partial s}\right|_{(32,20)} = 0.16\left[-35.75 + 0.4275\,(32)\right]20^{-0.84} \approx -0.285$; that is, the wind chill will drop by 0.3° for each 1 mph increase in wind speed.

55. $E = \left(1 - \dfrac{v}{V}\right)^{0.4}$.

a. $\dfrac{\partial E}{\partial V} = 0.4\left(1 - \dfrac{v}{V}\right)^{-0.6}\dfrac{\partial}{\partial V}\left(-\dfrac{v}{V}\right) = 0.4\left(1 - \dfrac{v}{V}\right)^{-0.6}\left(\dfrac{v}{V^2}\right) = \dfrac{0.4v}{V^2\left(1 - \dfrac{v}{V}\right)^{0.6}} > 0$. This says that for constant v, an increase in V increases the engine efficiency, which is to be expected.

b. $\dfrac{\partial E}{\partial v} = 0.4\left(1 - \dfrac{v}{V}\right)^{-0.6}\dfrac{\partial}{\partial v}\left(-\dfrac{v}{V}\right) = -\dfrac{0.4v}{V\left(1 - \dfrac{v}{V}\right)^{0.6}} < 0$. This says that for constant V, an increase in v decreases the engine efficiency.

56. $V = \dfrac{30.9T}{P}$, so $\dfrac{\partial V}{\partial T} = \dfrac{30.9}{P}$ and $\dfrac{\partial V}{\partial P} = -\dfrac{30.9T}{P^2}$. Therefore, $\left.\dfrac{\partial V}{\partial T}\right|_{T=300,\,P=800} = \dfrac{30.9}{800} = 0.039$, or 0.039 liters per degree. $\left.\dfrac{\partial V}{\partial P}\right|_{T=300,\,P=800} = -\dfrac{(30.9)\,(300)}{800^2} \approx -0.014$, or approximately -0.014 liters per millimeter of mercury.

57. $f\,(p,q) = 10{,}000 - 10p + 0.2q^2$, so $g\,(p,q) = 5000 + 0.8p^2 - 20q$. $\dfrac{\partial f}{\partial q} = 0.4q > 0$ and $\dfrac{\partial g}{\partial p} = 1.6p > 0$, and so the two products are substitute commodities.

58. $f\,(p,q) = 10{,}000 - 10p - e^{0.5q}$ and $g\,(p,q) = 50{,}000 - 4000q - 10p$. Thus, $\dfrac{\partial f}{\partial q} = -0.5e^{0.5q} < 0$ and $\dfrac{\partial g}{\partial p} = -10 < 0$, and so the two commodities are complementary commodities.

59. We have $\frac{1}{5}x + \frac{1}{10}y = 200 - p$ and $x + \frac{1}{2}y = 1000 - 5p$. Thus, $\frac{1}{10}x + \frac{1}{4}y = 160 - q$ and $x + \frac{5}{2}y = 1600 - 10q$. Then $2y = 600 + 5p - 10q$, so $y = \frac{1}{2}\,(600 + 5p - 10q) = g\,(p,q)$. Also, $x = 1000 - 5p - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(600 + 5p - 10q) = 1000 - 5p - 150 - \frac{5}{4}p + \frac{5}{2}q = 850 - \frac{25}{4}p + \frac{5}{2}q = f\,(p,q)$. Next, we compute $\dfrac{\partial f}{\partial q} = \dfrac{5}{2} > 0$ and $\dfrac{\partial g}{\partial p} = \dfrac{5}{2} > 0$, and so they are substitute commodities.

60. We have $\frac{\partial x}{\partial p} = -16p$ and $\frac{\partial x}{\partial q} = 0.4$. If $p = 6$ and $q = 40$, we have $x = 400 - 8(6)^2 + 0.4(40) = 128$, $\frac{\partial x}{\partial p} = -16(6) = -96$ and $\frac{\partial x}{\partial q} = 0.4$. Therefore, $E_p = -\dfrac{p\frac{\partial x}{\partial p}}{x} = -\dfrac{6\,(-96)}{128} = 4.5$ and $E_q = -\dfrac{q\frac{\partial x}{\partial q}}{x} = -\dfrac{40\,(0.4)}{128} = -0.125$. Our results show that an approximate increase of 1% in the unit price of product A will result in a 4.5% drop in the demand for that product, while a 1% increase in the unit price of product B will result in a 0.125% increase in the demand for product A.

61. We have $\frac{\partial x}{\partial p} = \frac{\partial}{\partial p}\left[(3q)\left(1+p^2\right)^{-1}\right] = (3q)(-1)\left(1+p^2\right)^{-2}(2p) = -\frac{6pq}{(1+p^2)^2}$ and $\frac{\partial x}{\partial q} = \frac{3}{1+p^2}$. If $p = 5$ and $q = 4$, we have $x = f(5,4) = \frac{3(4)}{1+5^2} = \frac{12}{26} = \frac{6}{13}$, $\frac{\partial x}{\partial p} = -\frac{6(5)(4)}{\left(1+5^2\right)^2} = -\frac{30}{169}$, and $\frac{\partial x}{\partial q} = \frac{3}{1+5^2} = \frac{3}{26}$. Therefore, $E_p = -\frac{p\frac{\partial x}{\partial p}}{x} = -\frac{5\left(-\frac{30}{169}\right)}{\frac{6}{13}} \approx 1.923$ and $E_q = -\frac{q\frac{\partial x}{\partial q}}{x} = -\frac{4\left(\frac{3}{26}\right)}{\frac{6}{13}} = -1$. Thus, an increase of 1% in the price of butter will result in a drop of approximately 1.9% in demand for butter (with the price of margarine held fixed at \$4/lb), while an increase of 1% in the price of margarine will result in the same percentage increase in the demand for butter (with the price of butter held fixed at \$5/lb).

62. We have $\frac{\partial x}{\partial p} = \frac{\partial}{\partial p}\left[2q\left(1+p^{1/2}\right)^{-1}\right] = (2q)(-1)\left(1+p^{1/2}\right)^{-2}\left(\frac{1}{2}p^{-1/2}\right) = -\frac{q}{\sqrt{p}(1+\sqrt{p})^2}$ and $\frac{\partial x}{\partial q} = \frac{2}{1+\sqrt{p}}$. If $p = 4$ and $q = 5$, we have $x = \frac{2(5)}{1+\sqrt{4}} = \frac{10}{3}$, $\frac{\partial x}{\partial p} = -\frac{5}{\sqrt{4}\left(1+\sqrt{4}\right)^2} = -\frac{5}{18}$, and $\frac{\partial x}{\partial q} = \frac{2}{1+\sqrt{4}} = \frac{2}{3}$. Therefore, $E_p = -\frac{p\frac{\partial x}{\partial p}}{x} = -\frac{4\left(-\frac{5}{18}\right)}{\frac{10}{3}} = \frac{1}{3}$ and $E_q = -\frac{q\frac{\partial x}{\partial q}}{x} = -\frac{5\left(\frac{2}{3}\right)}{\frac{10}{3}} = -1$. So an increase of 1% in the price of margarine will result in a decrease of approximately $\frac{1}{3}$% in demand for margarine (with the price of butter held fixed at \$5/lb), while an increase of 1% in the price of butter will result in the same percentage increase in the demand for margarine (with the price of margarine held fixed at \$4/lb).

63. $V = \frac{kT}{P}$, so $\frac{\partial V}{\partial T} = \frac{k}{P}$; $T = \frac{VP}{k}$, so $\frac{\partial T}{\partial P} = \frac{V}{k} = \frac{T}{P}$; and $P = \frac{kT}{V}$, so $\frac{\partial P}{\partial V} = -\frac{kT}{V^2} = -kT\frac{P^2}{(kT)^2} = -\frac{P^2}{kT}$. Therefore $\frac{\partial V}{\partial T}\cdot\frac{\partial T}{\partial P}\cdot\frac{\partial P}{\partial V} = \frac{k}{P}\left(\frac{T}{P}\right)\left(-\frac{P^2}{kT}\right) = -1$.

64. $\frac{\partial K}{\partial m} = \frac{1}{2}v^2$, $\frac{\partial K}{\partial v} = mv$, and $\frac{\partial^2 K}{\partial v^2} = m$. Therefore, $\frac{\partial K}{\partial m}\cdot\frac{\partial^2 K}{\partial v^2} = \left(\frac{1}{2}v^2\right)(m) = \frac{1}{2}mv^2 = K$, as was to be shown.

65. $\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}kx^{\alpha}y^{1-\alpha} = k\alpha x^{\alpha-1}y^{1-\alpha} = k\alpha\left(\frac{y}{x}\right)^{1-\alpha}$ and $\frac{\partial P}{\partial y} = k(1-\alpha)x^{\alpha}y^{-\alpha} = k(1-\alpha)\left(\frac{x}{y}\right)^{\alpha}$. Therefore, $x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} = \frac{k\alpha xy^{1-\alpha}}{x^{1-\alpha}} + \frac{k(1-\alpha)yx^{\alpha}}{y^{\alpha}} = k\alpha x^{\alpha}y^{1-\alpha} + k(1-\alpha)x^{\alpha}y^{1-\alpha} = kx^{\alpha}y^{1-\alpha} = P$, as was to be shown.

66. $f(x,y,z) = x^2y + xy^2 + yz^3 + xye^{2z}$, so $f_x = 2xy + y^2 + ye^{2z}$ and $f_{xz} = 2ye^{2z}$; $f_y = x^2 + 2xy + z^3 + xe^{2z}$ and $f_{yz} = 3z^2 + 2xe^{2z}$; $f_z = 3yz^2 + 2xye^{2z}$ and $f_{zz} = 6yz + 4xye^{2z}$.

67. False. Let $f(x,y) = xy^{1/2}$. Then $f_x = y^{1/2}$ is defined at (0, 0), but $f_y = \frac{1}{2}xy^{-1/2} = \frac{x}{2y^{1/2}}$ is not defined at (0, 0).

68. False. Consider $f(x,y) = |x| + |y|$. Then f is continuous at (0, 0), but neither $f_x(0,0)$ nor $f_y(0,0)$ exists. See Section 2.6.

69. True. Since $f_x(x,y) = 0$ for all x and y, we see that f is independent of x. So there exists a function $g(y)$ such that $f(x,y) = g(y)$. Then $f_y(x,y) = \frac{d}{dy}g(y) = g'(y) = 0$ for all y, implying that $g(y) = C$, a constant. Thus, $f(x,y) = C$.

70. True. This is a consequence of the definition of $f_x(a,b)$ as the rate of change of f in the x-direction at (a,b) with y held fixed.

71. True. See page 580 of the text.

72. False. Let $f(x,y) = xy^{5/3}$. Then $f_{xy} = \frac{5}{3}y^{2/3} = f_{yx}$, so both f_{xy} and f_{yx} exist at $(0,0)$. However, $f_{yy} = \dfrac{10x}{9y^{1/3}}$ is not defined at $(0,0)$.

Using Technology page 587

1. 1.3124, 0.4038.

2. 9.4548, 2.3637.

3. −1.8889, 0.7778.

4. 0.0370, 0.7407.

5. −0.3863, −0.8497.

6. −0.0255, 0.0316.

8.3 Maxima and Minima of Functions of Several Variables

Concept Questions page 594

1. a. A function $f(x,y)$ has a relative maximum at (a,b) if $f(a,b)$ is the largest value of $f(x,y)$ for all (x,y) near (a,b).

b. $f(a,y)$ has an absolute maximum at (a,b) if $f(a,b)$ is the largest value of $f(x,y)$ for all (x,y) in the domain of f.

2. a. See the definition on page 590 of the text.

b. If f has a relative extremum at (a,b), then (a,b) must be a critical point of f. The converse is false.

3. See the procedure on page 590 of the text.

4. a. $D(a,b) = f_{xx}(a,b)\,f_{yy}(a,b) - f_{xy}^2(a,b) = (-2)(-5) - 3^2 = 1 > 0$, so f has a relative extremum at (a,b). Since $f_{xx}(a,b) = -2 < 0$, we conclude that the relative extremum is, in fact, a relative maximum.

b. $D(a,b) = (3)(2) - 3^2 = -3 < 0$, so f has a saddle point at (a,b).

c. $D(a,b) = (1)(4) - 2^2 = 0$. The test is inconclusive and further investigation is required.

d. $D(a,b) = 2(4) - 2^2 = 4 > 0$. Since $f_{xx}(a,b) = 2 > 0$,we conclude that f has a relative minimum at (a,b).

Exercises page 595

1. $f(x,y) = 1 - 2x^2 - 3y^2$. To find the critical points of f, we solve the system $\begin{cases} f_x = -4x = 0 \\ f_y = -6y = 0 \end{cases}$ obtaining $(0,0)$ as the only critical point of f. Next, $f_{xx} = -4$, $f_{xy} = 0$, and $f_{yy} = -6$. In particular, $f_{xx}(0,0) = -4$, $f_{xy}(0,0) = 0$, and $f_{yy}(0,0) = -6$, giving $D(0,0) = (-4)(-6) - 0^2 = 24 > 0$. Because $f_{xx}(0,0) < 0$, the Second Derivative Test implies that $(0,0)$ gives rise to a relative maximum of f. Finally, the relative maximum value of f is $f(0,0) = 1$.

2. $f(x,y)=x^2-xy+y^2+1$. To find the critical points of f, we solve the system $\begin{cases} f_x=2x-y=0 \\ f_y=-x+2y=0 \end{cases}$ obtaining $x=0$ and $y=0$, so $(0,0)$ is the only critical point. Next, $f_{xx}=2$, $f_{xy}=-1$, and $f_{yy}=2$, so $D(x,y)=f_{xx}f_{yy}-f_{xy}^2=3$. In particular, since $D(0,0)=3>0$ and $f_{xx}(0,0)=2>0$, we see that $(0,0)$ gives rise to a relative minimum. The relative minimum value is $f(0,0)=1$.

3. $f(x,y)=x^2-y^2-2x+4y+1$. To find the critical points of f, we solve the system $\begin{cases} f_x=\ \ 2x-2=0 \\ f_y=-2y+4=0 \end{cases}$ obtaining $x=1$ and $y=2$, so $(1,2)$ is the only critical point of f. $f_{xx}=2$, $f_{xy}=0$, and $f_{yy}=-2$, so $D(x,y)=f_{xx}f_{yy}-f_{xy}^2=-4$. In particular, $D(1,2)=-4<0$, so $(1,2)$ gives a saddle point of f. Because $f(1,2)=4$, the saddle point is $(1,2,4)$.

4. $f(x,y)=2x^2+y^2-4x+6y+3$. To find the critical points of f, we solve the system $\begin{cases} f_x=4x-4=0 \\ f_y=2y+6=0 \end{cases}$ obtaining $(1,-3)$ as the only critical point of f. Next, $f_{xx}=4$, $f_{xy}=0$, and $f_{yy}=2$, so $D(1,-3)=f_{xx}(1,-3)f_{yy}(1,-3)-f_{xy}^2(1,-3)=4\cdot 2-0=8>0$. Because $f_{xx}(1,-3)>0$, $(1,-3)$ gives rise to a relative minimum of f. Finally, $f(1,-3)=2+9-4+6(-3)+3=-8$ is the relative minimum value.

5. $f(x,y)=x^2+2xy+2y^2-4x+8y-1$. To find the critical points of f, we solve the system $\begin{cases} f_x=2x+2y-4=0 \\ f_y=2x+4y+8=0 \end{cases}$ obtaining $(8,-6)$ as the critical point of f. Next, $f_{xx}=2$, $f_{xy}=2$, and $f_{yy}=4$. In particular, $f_{xx}(8,-6)=2$, $f_{xy}(8,-6)=2$, and $f_{yy}(8,-6)=4$, giving $D=2(4)-4=4>0$. Because $f_{xx}(8,-6)>0$, $(8,-6)$ gives rise to a relative minimum of f. The relative minimum value of f is $f(8,-6)=-41$.

6. $f(x,y)=x^2-4xy+2y^2+4x+8y-1$. To find the critical points of f, we solve the system $\begin{cases} f_x=\ \ 2x-4y+4=0 \\ f_y=-4x+4y+8=0 \end{cases}$ obtaining $x=6$ and $y=4$, so $(6,4)$ is the only critical point. Next, $f_{xx}=2$, $f_{xy}=-4$, and $f_{yy}=4$, so $D(x,y)=f_{xx}f_{yy}-f_{xy}^2=8-(16)=-8$. In particular, $D(6,4)=-8<0$ and so $(6,4)$ gives a saddle point of f. Because $f(6,4)=27$, the saddle point is $(6,4,27)$.

7. $f(x,y)=2x^3+y^2-9x^2-4y+12x-2$. To find the critical points of f, we solve the system $\begin{cases} f_x=6x^2-18x+12=0 \\ f_y=2y-4=0 \end{cases}$ The first equation is equivalent to $x^2-3x+2=0$, or $(x-2)(x-1)=0$, giving $x=1$ or 2. The second equation of the system gives $y=2$. Therefore, there are two critical points, $(1,2)$ and $(2,2)$. Next, we compute $f_{xx}=12x-18=6(2x-3)$, $f_{xy}=0$, and $f_{yy}=2$.

At the point $(1,2)$, $f_{xx}(1,2)=6(2-3)=-6$, $f_{xy}(1,2)=0$, and $f_{yy}(1,2)=2$, so $D(1,2)=(-6)(2)-0=-12<0$ and we conclude that $(1,2)$ gives a saddle point of f. Because $f(1,2)=-1$, the saddle point is $(1,2,-1)$.

At the point $(2,2)$, $f_{xx}(2,2)=6(4-3)=6$, $f_{xy}(2,2)=0$, and $f_{yy}(2,2)=2$, so $D(2,2)=(6)(2)-0=12>0$. Because $f_{xx}(2,2)>0$, we see that $(2,2)$ gives a relative minimum with value $f(2,2)=-2$.

8. $f(x,y) = 2x^3 + y^2 - 6x^2 - 4y + 12x - 2$. To find the critical points of f, we solve the system

$$\begin{cases} f_x = 6x^2 - 12x + 12 = 0 \\ f_y = 2y - 4 = 0 \end{cases}$$

The first equation is equivalent to $x^2 - 2x + 2 = 0$. Using the quadratic formula, we find that the equation has no real solution. Therefore, there is no critical point.

9. $f(x,y) = x^3 + y^2 - 2xy + 7x - 8y + 4$. To find the critical points of f, we solve the system

$$\begin{cases} f_x = 3x^2 - 2y + 7 = 0 \\ f_y = 2y - 2x - 8 = 0 \end{cases}$$

Adding the two equations gives $3x^2 - 2x - 1 = (3x+1)(x-1) = 0$. Therefore, $x = -\frac{1}{3}$ or 1. Substituting each of these values of x into the second equation gives $y = \frac{11}{3}$ and $y = 5$, respectively. Therefore, $\left(-\frac{1}{3}, \frac{11}{3}\right)$ and $(1,5)$ are critical points of f. Next, $f_{xx} = 6x$, $f_{xy} = -2$, and $f_{yy} = 2$, so $D(x,y) = 12x - 4 = 4(3x-1)$. Then $D\left(-\frac{1}{3}, \frac{11}{3}\right) = 4(-1-1) = -8 < 0$, and so $\left(-\frac{1}{3}, \frac{11}{3}\right)$ gives a saddle point. Because $f\left(-\frac{1}{3}, \frac{11}{3}\right) = -\frac{319}{27}$, the saddle point is $\left(-\frac{1}{3}, \frac{11}{3}, -\frac{319}{27}\right)$. Next, $D(1,5) = 4(3-1) = 8 > 0$, and since $f_{xx}(1,5) = 6 > 0$, we see that $(1,5)$ gives a relative minimum with value $f(1,5) = -13$.

10. $f(x,y) = 2y^3 - 3y^2 - 12y + 2x^2 - 6x + 2$. To find the critical points of f, we solve the system

$$\begin{cases} f_x = 4x - 6 = 0 \\ f_y = 6y^2 - 6y - 12 = 0 \end{cases}$$

We find $x = \frac{3}{2}$ and $y = -1$ or 2. Therefore, $\left(\frac{3}{2}, -1\right)$ and $\left(\frac{3}{2}, 2\right)$ are critical points of f. Next, we find $f_{xx} = 4$, $f_{xy} = 0$ and $f_{yy} = 12y - 6 = 6(2y-1)$.

At the point $\left(\frac{3}{2}, -1\right)$, $f_{xx}\left(\frac{3}{2}, -1\right) = 4$, $f_{xy}\left(\frac{3}{2}, -1\right) = 0$, and $f_{yy}\left(\frac{3}{2}, -1\right) = 6(-2-1) = -18$. Therefore, $D\left(\frac{3}{2}, -1\right) = (4)(-18) - 0 = -72 < 0$, and $\left(\frac{3}{2}, -1\right)$ gives a saddle point of f. Because $f\left(\frac{3}{2}, -1\right) = \frac{9}{2}$, the saddle point is $\left(\frac{3}{2}, -1, \frac{9}{2}\right)$.

At the point $\left(\frac{3}{2}, 2\right)$, $f_{xx}\left(\frac{3}{2}, 2\right) = 4$, $f_{xy}\left(\frac{3}{2}, 2\right) = 0$, and $f_{yy}\left(\frac{3}{2}, 2\right) = 6(4-1) = 18$. Because $D\left(\frac{3}{2}, 2\right) > 0$ and $f_{xx}\left(\frac{3}{2}, 0\right) > 0$, we conclude that $\left(\frac{3}{2}, 2\right)$ gives a relative minimum of f.

$f\left(\frac{3}{2}, 2\right) = 2(2)^3 - 3(2)^2 - 12(2) + 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 2 = -\frac{45}{2}$, so the relative minimum value is $f\left(\frac{3}{2}, 2\right) = -\frac{45}{2}$.

11. $f(x,y) = x^3 - 3xy + y^3 - 2$. To find the critical points of f, we solve the system $\begin{cases} f_x = 3x^2 - 3y = 0 \\ f_y = -3x + 3y^2 = 0 \end{cases}$

The first equation gives $y = x^2$, and substituting this into the second equation gives $-3x + 3x^4 = 3x\left(x^3 - 1\right) = 0$. Therefore, $x = 0$ or 1. Substituting these values of x into the first equation gives $y = 0$ and $y = 1$, respectively. Therefore, $(0,0)$ and $(1,1)$ are critical points of f. Next, we find $f_{xx} = 6x$, $f_{xy} = -3$, and $f_{yy} = 6y$, so $D = f_{xx}f_{yy} - f_{xy}^2 = 36xy - 9$. Because $D(0,0) = -9 < 0$, we see that $(0,0)$ gives a saddle point of f. Because $f(0,0) = -2$, the saddle point is $(0,0,-2)$. Next, $D(1,1) = 36 - 9 = 27 > 0$, and since $f_{xx}(1,1) = 6 > 0$, we see that $f(1,1) = -3$ is a relative minimum value of f.

12. $f(x,y) = x^3 - 2xy + y^2 + 5$. We set $f_x = 3x^2 - 2y = 0$ and $f_y = -2x + 2y = 0$, so $3x^2 - 2x = x(3x - 2) = 0$, giving $x = 0$ or $x = \frac{2}{3}$. Therefore, $y = 0$ or $y = \frac{2}{3}$ respectively. Thus, the critical points of f are $(0,0)$ and $\left(\frac{2}{3}, \frac{2}{3}\right)$. Next, $f_{xx} = 6x$, $f_{xy} = -2$, and $f_{yy} = 2$, so $D = 12x - 4 = 4(3x - 1)$. At the point $(0,0)$, $D(0,0) = -4 < 0$, so $(0,0)$ gives a saddle point. Because $f(0,0) = 5$, the saddle point is $(0,0,5)$. At the point $\left(\frac{2}{3}, \frac{2}{3}\right)$, $D\left(\frac{2}{3}, \frac{2}{3}\right) = 4 > 0$ and $f_{xx}\left(\frac{2}{3}, \frac{2}{3}\right) > 0$, so $f\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{8}{27} - \frac{8}{9} + \frac{4}{9} + 5 = \frac{131}{27}$ is a relative minimum value.

13. $f(x,y) = xy + \dfrac{4}{x} + \dfrac{2}{y}$. Solving the system of equations $\begin{cases} f_x = y - \dfrac{4}{x^2} = 0 \\ f_y = x - \dfrac{2}{y^2} = 0 \end{cases}$ we obtain $y = \dfrac{4}{x^2}$.

Therefore, $x - 2\left(\dfrac{x^4}{16}\right) = 0$ and $8x - x^4 = x\left(8 - x^3\right) = 0$, so $x = 0$ or $x = 2$. Because $x = 0$ is not in the domain of f, $(2,1)$ is the only critical point of f. Next, $f_{xx} = \dfrac{8}{x^3}$, $f_{xy} = 1$, and $f_{yy} = \dfrac{4}{y^3}$. Therefore, $D(2,1) = \left.\left(\dfrac{32}{x^3y^3} - 1\right)\right|_{(2,1)} = 4 - 1 = 3 > 0$ and $f_{xx}(2,1) = 1 > 0$, so the relative minimum value of f is $f(2,1) = 2 + \frac{4}{2} + \frac{2}{1} = 6$.

14. $f(x,y) = \dfrac{x}{y^2} + xy$. Setting $f_x = \dfrac{1}{y^2} + y = 0$ gives $1 + y^3 = 0$, so $y = -1$, and setting $f_y = -\dfrac{2x}{y^3} + x = 0$ gives $\dfrac{2x}{1} + x = 0$, so $x = 0$. Thus, $(0,-1)$ is a critical point. Next, $f_{xx} = 0$, $f_{xy} = -\dfrac{2}{y^3} + 1$, and $f_{yy} = \dfrac{6x}{y^4}$, so $D(0,-1) = 0 - 9 < 0$, and $f(0,-1)$ gives rise to a saddle point. Because $f(0,-1) = 0$, the saddle point is $(0,-1,0)$.

15. $f(x,y) = x^2 - e^{y^2}$. Solving the system of equations $\begin{cases} f_x = 2x = 0 \\ f_y = -2ye^{y^2} = 0 \end{cases}$ we obtain $x = 0$ and $y = 0$.

Therefore, $(0,0)$ is the only critical point of f. Next, $f_{xx} = 2$, $f_{xy} = 0$, and $f_{yy} = -2e^{y^2} - 4y^2e^{y^2}$, so $D(0,0) = \left[-4e^{y^2}\left(1 + 2y^2\right)\right]_{(0,0)} = -4(1) < 0$, and we conclude that $(0,0)$ gives a saddle point. Because $f(0,0) = -1$, the saddle point is $(0,0,-1)$.

16. $f(x,y) = e^{x^2-y^2}$. To find the critical points of f, we solve the system $\begin{cases} f_x = 2xe^{x^2-y^2} = 0 \\ f_y = -2ye^{x^2-y^2} \end{cases}$ obtaining $x = 0$ and $y = 0$. Therefore, $(0,0)$ is the only critical point of f. $f_{xx} = 2e^{x^2-y^2} + 4x^2e^{x^2-y^2} = 2\left(1 + 2x^2\right)e^{x^2-y^2}$, $f_{xy} = -4xye^{x^2-y^2}$, and $f_{yy} = -2e^{x^2-y^2} + 4y^2e^{x^2-y^2} = -2\left(1 - 2y^2\right)e^{x^2-y^2}$. At $(0,0)$, we see that $f_{xx}(0,0) = 2$, $f_{xy}(0,0) = 0$, and $f_{yy}(0,0) = -2$. Because $D(0,0) = f_{xx}(0,0)\,f_{yy}(0,0) - f_{xy}^2(0,0) = -4 < 0$, we see that $(0,0)$ gives a saddle point of f. Because $f(0,0) = 1$, the saddle point is $(0,0,1)$.

17. $f(x,y) = e^{x^2+y^2}$. Solving the system $\begin{cases} f_x = 2xe^{x^2+y^2} = 0 \\ f_y = 2ye^{x^2+y^2} = 0 \end{cases}$ we see that $x = 0$ and $y = 0$ (recall that $e^{x^2+y^2} \neq 0$). Therefore, $(0,0)$ is the only critical point of f. Next, we compute $f_{xx} = 2e^{x^2+y^2} + 2x(2x)e^{x^2+y^2} = 2(1+2x^2)e^{x^2+y^2}$, $f_{xy} = 2x(2y)e^{x^2+y^2} = 4xye^{x^2+y^2}$, and $f_{yy} = 2(1+2y^2)e^{x^2+y^2}$. In particular, at the point $(0,0)$, $f_{xx}(0,0) = 2$, $f_{xy}(0,0) = 0$, and $f_{yy}(0,0) = 2$. Therefore, $D(0,0) = (2)(2) - 0 = 4 > 0$. Because $f_{xx}(0,0) > 0$, we conclude that $(0,0)$ gives rise to a relative minimum of f. The relative minimum value is $f(0,0) = 1$.

18. $f(x,y) = e^{xy}$. To find the critical points of f, we solve the system $\begin{cases} f_x = ye^{xy} = 0 \\ f_y = xe^{xy} = 0 \end{cases}$ obtaining $y = 0$ and $x = 0$, so $(0,0)$ is a critical point of f. Next, $f_{xx} = y^2e^{xy}$, $f_{xy} = e^{xy} + xye^{xy} = (1+xy)e^{xy}$, and $f_{yy} = x^2e^{xy}$, so $D(x,y) = x^2y^2e^{2xy} - (1+xy)^2e^{2xy}$. In particular, $D(0,0) = -1 < 0$, and so $(0,0)$ gives rise to a saddle point of f. The saddle point is $(0,0,1)$.

19. $f(x,y) = \ln(1+x^2+y^2)$. We solve the system of equations $\begin{cases} f_x = \dfrac{2x}{1+x^2+y^2} = 0 \\ f_y = \dfrac{2y}{1+x^2+y^2} = 0 \end{cases}$ obtaining $x = 0$ and $y = 0$. Therefore, $(0,0)$ is the only critical point of f. Next, $f_{xx} = \dfrac{(1+x^2+y^2)2 - (2x)(2x)}{(1+x^2+y^2)^2} = \dfrac{2+2y^2-2x^2}{(1+x^2+y^2)^2}$, $f_{yy} = \dfrac{(1+x^2+y^2)2 - (2y)(2y)}{(1+x^2+y^2)^2} = \dfrac{2+2x^2-2y^2}{(1+x^2+y^2)^2}$, and $f_{xy} = -2x(1+x^2+y^2)^{-2}(2y) = -\dfrac{4xy}{(1+x^2+y^2)^2}$. Therefore, $D(x,y) = \dfrac{(2+2y^2-2x^2)(2+2x^2-2y^2)}{(1+x^2+y^2)^4} - \dfrac{16x^2y^2}{(1+x^2+y^2)^4}$. Because $D(0,0) = 4 > 0$ and $f_{xx}(0,0) = 2 > 0$, $f(0,0) = 0$ is a relative minimum value.

20. $f(x,y) = xy + \ln x + 2y^2$. We solve the system of equations $\begin{cases} f_x = y + \dfrac{1}{x} = 0 \\ f_y = x + 4y = 0 \end{cases}$ giving $y = -\dfrac{1}{x}$, $x + 4\left(-\dfrac{1}{x}\right) = 0$, and so $x^2 - 4 = 0$ and $x = 2$. (Remember that x must be positive.) Therefore, $y = -\frac{1}{2}$, and $\left(2, -\frac{1}{2}\right)$ is a critical point. Next, $f_{xx} = -\dfrac{1}{x^2}$, $f_{xy} = 1$, and $f_{yy} = 4$, so $D\left(2, -\frac{1}{2}\right) = -\frac{1}{4}(4) - 1 < 0$ and thus $\left(2, -\frac{1}{2}\right)$ gives a saddle point. Because $f\left(2, -\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right) + \ln 2 + 2\left(-\frac{1}{2}\right)^2 = \ln 2 - \frac{1}{2}$, the saddle point is $\left(2, -\frac{1}{2}, \ln 2 - \frac{1}{2}\right)$.

21. $P(x) = -0.2x^2 - 0.25y^2 - 0.2xy + 200x + 160y - 100x - 70y - 4000$

$= -0.2x^2 - 0.25y^2 - 0.2xy + 100x + 90y - 4000.$

Thus, $\begin{cases} P_x = -0.4x - 0.2y + 100 = 0 \\ P_y = -0.5y - 0.2x + 90 = 0 \end{cases}$ implies that $\begin{cases} 4x + 2y = 1000 \\ 2x + 5y = 900 \end{cases}$ Solving, we find $x = 200$ and $y = 100$. Next, $P_{xx} = -0.4$, $P_{yy} = -0.5$, $P_{xy} = -0.2$, and $D(200, 100) = (-0.4)(-0.5) - (-0.2)^2 > 0$. Because $P_{xx}(200, 100) < 0$, we conclude that $(200, 100)$ is a relative maximum of P. Thus, the company should manufacture 200 finished and 100 unfinished units per week. The maximum profit is $P(200, 100) = -0.2(200)^2 - 0.25(100)^2 - 0.2(100)(200) + 100(200) + 90(100) - 4000 = 10{,}500$, or \$10,500.

22. $P(x, y) = -0.005x^2 - 0.003y^2 - 0.002xy + 20x + 15y - 6x - 3y - 200$

$= -0.005x^2 - 0.003y^2 - 0.002xy + 14x + 12y - 200.$

Next, $\begin{cases} P_x = -0.01x - 0.002y + 14 = 0 \\ P_y = -0.006y - 0.002x + 12 = 0 \end{cases}$ Therefore, $\begin{cases} 10x + 2y = 14{,}000 \\ 2x + 6y = 12{,}000 \end{cases}$

Solving, we find $x \approx 1071$ and $y \approx 1643$. Next, $P_{xx} = -0.01$, $P_{xy} = -0.002$, and $P_{yy} = -0.006$, so $D(1071, 1643) = (-0.01)(-0.006) - (-0.002)^2 = 0.000056 > 0$. Because $P_{xx}(1071, 1643) < 0$, we conclude that $(1071, 1643)$ is a relative maximum of P. Thus, the company should publish 1071 deluxe and 1643 standard copies. The maximum profit is

$P(1071, 1643) = -0.005(1071)^2 - 0.003(1643)^2 - 0.002(1071)(1643) + 14(1071) + 12(1643) - 200$

$= 17{,}157.14$, or \$17,157.14.

23. $p(x, y) = 200 - 10\left(x - \frac{1}{2}\right)^2 - 15(y - 1)^2$. Solving the system of equations $\begin{cases} p_x = -20\left(x - \frac{1}{2}\right) = 0 \\ p_y = -30(y - 1) = 0 \end{cases}$ we obtain $x = \frac{1}{2}$ and $y = 1$. We conclude that the only critical point of f is $\left(\frac{1}{2}, 1\right)$. Next, $p_{xx} = -20$, $p_{xy} = 0$, and $p_{yy} = -30$, so $D\left(\frac{1}{2}, 1\right) = (-20)(-30) = 600 > 0$. Because $p_{xx} = -20 < 0$, we conclude that $f\left(\frac{1}{2}, 1\right)$ gives a relative maximum. We conclude that the price of land is highest at $\left(\frac{1}{2}, 1\right)$.

24. We wish to maximize the function

$P(x) = R(x) - C(x)$

$= (2000 - 150p + 100q)p + (1000 + 80p - 120q)q - 4(2000 - 150p + 100q) - 3(1000 + 80p - 120q)$

$= 2360p - 150p^2 + 180pq + 960q - 120q^2 - 11{,}000.$

Then $\begin{cases} P_p = 2360 - 300p + 180q \\ P_q = 180p + 960 - 240q \end{cases}$ Solving the system $\begin{cases} 300p - 180q = 2360 \\ -180p + 240q = 960 \end{cases}$ we find that $\left(\frac{56}{3}, 18\right)$ is a critical point. Next, $P_{pp} = -300$, $P_{qq} = -240$, and $P_{pq} = 180$, so $D\left(\frac{56}{3}, 18\right) = (-300)(-240) - (180)^2 = 39{,}600 > 0$. Because $P_{pp} < 0$, we conclude that P is maximized at $\left(\frac{56}{3}, 18\right)$ and the company should therefore sell the German wine at \$18.67 per bottle and the Italian wine at \$18 per bottle.

25. a. $R(p,q) = xp + yq = (6400 - 400p - 200q)\,p + (5600 - 200p - 400q)\,q = 400\left(-p^2 - q^2 - pq + 16p + 14q\right)$.

b. To find the critical point of R, we solve the system $\begin{cases} R_p = 400(-2p - q + 16) = 0 \\ R_q = 400(-p - 2q + 14) = 0 \end{cases}$ obtaining $p = 6$ and $q = 4$. So the sole critical point of R is $(6,4)$. Next, we find $R_{pp} = -800$, $R_{pq} = -400$, and $R_{qq} = -800$. Then $D(6,4) = (-800)(-800) - (-400)^2 = 480{,}000 > 0$. Since $R_{pp}(6,4) < 0$, we see that $(6,4)$ does yield a relative (and therefore, the absolute) maximum of R. We conclude that the supermarket should charge \$6/lb for the ground sirloin and \$4/lb for the ground beef. The quantity of sirloin sold per week will be $x = 6400 - 400(6) - 200(4) = 3200$, or 3200 lb, and the quantity of ground beef sold per week will be $y = 5600 - 200(6) - 400(4) = 2800$, or 2800 lb. The maximum revenue is $R(6,4) = 400\left[-(6)^2 - (4)^2 - 4(6)(4) + 16(6) + 14(4)\right] = 30{,}400$, or \$30,400.

26. a. The revenue function is $R(x,y) = px + qy = (80 - 0.01x - 0.005y)\,x + (60 - 0.005x - 0.015y)\,y = -0.01x^2 - 0.015y^2 - 0.01xy + 80x + 60y$.
The total cost function is $C(x,y) = 8x + 12y + 10{,}000$, and
$P(x,y) = R(x,y) - C(x,y) = -0.01x^2 - 0.015y^2 - 0.01xy + 72x + 48y - 10$.

b. To find the critical point of P, we solve $\begin{cases} P_x = -0.02x - 0.01y + 72 = 0 \\ P_y = -0.01x - 0.03y + 48 = 0 \end{cases}$ and find $x = 3360$ and $y = 480$. Next, we find $P_{xx} = -0.02$, $P_{xy} = -0.01$, and $P_{yy} = -0.03$. Then $D(3360, 480) = (-0.02)(-0.03) - (-0.01)^2 = 0.0005 > 0$. Since $P_{xx}(3360, 480) < 0$, we see that $(3360, 480) < 0$, and so $(3360, 480)$ does yield a relative (and therefore absolute) maximum of P. So the division should produce 336,000 bars of 3.5-oz soap and 48,000 bars of 5-oz soap. (Recall that x and y are measured in hundreds.)
The maximum weekly profit will be $P(3360, 480) = -0.01(3360)^2 - 0.015(480)^2 - 0.01(3360)(480) + 72(3360) + 48(480) - 10{,}000 = 122{,}480$, or \$122,480. (Recall that p and q are measured in dollars per hundred bars.)

27. a. $R(x,y) = px + qy = (3000 - 20x - 10y)\,x + (4000 - 10x - 30y)\,y = -20x^2 - 30y^2 - 20xy + 3000x + 4000y$.

b. $C(x,y) = 400x + 500y + 20{,}000$.

c. $P(x,y) = R(x,y) - C(x,y) = \left(-20x^2 - 30y^2 - 20xy + 3000x + 4000y\right) - (400x + 500y + 20{,}000) = -20x^2 - 30y^2 - 20xy + 2600x + 3500y - 20{,}000$.

d. To find the critical point of P, we solve $\begin{cases} P_x = -40x - 20y + 2600 = 0 \\ P_y = -20x - 60y + 3500 = 0 \end{cases}$ or $\begin{cases} 4x + 2y = 260 \\ 2x + 6y = 350 \end{cases}$ finding $x = 43$ and $y = 44$. Next, we find $P_{xx} = -40$, $P_{xy} = -20$, and $P_{yy} = -60$. Thus, $D(43,44) = (-40)(-60) - (-20)^2 = 2000 > 0$. Since $D(43,44) > 0$ and $P_{xx}(43,44) < 0$, we see that $(43,44)$ yields a relative (and therefore absolute) maximum of P. So the company should produce 43,000 tubes of regular and 44,000 tubes of whitening toothpaste. (Recall that x and y are measured in thousands .)
The maximum weekly profit will be $P(43,44) = -20(43)^2 - 30(44)^2 - 20(43)(44) + 2600(43) + 3500(44) - 20{,}000 = 112{,}900$, or \$112,900.

28. We want to minimize $f(x,y) = D^2 = (x-5)^2 + (y-2)^2 + (x+4)^2 + (y-4)^2 + (x+1)^2 + (y+3)^2$. We calculate $\begin{cases} f_x = 2(x-5) + 2(x+4) + 2(x+1) = 6x = 0, \\ f_y = 2(y-2) + 2(y-4) + 2(y+3) = 6y - 6 = 0 \end{cases}$ and conclude that $x = 0$ and $y = 1$. Also, $f_{xx} = 6$, $f_{xy} = 0$, $f_{yy} = 6$, and $D(x,y) = (6)(6) = 36 > 0$. Because $f_{xx} > 0$, we conclude that the function is minimized at (0, 1), the desired location.

29. The sums of the squares of the distances from the proposed site of the radio station to the three communities is

$$\begin{aligned} D = f(x,y) &= [d(P,A)]^2 + [d(P,B)]^2 + [d(P,C)]^2 \\ &= \left[(x-2)^2 + (y-4)^2\right] + \left[(x-20)^2 + (y-8)^2\right] + \left[(x-4)^2 + (y-24)^2\right] \end{aligned}$$

$$\left.\begin{aligned} f_x(x,y) &= 2(x-2) + 2(x-20) + 2(x-4) = 0 \\ f_y(x,y) &= 2(y-4) + 2(y-8) + 2(y-24) = 0 \end{aligned}\right\} \Rightarrow x = \tfrac{26}{3} \text{ and } y = 12 \text{, so } \left(\tfrac{26}{3}, 12\right) \text{ is a critical point of } f.$$

Since it is clear that D must attain a minimum, we see that the station should be located at $\left(\frac{26}{3}, 12\right)$.

30. We want to maximize $V = \pi r^2 \ell$. But we have $2\pi r + \ell = 130 \Rightarrow \ell = 130 - 2\pi r$, so we need to maximize $V = f(r) = \pi r^2(130 - 2\pi r) = -2\pi^2 r^3 + 130\pi r^2$. Now $V' = f'(r) = -6\pi^2 r^2 + 260\pi r = -2\pi r(3\pi r - 130) = 0 \Rightarrow r = 0$ or $r = \frac{130}{3\pi}$. Since $f''\left(\frac{130}{3\pi}\right) = \left(-12\pi^2 r + 260\pi\right)\big|_{130/(3\pi)} = -12\pi^2\left(\frac{130}{3\pi}\right) + 260\pi = -260\pi < 0$, we conclude that $r = \frac{130}{3\pi}$ does yield the absolute maximum for f. We find $\ell = 130 - 2\pi\left(\frac{130}{3\pi}\right) = \frac{130}{3}$ or $43\frac{1}{3}''$.

$V = \pi r^2 \ell = \pi\left(\frac{130}{3\pi}\right)^2\left(\frac{130}{3}\right) = \frac{2{,}197{,}000}{27\pi}$ in^3.

31. Solving the equation $xyz = 108$ for x, we have $z = 108/(xy)$. Substituting this value of z into the expression for S, we obtain $S = f(x,y) = xy + 2y\left[108/(xy)\right] + 2x\left[108/(xy)\right] = xy + (216/x) + (216/y)$. To minimize f, we first find the critical points of f. To do this, we solve the system $\begin{cases} f_x = y - \left(216/x^2\right) = 0 \\ f_y = x - \left(216/y^2\right) = 0 \end{cases}$ Solving the first equation for y, we obtain $y = 216/x^2$. Substituting this value into the second equation then yields $x - 216\left(x^2/216\right)^2 = 0$, $x - \left(x^4/216\right) = 0$, and $x\left(216 - x^3\right) = 0$, from which we deduce that $x = 0$, or $x = 6$. We reject the root $x = 0$, since it is not in the domain of f. Next, substituting $x = 6$ into the expression for y obtained earlier, we find $y = 6$. Thus, the point (6, 6) is the only critical point of f. We then compute $f_{xx} = 432/x^3$, $f_{xy} = 1$, $f_{yy} = 432/y^3$. In particular, $f_{xx}(6,6) = 2$, $f_{xy}(6,6) = 1$, and $f_{yy}(6,6) = 2$. Thus, $D = (2)(2) - 1 = 3 > 0$. Because $f_{xx}(6,6) > 0$, we conclude that (6, 6) gives rise to a relative minimum of f. Substituting these values of x and y into the expression for z yields $z = \frac{108}{6 \cdot 6} = 3$. Therefore, the required dimensions of the box are $6'' \times 6'' \times 3''$.

32. Refer to the figure in the text. $xy + 2xz + 2yz = 300$, so $z\left(2x + 2y\right) = 300 - xy$, The volume is given by $V = xyz = xy\dfrac{300 - xy}{2\left(x + y\right)} = \dfrac{300xy - x^2y^2}{2\left(x + y\right)}$.We find

$$\frac{\partial V}{\partial x} = \frac{1}{2}\frac{\left(x + y\right)\left(300y - 2xy^2\right) - \left(300xy - x^2y^2\right)}{\left(x + y\right)^2} = \frac{300xy - 2x^2y^2 + 300y^2 - 2xy^3 - 300xy + x^2y^2}{2\left(x + y\right)^2}$$

$$= \frac{300y^2 - 2xy^3 - x^2y^2}{2\left(x + y\right)^2} = \frac{y^2\left(300 - 2xy - x^2\right)}{2\left(x + y\right)^2}$$

and similarly $\dfrac{\partial V}{\partial y} = \dfrac{x^2\left(300 - 2xy - y^2\right)}{2\left(x + y\right)^2}$. Setting both $\dfrac{\partial V}{\partial x}$ and $\dfrac{\partial V}{\partial y}$ equal to 0 and observing that both $x > 0$ and $y > 0$, we have the system $\begin{cases} 2yx + x^2 = 300 \\ 2yx + y^2 = 300 \end{cases}$ Subtracting, we find $y^2 - x^2 = 0$, so $\left(y - x\right)\left(y + x\right) = 0$. Thus, $y = x$ or $y = -x$. The latter is not possible since x and y are both positive**.** Therefore, $y = x$. Substituting this value into the first equation in the system gives $2x^2 + x^2 = 300$, so $x^2 = 100$ and $x = y = 10$. Substituting these values into the expression for z gives $z = \dfrac{300 - 10^2}{2\left(10 + 10\right)} = 5$, so the dimensions are $10'' \times 10'' \times 5''$ and the volume is 500 in^3.

33. The volume is given by $V = xyz = xz\left(130 - 2x - 2z\right) = 130xz - 2x^2z - 2xz^2$. Solving the system of equations $\begin{cases} V_x = 130z - 4xz - 2z^2 = 0 \\ V_z = 130x - 2x^2 - 4xz = 0 \end{cases}$ we obtain $\left(130 - 4x - 2z\right)z = 0$, giving $130 - 4x - 2z = 0$, and $\left(130 - 4z - 2x\right)x = 0$, giving $130 - 2x - 4z = 0$. Thus, we have $\begin{cases} 130 - 4x - 2z = 0 \\ 260 - 4x - 8z = 0 \end{cases}$ giving $130 - 6z = 0$ and $z = \frac{65}{3}$. Therefore, $x = \frac{1}{4}\left(130 - 2 \cdot \frac{65}{3}\right) = \frac{65}{3}$ and $y = 130 - 2x - 2z = 130 - 2\left(\frac{65}{3}\right) - 2\left(\frac{65}{3}\right) = \frac{130}{3}$, and so $\left(x, z\right) = \left(\frac{65}{3}, \frac{65}{3}\right)$ is the critical point of V. Next, $V_{xx} = -4z$, $V_{zz} = -4x$, $V_{xz} = 130 - 4x - 4z$, and $D\left(\frac{65}{3}, \frac{65}{3}\right) = -4\left(\frac{65}{3}\right)\left(-4\right)\left(\frac{65}{3}\right) - \left(130 - 4 \cdot \frac{65}{3} - 4 \cdot \frac{65}{3}\right)^2 > 0$ and $V_{xx}\left(\frac{65}{3}, \frac{65}{3}\right) < 0$. We conclude that the dimensions yielding the maximum volume are $21\frac{2}{3}'' \times 43\frac{1}{3}'' \times 21\frac{2}{3}''$.

34. The heating cost is $C = 2xy + 8xz + 6yz$. But $xyz = 12{,}000$, so $z = \dfrac{12{,}000}{xy}$. Therefore, $C = f\left(x, y\right) = 2xy + 8x\left(\dfrac{12{,}000}{xy}\right) + 6y\left(\dfrac{12{,}000}{xy}\right) = 2xy + \dfrac{96{,}000}{y} + \dfrac{72{,}000}{x}$. To find the minimum of f, we find the critical point of f by solving the system $\begin{cases} f_x = 2y - \dfrac{72{,}000}{x^2} = 0 \\ f_y = 2x - \dfrac{96{,}000}{y^2} = 0 \end{cases}$ The first equation gives $y = \dfrac{36{,}000}{x^2}$, which when substituted into the second equation yields $2x - 96{,}000\left(\dfrac{x^2}{36{,}000}\right)^2 = 0$, so $\left(36{,}000\right)^2 x - 48{,}000x^4 = 0$ and $x\left(27{,}000 - x^3\right) = 0$. Solving this equation, we have $x = 0$ or $x = 30$. We reject the first root because $x = 0$ lies outside the domain of f. With $x = 30$, we find $y = 40$ and $z = 10$. Next, $f_{xx} = \dfrac{144{,}000}{x^3}$ and $f_{xy} = 2$. In particular, $f_{xx}\left(30, 40\right) \approx 5.33$, $f_{xy} = \left(30, 40\right) = 2$, and $f_{yy}\left(30, 40\right) = 3$.

Thus, $D(30,40) \approx (5.33)(3) - 4 = 11.99 > 0$, and since $f_{xx}(30,40) > 0$, we see that $(30,40)$ gives a relative minimum. Physical considerations tell us that this is an absolute minimum. The minimal annual heating cost is $f(30,40) = 2(30)(40) + \dfrac{96{,}000}{40} + \dfrac{72{,}000}{30} = 7200$, or \$7,200.

35. Because $V = xyz$, $z = \dfrac{48}{xy}$. Then the amount of material used in the box is given by $S = xy + 2xz + 3yz = xy + \dfrac{48}{xy}(2x+3y) = xy + \dfrac{96}{y} + \dfrac{144}{x}$. Solving the system of equations $\begin{cases} S_x = y - \dfrac{144}{x^2} = 0 \\ S_y = x - \dfrac{96}{y^2} = 0 \end{cases}$ we have $y = \dfrac{144}{x^2}$. Therefore, $x - \dfrac{96x^4}{144^2} = 0$, $144^2x - 96x^4 = 0$, $96x\left(216 - x^3\right) = 0$, and so $x = 0$ or $x = 6$. We reject $x = 0$, so $x = 6$ and $y = \frac{144}{36} = 4$. Next, $S_{xx} = \dfrac{288}{x^3}$, $S_{yy} = \dfrac{192}{y^3}$, and $S_{xy} = 1$. At the point $(6,4)$, $D(x,y) = \left.\left(\dfrac{288 \cdot 192}{x^3y^3} - 1\right)\right|_{(6,4)} = \dfrac{288(192)}{216(64)} - 1 = 3 > 0$ and $S_{xx} > 0$. We conclude that the function is minimized when the dimensions of the box are 6″ × 4″ × 2″.

36. False. Let $f(x,y) = xy$. Then $f_x(0,0) = 0$ and $f_y(0,0) = 0$, but $(0,0)$ does not give a relative extremum of $(0,0)$. In fact, $f_{xx} = 0$, $f_{yy} = 0$, and $f_{xy} = 1$, so $D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = -1$ and $D(0,0) = -1$, showing that $(0,0,0)$ is a saddle point.

37. False. Let $f(x,y) = -x^2 - y^2 + 4xy$. Then setting $\begin{cases} f_x(x,y) = -2x + 4y = 0 \\ f_y(x,y) = -2y + 4x = 0 \end{cases}$ we find that $(0,0)$ is the only critical point of f. Next, $f_{xx} = -2$, $f_{xy} = 4$, and $f_{yy} = -2$. Because $D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - 4^2 = -12 < 0$, we see that $(0,0,0)$ is a saddle point.

38. False. $f_x(a,b)$ and/or $f_y(a,b)$ may be undefined.

39. False. Take $f(x) = \dfrac{1}{x^2+1}$ and $g(y) = -y^2 - 1$. Then $h(x,y) = \dfrac{1}{x^2+1} - y^2 - 1$, so $h_x = -\dfrac{2x}{\left(x^2+1\right)^2}$ and $h_y = -2y$. Therefore, $(0,0)$ is a critical point of h. $h_{xx} = -2\left[\dfrac{1}{\left(x^2+1\right)^2} - \dfrac{4x^3}{\left(x^2+1\right)^3}\right]$, $h_{xy} = 0$, and $h_{yy} = -2$, so $D(0,0) = h_{xx}(0)\,h_{yy}(0) - h_{xy}^2(0,0) = (-2)(-2) - 0 = 4 > 0$. Because $h_{yy}(0,0) = -2 < 0$, we see that $(0,0)$ gives a relative maximum of h.

40. True. Here $f_{xx}(a,b) = -f_{yy}(a,b)$, so $D(a,b) = f_{xx}(a,b)\,f_{yy}(a,b) - f_{xy}^2(a,b) = f_{xx}(a,b)\left[-f_{xx}(a,b)\right] - 0 = -f_{xx}^2(a,b) < 0$, and so f cannot have a relative extremum at (a,b).

41. True. $h(x,y) = f(x) + g(y)$, so $h_x(x,y) = f'(x)$ and $h_y(x,y) = g'(y)$. Because a and b are critical numbers of f and g, respectively, $f'(a) = g'(b) = 0$ and (a,b) is a critical point of h. Next, $h_{xx}(x,y) = f''(x)$, $h_{xy}(x,y) = 0$, and $h_{yy}(x,y) = g''(y)$, so $D(a,b) = f''(a)\,g''(b) - 0 > 0$ and $h_{xx}(a,b) = f''(a) \neq 0$. Therefore, h has a relative extremum.

42. True. Since $f_{xx}(a,b)$ and $f_{yy}(a,b)$ have opposite signs, $f_{xx}(a,b)\,f_{yy}(a,b) < 0$, and therefore, $D(a,b) = f_{xx}(a,b)\,f_{yy}(a,b) - f_{xy}^2(a,b) < 0$.

8.4 The Method of Least Squares

Concept Questions page 604

1. a. A scatter diagram is a graph showing the data points that describe the relationship between the two variables x and y.

b. The least squares line is the straight line that best fits a set of data points when the points are scattered about a straight line.

2. See page 599 of the text.

3. a. Since an error term in the sum could be nonnegative or nonpositive, the sum of the errors could be very small, and even be zero, even if the points are widely scattered about the least-squares line.

b. Using the absolute values of the errors does eliminate the problems mentioned in part (a), but introduces another disadvantage—the absolute value function is not differentiable.

4. Dividing both sides of Equation (9) by n, we obtain $\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\,m + \frac{1}{n}(nb) = \frac{1}{n}(y_1 + y_2 + \cdots + y_n)$ or $m\overline{x} + b = \overline{y}$. This shows that the point $(\overline{x}, \overline{y})$ lies on the least-squares line $y = mx + b$.

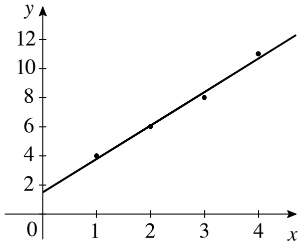
Exercises page 604

1. a. We first summarize the data.

	x	y	x^2	xy
	1	4	1	4
	2	6	4	12
	3	8	9	24
	4	11	16	44
Sum	10	29	30	84

The normal equations are $4b + 10m = 29$ and $10b + 30m = 84$. Solving this system of equations, we obtain $m = 2.3$ and $b = 1.5$, so an equation is $y = 2.3x + 1.5$.

b.

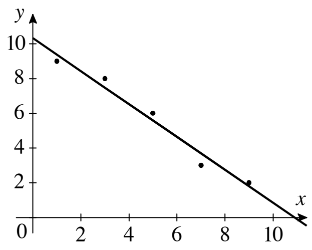


2. **a.** We first summarize the data.

	x	y	x^2	xy
	1	9	1	9
	3	8	9	24
	5	6	25	30
	7	3	49	21
	9	2	81	18
Sum	25	28	165	102

The normal equations are $165m + 25b = 102$ and $25m + 5b = 28$. Solving, we find $m = -0.95$ and $b = 10.35$, so the required equation is $y = -0.95x + 10.35$.

b.

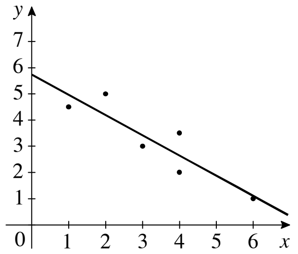


3. **a.** We first summarize the data.

	x	y	x^2	xy
	1	4.5	1	4.5
	2	5	4	10
	3	3	9	9
	4	2	16	8
	4	3.5	16	14
	6	1	36	6
Sum	20	19	82	51.5

The normal equations are $6b + 20m = 19$ and $20b + 82m = 51.5$. The solutions are $m \approx -0.7717$ and $b \approx 5.7391$, so the required equation is $y = -0.772x + 5.739$.

b.

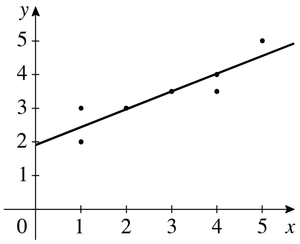


4. **a.** We first summarize the data:

	x	y	x^2	xy
	1	2	1	2
	1	3	1	3
	2	3	4	6
	3	3.5	9	10.5
	4	3.5	16	14
	4	4	16	16
	5	5	25	25
Sum	20	24	72	76.5

The normal equations are $72m + 20b = 76.5$ and $20m + 7b = 24$. Solving, we find $m = 0.53$ and $b = 1.91$. The required equation is $y = 0.53x + 1.91$.

b.

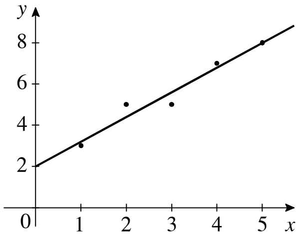


5. a. We first summarize the data:

	x	y	x^2	xy
	1	3	1	3
	2	5	4	10
	3	5	9	15
	4	7	16	28
	5	8	25	40
Sum	15	28	55	96

The normal equations are $55m + 15b = 96$ and $15m + 5b = 28$. Solving, we find $m = 1.2$ and $b = 2$, so the required equation is $y = 1.2x + 2$.

b.

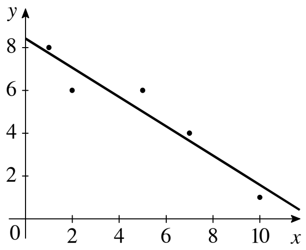


6. a. We first summarize the data:

	x	y	x^2	xy
	1	8	1	8
	2	6	4	12
	5	6	25	30
	7	4	49	28
	10	1	100	10
Sum	25	25	179	88

The normal equations are $5b + 25m = 25$ and $25b + 179m = 88$. The solutions are $m = -0.68519$ and $b = 8.4259$, so the required equation is $y = -0.685x + 8.426$.

b.

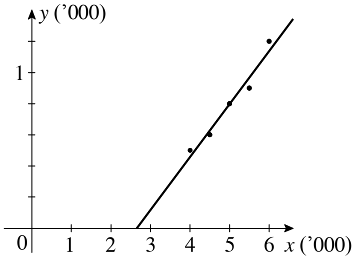


7. a. We first summarize the data:

	x	y	x^2	xy
	4	0.5	16	2
	4.5	0.6	20.25	2.7
	5	0.8	25	4
	5.5	0.9	30.25	4.95
	6	1.2	36	7.2
Sum	25	4	127.5	20.85

The normal equations are $5b + 25m = 4$ and $25b + 127.5m = 20.85$. The solutions are $m = 0.34$ and $b = -0.9$, so the required equation is $y = 0.34x - 0.9$.

b.



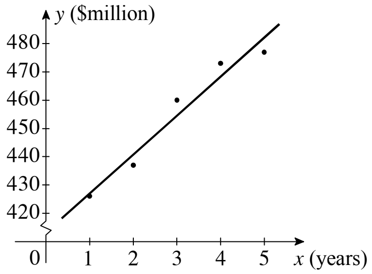
c. If $x = 6.4$, then $y = 0.34\,(6.4) - 0.9 = 1.276$, and so 1276 completed applications can be expected.

8. a. We first summarize the data:

	x	y	x^2	xy
	1	426	1	426
	2	437	4	874
	3	460	9	1380
	4	473	16	1892
	5	477	25	2385
Sum	15	2273	55	6957

The normal equations are $55m + 15b = 6957$ and $15m + 5b = 2273$. Solving, we find $m = 13.8$ and $b = 413.2$, so the required equation is $y = 13.8x + 413.2$.

b.



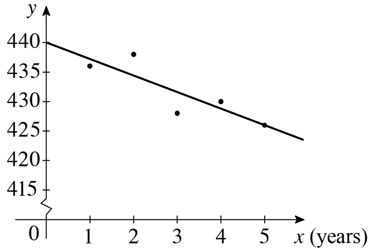
c. When $x = 6$, $y = 13.8\,(6) + 413.2 = 496$, so the predicted net sales for the upcoming year are \$496 million.

9. a. We first summarize the data:

	x	y	x^2	xy
	1	436	1	436
	2	438	4	876
	3	428	9	1284
	4	430	16	1720
	5	426	25	2138
Sum	15	2158	55	6446

The normal equations are $5b + 15m = 2158$ and $15b + 55m = 6446$. Solving this system, we find $m = -2.8$ and $b = 440$. Thus, the equation of the least-squares line is $y = -2.8x + 440$.

b.



c. Two years from now, the average SAT verbal score in that area will be $y = -2.8\,(7) + 440 = 420.4$.

10. a. We first summarize the data:

	x	y	x^2	xy
	1	2.1	1	2.1
	2	2.4	4	4.8
	3	2.7	9	8.1
Sum	6	7.2	14	15.0

The normal equations are $3b + 6m = 7.2$ and $6b + 14m = 15$. Solving the system, we find $m = 0.3$ and $b = 1.8$. Thus, the equation of the least-squares line is $y = 0.3x + 1.8$.

b. The amount of money that Hollywood is projected to spend in 2015 is approximately $0.3\,(5) + 1.8 = 3.3$, or \$3.3 billion.

11. a.

	x	y	x^2	xy
	0	154.5	0	0
	1	381.8	1	381.8
	2	654.5	4	1309
	3	845	9	2535
Sum	6	2035.8	14	4225.8

The normal equations are $4b + 6m = 2035.8$ and $6b + 14m = 4225.8$. The solutions are $m = 234.42$ and $b = 157.32$, so the required equation is $y = 234.4x + 157.3$.

b. The projected number of Facebook users is $f(7) = 234.4(7) + 157.3 = 1798.1$, or approximately 1798.1 million.

12. a.

	x	y	x^2	xy
	0	25.3	0	0
	1	33.4	1	33.4
	2	39.5	4	79
	3	50	9	150
	4	59.6	16	238.4
Sum	10	207.8	30	500.8

The normal equations are $5b + 10m = 207.8$ and $10b + 30m = 500.8$. The solutions are $m = 8.52$ and $b = 24.52$, so the required equation is $y = 8.52x + 24.52$.

b. The average rate of growth of the number of e-book readers between 2011 and 2015 is projected to be approximately 8.52 million per year.

13. a.

	x	y	x^2	xy
	1	20	1	20
	2	24	4	48
	3	26	9	78
	4	28	16	112
	5	32	25	160
Sum	15	130	55	418

The normal equations are $5b + 15m = 130$ and $15b + 55m = 418$. The solutions are $m = 2.8$ and $b = 17.6$, and so an equation of the line is $y = 2.8x + 17.6$.

b. When $x = 8$, $y = 2.8(8) + 17.6 = 40$. Hence, the state subsidy is expected to be \$40 million for the eighth year.

14. a.

	x	y	x^2	xy
	0	26.2	0	0
	1	26.8	1	26.8
	2	27.5	4	55.0
	3	28.3	9	84.9
	4	28.7	16	114.8
Sum	10	137.5	30	281.5

The normal equations are $5b + 10m = 137.5$ and $10b + 30m = 281.5$. Solving this system, we find $m = 0.65$ and $b = 26.2$. Thus, an equation of the least-squares line is $y = 0.65x + 26.2$.

b. The percentage of the population enrolled in college in 2014 is projected to be $0.65(7) + 26.2 = 30.75$, or 30.75 million.

15. **a.**

	x	y	x^2	xy
	4	1.42	16	5.68
	5	1.73	25	8.65
	6	1.98	36	11.88
	7	2.32	49	16.24
	8	2.65	64	21.2
Sum	30	10.1	190	63.65

The normal equations are $5b + 30m = 10.1$ and $30b + 190m = 63.65$. The solutions are $m \approx 0.305$ and $b \approx 0.19$, so the required equation is $y = 0.305x + 0.19$.

b. The rate of change is given by the slope of the least-squares line, that is, approximately \$0.305 billion/yr.

c. $f(10) = 0.305(10) + 0.19 = 3.24$, or \$3.24 billion

16. **a.**

	x	y	x^2	xy
	0	34.4	0	0
	1	34.1	1	34.1
	2	33.4	4	66.8
	3	33.1	9	99.3
	4	32.7	16	130.8
Sum	10	167.7	30	331.0

The normal equations are $5b + 10m = 167.7$ and $10b + 30m = 331$. Solving this system, we find $m = -0.44$ and $b = 34.42$. Thus, an equation of the least-squares line is $y = -0.44x + 34.42$.

b. The percentage of households in which someone is under 18 years old in 2013 is projected to be $-0.44(6) + 34.42 = 31.78$, or 31.78%.

17.

	x	y	x^2	xy
	0	82.0	0	0
	1	84.7	1	84.7
	2	86.8	4	173.6
	3	89.7	9	269.1
	4	91.8	16	367.2
Sum	10	435	30	894.6

The normal equations are $5b + 10m = 435$ and $10b + 30m = 894.6$. The solutions are $m = 2.46$ and $b = 82.08$, so the required equation is $y = 2.46x + 82.1$.

b. The estimated number of credit union members in 2013 is $f(5) = 2.46(5) + 82.1 = 94.4$, or approximately 94.4 million.

18. **a.**

	x	y	x^2	xy
	1	95.9	1	95.9
	2	91.7	4	183.4
	3	83.8	9	251.4
	4	78.2	16	312.8
	5	73.5	25	367.5
Sum	15	423.1	55	1211.0

The normal equations are $5b + 15m = 423.1$ and $15b + 55m = 1211$. Solving this system, we find $m \approx -5.83$ and $b \approx 102.11$. Thus, an equation of the least-squares line is $y = -5.83x + 102.11$.

b. The volume of first-class mail in 2014 is projected to be $-5.83(8) + 102.11 = 55.47$, or approximately 55.47 billion pieces.

19. a.

	x	y	x^2	xy
	0	29.4	0	0
	1	32.2	1	32.2
	2	34.8	4	69.6
	3	37.7	9	113.1
	4	40.4	16	161.6
Sum	10	174.5	30	376.5

The normal equations are $5b + 10m = 174.5$ and $10b + 30m = 376.5$. The solutions are $m = 2.75$ and $b = 29.4$, so $y = 2.75x + 29.4$.

b. The average rate of growth of the number of subscribers from 2006 through 2010 was 2.75 million per year.

20. a.

	x	y	x^2	xy
	0	2.0	0	0
	1	3.1	1	3.1
	2	4.5	4	9.0
	3	6.3	9	18.9
	4	7.8	16	31.2
	5	9.3	25	46.5
Sum	15	33.0	55	108.7

The normal equations are $6b + 15m = 33$ and $15b + 55m = 108.7$. Solving this system, we find $m \approx 1.50$ and $b \approx 1.76$, so an equation of the least-squares line is $y = 1.5x + 1.76$.

b. The rate of growth of video advertising spending between 2011 and 2016 is projected to be the slope of the least-squares line, that is \$1.5 billion/yr.

21. a.

	x	y	x^2	xy
	0	6.4	0	0
	1	6.8	1	6.8
	2	7.1	4	14.2
	3	7.4	9	22.2
	4	7.6	16	30.4
Sum	10	35.3	30	73.6

The normal equations are $5b + 10m = 35.3$ and $10b + 30m = 73.6$. The solutions are $m = 0.3$ and $b = 6.46$, so the required equation is $y = 0.3x + 6.46$.

b. The rate of change is given by the slope of the least-squares line, that is, approximately \$0.3 billion/yr.

22. a.

	x	y	x^2	xy
	0	12.9	0	0
	1	13.9	1	13.9
	2	14.65	4	29.3
	3	15.25	9	45.75
	4	15.85	16	63.4
Sum	10	72.55	30	152.35

The normal equations are $5b + 10m = 72.55$ and $10b + 30m = 152.35$. The solutions are $m \approx 0.725$ and $b \approx 13.06$, so the required equation is $y = 0.725x + 13.06$.

b. $y = 0.725\,(5) + 13.06 = 16.685$, or approximately \$16.685 million.

23. **a.** We summarize the data at right. The normal equations are $6b + 39m = 195.5$ and $39b + 271 = 1309$. The solutions are $b = 18.38$ and $m = 2.19$, so the required least-squares line is given by $y = 2.19x + 18.38$.

	x	y	x^2	xy
	4	27.5	16	110
	5	29	25	145
	6	31	36	186
	7	34	49	238
	8	36	64	288
	9	38	81	342
Sum	39	195.5	271	1309

b. The average rate of increase is given by the slope of the least-squares line, namely \$2.19 billion/yr.

c. The revenue from overdraft fees in 2011 is $y = 2.19(11) + 18.38 = 42.47$, or approximately \$42.47 billion.

24. **a.**

	x	y	x^2	xy
	0	15.9	0	0
	10	16.8	100	168
	20	17.6	400	352
	30	18.5	900	555
	40	19.3	1600	772
	50	20.3	2500	1015
Sum	150	108.4	5500	2862

The normal equations are $6b + 150m = 108.4$ and $150b + 5500m = 2862$. The solutions are $b = 15.90$ and $m = 0.09$, so $y = 0.09x + 15.9$.

b. The life expectancy at 65 of a male in 2040 is $y = 0.09(40) + 15.9 = 19.5$, or 19.5 years.

c. The life expectancy at 65 of a male in 2030 is $y = 0.09(30) + 15.9 = 18.6$, or 18.6 years.

25. **a.**

	x	y	x^2	xy
	0	60	0	0
	2	74	4	148
	4	90	16	360
	6	106	36	636
	8	118	64	944
	10	128	100	1280
	12	150	144	1800
Sum	42	726	364	5168

The normal equations are $7b + 42m = 726$ and $42b + 364m = 5168$. The solutions are $m \approx 7.25$ and $b \approx 60.21$, so the required equation is $y = 7.25x + 60.21$.

b. $y = 7.25(11) + 60.21 = 139.96$, or \$139.96 billion.

c. \$7.25 billion/yr.

26. a.

	t	y	t^2	ty
	0	1.38	0	0
	1	1.44	1	1.44
	2	1.49	4	2.98
	3	1.56	9	4.68
	4	1.61	16	6.44
	5	1.67	25	8.35
	6	1.74	36	10.44
	7	1.78	49	12.46
Sum	28	12.67	140	46.79

The normal equations are $8b + 28m = 12.67$ and $28b + 140 = 46.79$. The solutions are $m \approx 0.058$ and $b \approx 138$, so the required equation is $y = 0.058t + 138$.

b. The rate of change is given by the slope of the least-squares line, that is, approximately \$0.058 trillion/yr, or \$58 billion/yr.

c. $y = 0.058\,(10) + 1.38 = 1.96$, or \$1.96 trillion.

27. False. See Example 1 on page 600 of the text.

28. True. The error involves the sum of the squares of the form $\left[f\,(x_i) - y_i\right]^2$, where f is the least-squares function and y_i is a data point. Thus, the error is zero if and only if $f\,(x_i) = y_i$ for each $1 \le i \le n$.

29. True.

30. True.

Technology Exercises page 609

1. $y = 2.3596x + 3.8639$

2. $y = 1.4068x - 2.1241$

3. $y = -1.1948x + 3.5525$

4. $y = -2.07715x + 5.23847$

5. a. $22.3x + 143.5$ **b.** \$22.3 billion/yr **c.** \$366.5 billion

6. a. $y = 5.5x + 15.7$ **b.** 70.7%

7. a. $y = 1.5857t + 6.6857$ **b.** \$19.4 billion

8. a. $y = 46.6x + 495$ **b.** \$46.60/buyer/yr

9. a. $y = 1.751x + 7.9143$ **b.** \$22 billion

8.5 Constrained Maxima and Minima and the Method of Lagrange Multipliers

Concept Questions page 618

1. A constrained relative extremum of f is an extremum of f subject to a constraint of the form $g\,(x, y) = 0$.

2. See the procedure given on page 612 of the text.

Exercises

page 618

1. $f(x,y) = x^2 + 3y^2$. We form the Lagrangian function $F(x,y,\lambda) = x^2 + 3y^2 + \lambda(x+y-1)$ and solve the system $\begin{cases} F_x = 2x + \lambda = 0 \\ F_y = 6y + \lambda = 0 \\ F_\lambda = x + y - 1 = 0 \end{cases}$ Solving the first and the second equations for x and y in terms of λ, we obtain $x = -\frac{\lambda}{2}$ and $y = -\frac{\lambda}{6}$ which, upon substitution into the third equation, yields $-\frac{\lambda}{2} - \frac{\lambda}{6} - 1 = 0$ or $\lambda = -\frac{3}{2}$. Therefore, $x = \frac{3}{4}$ and $y = \frac{1}{4}$, which gives the point $\left(\frac{3}{4}, \frac{1}{4}\right)$ as the sole critical point of f. Thus, $\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{3}{4}$ is a minimum of f.

2. $f(x,y) = x^2 + y^2 - xy$. We form the Lagrangian function $F(x,y,\lambda) = x^2 + y^2 - xy + \lambda(x+2y-14)$ and solve the system $\begin{cases} F_x = 2x - y + \lambda = 0 \\ F_y = 2y - x + 2\lambda = 0 \\ F_\lambda = x + 2y - 14 = 0 \end{cases}$ Then $\begin{cases} 4x - 2y + 2\lambda = 0 \\ -x + 2y + 2\lambda = 0 \end{cases}$ and $\begin{cases} 5x - 4y = 0 \\ 5x + 10y = 70 \end{cases}$ Thus, $14y = 70$, so $y = 5$ and $x = 4$, and the minimum value is $f(4,5) = 21$.

3. $f(x,y) = 2x + 3y - x^2 - y^2$. We form the Lagrangian function $F(x,y,\lambda) = 2x + 3y - x^2 - y^2 + \lambda(x+2y-9)$ and solve the system $\begin{cases} F_x = 2 - 2x + \lambda = 0 \\ F_y = 3 - 2y + 2\lambda = 0 \\ F_\lambda = x + 2y - 9 = 0 \end{cases}$ Solving the first equation for λ, we obtain $\lambda = 2x - 2$. Substituting into the second equation, we have $3 - 2y + 4x - 4 = 0$, or $4x - 2y - 1 = 0$. Adding this equation to the third equation in the system, we have $5x - 10 = 0$, or $x = 2$. Therefore, $y = \frac{7}{2}$ and $f\left(2, \frac{7}{2}\right) = -\frac{7}{4}$ is the maximum value of f.

4. $f(x,y) = 16 - x^2 - y^2$. We form the Lagrangian function $F(x,y,\lambda) = 16 - x^2 - y^2 + \lambda(x+y-6)$ and solve the system $\begin{cases} F_x = -2x + \lambda = 0 \\ F_y = -2y + \lambda = 0 \\ F_\lambda = x + y - 6 = 0 \end{cases}$ The first two equations imply that $x = y$. Substituting into the third equation, we have $2x - 6 = 0$, or $x = 3$. So $y = 3$ and $f(3,3) = -2$ is the maximum value of f.

5. $f(x,y) = x^2 + 4y^2$. We form the Lagrangian function $F(x,y,\lambda) = x^2 + 4y^2 + \lambda(xy-1)$ and solve the system $\begin{cases} F_x = 2x + \lambda y = 0 \\ F_y = 8y + \lambda x = 0 \\ F_\lambda = xy - 1 = 0 \end{cases}$ Multiplying the first and second equations by x and y, respectively, and subtracting the resulting equations, we obtain $2x^2 - 8y^2 = 0$, or $x = \pm 2y$. Substituting this into the third equation gives $2y^2 - 1 = 0$ or $y = \pm\frac{\sqrt{2}}{2}$. We conclude that $f\left(-\sqrt{2}, -\frac{\sqrt{2}}{2}\right) = f\left(\sqrt{2}, \frac{\sqrt{2}}{2}\right) = 4$ is the minimum value of f.

6. $f(x,y)=xy$. We form the Lagrangian function $F(x,y,\lambda)=xy-\lambda\left(x^2+4y^2-4\right)=0$ and solve the system $\begin{cases} F_x=y-2\lambda x=0\\ F_y=x-8\lambda y=0\\ F_\lambda=x^2+4y^2-4=0\end{cases}$ Multiplying the first and second equations by $4y$ and x, respectively, and subtracting the resulting equations, we obtain $4y^2-x^2=0$ and so $x=\pm 2y$. Substituting this into the third equation, we find $4y^2+4y^2-4=0$, $8y^2=4$, or $y=\pm\frac{\sqrt{2}}{2}$. Therefore, $f\left(-\sqrt{2},\frac{\sqrt{2}}{2}\right)=f\left(\sqrt{2},-\frac{\sqrt{2}}{2}\right)=-1$ is the minimum value of f.

7. $f(x,y)=x+5y-2xy-x^2-2y^2$. We form the Lagrangian function $F(x,y,\lambda)=x+5y-2xy-x^2-2y^2+\lambda(2x+y-4)$ and solve the system $\begin{cases} F_x=1-2y-2x+2\lambda=0\\ F_y=5-2x-4y+\lambda=0\\ F_\lambda=2x+y-4=0\end{cases}$ Solving the last two equations for x and y in terms of λ, we obtain $y=\frac{1}{3}(1+\lambda)$ and $x=\frac{1}{6}(11-\lambda)$ which, upon substitution into the first equation, yields $1-\frac{2}{3}(1+\lambda)-\frac{1}{3}(11-\lambda)+2\lambda$, so $1-\frac{2}{3}-\frac{2}{3}\lambda-\frac{11}{3}+\frac{1}{3}\lambda+2\lambda=0$. Hence, $\lambda=2$, so $x=\frac{3}{2}$ and $y=1$. The maximum of f is $f\left(\frac{3}{2},1\right)=\frac{3}{2}+5-2\left(\frac{3}{2}\right)-\left(\frac{3}{2}\right)^2-2=-\frac{3}{4}$.

8. $f(x,y)=xy$. We form the Lagrangian function $F(x,y,\lambda)=xy+\lambda(2x+3y-6)$ and solve the system $\begin{cases} F_x=y+2\lambda=0\\ F_y=x+3\lambda=0\\ F_\lambda=2x+3y-6=0\end{cases}$ Then $\begin{cases} 3y+6\lambda=0\\ 2x+6\lambda=0\end{cases}$ and $2x+3y=6$. Hence, $x=\frac{3}{2}$ and $y=1$, and so the relative maximum of f is $f\left(\frac{3}{2},1\right)=\frac{3}{2}$.

9. $f(x,y)=xy^2$. We form the Lagrangian $F(x,y,\lambda)=xy^2+\lambda\left(9x^2+y^2-9\right)$ and solve the system $\begin{cases} F_x=y^2+18\lambda x=0\\ F_y=2xy+2\lambda y=0\\ F_\lambda=9x^2+y^2-9=0\end{cases}$ The first equation gives $\lambda=-\dfrac{y^2}{18x}$. Substituting into the second gives $2xy+2y\left(-\dfrac{y^2}{18x}\right)=0$, or $18x^2y-y^3=y\left(18x^2-y^2\right)=0$, giving $y=0$ or $y=\pm 3\sqrt{2}x$. If $y=0$, then the third equation gives $9x^2-9=0$, so $x=\pm 1$. Therefore, the points $(-1,0)$, $(1,0)$, $\left(-\frac{\sqrt{3}}{3},-\sqrt{6}\right)$, $\left(-\frac{\sqrt{3}}{3},\sqrt{6}\right)$, $\left(\frac{\sqrt{3}}{3},-\sqrt{6}\right)$ and $\left(\frac{\sqrt{3}}{3},\sqrt{6}\right)$ give extreme values of f subject to the given constraint. Evaluating $f(x,y)$ at each of these points, we see that $f\left(\frac{\sqrt{3}}{3},-\sqrt{6}\right)=f\left(\frac{\sqrt{3}}{3},\sqrt{6}\right)=2\sqrt{3}$ is the maximum value of f.

10. $f(x,y)=\sqrt{y^2-x^2}$. We form the Lagrangian function $F(x,y,\lambda)=\sqrt{y^2-x^2}+\lambda(x+2y-5)$ and solve the system $\begin{cases} F_x=-\dfrac{x}{\sqrt{y^2-x^2}}+\lambda=0\\ F_y=\dfrac{y}{\sqrt{y^2-x^2}}+2\lambda=0\\ F_\lambda=x+2y-5=0\end{cases}$ Then $-\dfrac{2x}{\sqrt{y^2-x^2}}+2\lambda=0$, $\dfrac{y}{\sqrt{y^2-x^2}}+2\lambda=0$, $-2x-y=0$, and $2x+4y=10$, so $y=\frac{10}{3}$ and $x=-\frac{5}{3}$. Therefore, the relative minimum of f is $f\left(-\frac{5}{3},\frac{10}{3}\right)=\frac{5\sqrt{3}}{3}$.

11. $f(x,y) = xy$. We form the Lagrangian function $F(x,y,\lambda) = xy + \lambda\left(x^2 + y^2 - 16\right)$ and solve the system

$$\begin{cases} F_x = y + 2\lambda x = 0 \\ F_y = x + 2\lambda y = 0 \\ F_\lambda = x^2 + y^2 - 16 = 0 \end{cases}$$

Solving the first equation for λ and substituting this value into the second equation yields $x - 2\left(\dfrac{y}{2x}\right)y = 0$, or $x^2 = y^2$. Substituting the last equation into the third equation in the system, yields $x^2 + x^2 - 16 = 0$, or $x^2 = 8$, that is, $x = \pm 2\sqrt{2}$. The corresponding values of y are $y = \pm 2\sqrt{2}$. Therefore the critical points of F are $\left(\pm 2\sqrt{2}, \pm 2\sqrt{2}\right)$ (all four possibilities). Evaluating f at each of these points, we find that $f\left(-2\sqrt{2}, 2\sqrt{2}\right) = f\left(2\sqrt{2}, -2\sqrt{2}\right) = -8$ are relative minima and $f\left(-2\sqrt{2}, -2\sqrt{2}\right) = f\left(2\sqrt{2}, 2\sqrt{2}\right) = 8$ are relative maxima.

12. $f(x,y) = e^{xy}$. We form the Lagrangian function $F(x,y,\lambda) = e^{xy} + \lambda\left(x^2 + y^2 - 8\right)$ and solve the system

$$\begin{cases} F_x = ye^{xy} - 2\lambda x = 0 \\ F_y = xe^{xy} - 2\lambda y = 0 \\ F_\lambda = x^2 + y^2 - 8 = 0 \end{cases}$$

Multiplying the first and second equations by x and y, respectively, and subtracting the resulting equations, we obtain $2\lambda\left(x^2 - y^2\right) = 0$. This gives $\lambda = 0$, or $y = \pm x$. But $\lambda \neq 0$; otherwise $x = y = 0$ and the third equation is not satisfied. If $y = \pm x$, then $x^2 + x^2 - 8 = 0$ and $x = \pm\, 2$. Therefore, the points $(-2,-2)$, $(-2,2)$, $(2,-2)$, and $(2,2)$ give extrema of f. The relative maximum value of f is e^4 and the relative minimum value is e^{-4}.

13. $f(x,y) = xy^2$. We form the Lagrangian function $F(x,y,\lambda) = xy^2 + \lambda\left(x^2 + y^2 - 1\right)$ and solve the system

$$\begin{cases} F_x = y^2 + 2x\lambda = 0 \\ F_y = 2xy + 2y\lambda = 0 \\ F_\lambda = x^2 + y^2 - 1 = 0 \end{cases}$$

We find that either $x = \pm\frac{\sqrt{3}}{3}$ and $y = \pm\frac{\sqrt{6}}{3}$, or $x = \pm 1$ and $y = 0$. Evaluating f at each of the critical points $\left(\pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{6}}{3}\right)$ (all four possibilities) and $(\pm 1, 0)$, we find that $f\left(-\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{6}}{3}\right) = -\frac{2\sqrt{3}}{9}$ are relative minima and $f\left(\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{6}}{3}\right) = \frac{2\sqrt{3}}{9}$ are relative maxima.

14. $f(x,y,z) = xyz$. We form the Lagrangian function $F(x,y,z,\lambda) = xyz + \lambda(2x + 2y + z - 84)$ and solve the system

$$\begin{cases} F_x = yz + 2\lambda = 0 \\ F_y = xz + 2\lambda = 0 \\ F_z = xy + \lambda = 0 \\ F_\lambda = 2x + 2y + z - 84 = 0 \end{cases}$$

From the third equation, we find that $\lambda = -xy$. The first two equations become $yz - 2xy = 0$ and $xz - 2xy = 0$, so $z(y - x) = 0$ and either $z = 0$ or $y = x$. If $z = 0$, then $\lambda = 0$ and $xy = 0$, so either $x = 42$ or $y = 42$. If $y = x$, then $x^2 = -\lambda$, $xz - 2x^2 = 0$, $x(z - 2x) = 0$, which implies that $z = 2x$ or $x = 0$. But, if $x = 0$, then $y = 42$ and $z = 0$ as above. Therefore, $z = 2x$ and $2x + 2x + 2x = 84$, so $x = \frac{84}{6} = 14 = y$ and $z = 28$. The relative minima of f are $f(0, 42, 0) = f(42, 0, 0) = 0$ and the relative maximum of f is $f(14, 14, 28) = 5488$.

15. $f(x,y,z) = x^2 + y^2 + z^2$. We form the Lagrangian function $F(x,y,z,\lambda) = x^2 + y^2 + z^2 + \lambda(3x + 2y + z - 6)$ and solve the system $\begin{cases} F_x = 2x + 3\lambda = 0 \\ F_y = 2y + 2\lambda = 0 \\ F_z = 2z + \lambda = 0 \\ F_\lambda = 3x + 2y + z - 6 = 0 \end{cases}$ The third equation gives $\lambda = -2z$. Substituting into the first two equations, we obtain $\begin{cases} 2x - 6z = 0 \\ 2y - 4z = 0 \end{cases}$ Thus, $x = 3z$ and $y = 2z$. Substituting into the fourth equation yields $9z + 4z + z - 6 = 0$, or $z = \frac{3}{7}$. Therefore, $x = \frac{9}{7}$ and $y = \frac{6}{7}$, and so $f\left(\frac{9}{7}, \frac{6}{7}, \frac{3}{7}\right) = \frac{18}{7}$ is the minimum value of f.

16. $f(x,y,z) = x + 2y - 3z$. Form the Lagrangian function $F(x,y,z,\lambda) = x + 2y - 3z + \lambda\left(4x^2 + y^2 - z\right)$ and solve the system $\begin{cases} F_x = 1 + 8\lambda x = 0 \\ F_y = 2 + 2\lambda y = 0 \\ F_z = -3 - \lambda = 0 \\ F_\lambda = 4x^2 + y^2 - z = 0 \end{cases}$ From the third equation, we find $\lambda = -3$. Substituting into the first two equations, we obtain $1 - 24x = 0$ or $x = \frac{1}{24}$ and $2 - 6y = 0$, so $y = \frac{1}{3}$. Substituting into the third equation gives $4\left(\frac{1}{24}\right)^2 + \frac{1}{9} - z = 0$, so $z = \frac{17}{144}$. Thus, $f\left(\frac{1}{24}, \frac{1}{3}, \frac{17}{144}\right) = \frac{17}{48} \approx 0.35$ is the maximum value of f.

17. $P(x,y) = -0.2x^2 - 0.25y^2 - 0.2xy + 100x + 90y - 4000$. We want to maximize P subject to the constraint $x + y = 200$. The Lagrangian function is $F(x,y,\lambda) = -0.2x^2 - 0.25y^2 - 0.2xy + 100x + 90y - 4000 + \lambda(x + y - 200)$. We solve the system $\begin{cases} F_x = -0.4x - 0.2y + 100 + \lambda = 0 \\ F_y = -0.5y - 0.2x + 90 + \lambda = 0 \\ F_\lambda = x + y - 200 = 0 \end{cases}$ Subtracting the first equation from the second yields $0.2x - 0.3y - 10 = 0$, or $2x - 3y - 100 = 0$. Multiplying the third equation in the system by 2 and subtracting the resulting equation from the last equation, we find $-5y + 300 = 0$, so $y = 60$. Thus, $x = 140$ and the company should make 140 finished and 60 unfinished units.

18. $P(x,y) = -0.005x^2 - 0.003y^2 - 0.002xy + 14x + 12y - 200$. We want to maximize the function $P(x,y)$ subject to the constraint $x + y = 400$. We form the Lagrangian function $F(x,y,\lambda) = -0.005x^2 - 0.003y^2 - 0.002xy + 14x + 12y - 200 + \lambda(x + y - 400)$. To find the critical points of F, we solve the system $\begin{cases} F_x = -0.01x - 0.002y + 14 + \lambda = 0 \\ F_y = -0.006y - 0.002x + 12 + \lambda = 0 \\ F_\lambda = x + y - 400 = 0 \end{cases}$ Solving the first equation for λ, we obtain $\lambda = 0.01x + 0.02y - 14$ which, upon substitution into the second equation, yields $-0.006y - 0.002x + 12 + 0.01x + 0.002y - 14 = 0$, $0.008x - 0.004y - 2 = 0$, and so $y = 2x - 500$. Substituting this value of y into the third equation in the system gives $x + 2x - 500 - 400 = 0$, so $x = 300$ and $y = 100$. Thus, the company should publish 100 deluxe and 300 standard editions.

19. Suppose each of the sides made of pine board is x feet long and those of steel are y feet long. Then $xy = 800$. The cost is $C = 12x + 3y$ and is to be minimized subject to the condition $xy = 800$. We form the Lagrangian function $F(x, y, \lambda) = 12x + 3y + \lambda(xy - 800)$ and solve the system $\begin{cases} F_x = 12 + \lambda y = 0 \\ F_y = 3 + \lambda x = 0 \\ F_\lambda = xy - 800 = 0 \end{cases}$ Multiplying the first equation by x and the second equation by y and subtracting the resulting equations, we obtain $12x - 3y = 0$, or $y = 4x$. Substituting this into the third equation of the system, we obtain $4x^2 - 800 = 0$, so $x = \pm 10\sqrt{2}$. Because x must be positive, we take $x = 10\sqrt{2}$, so $y = 40\sqrt{2}$ and the dimensions are approximately 14.14 ft by 56.56 ft.

20. Let the dimensions of the box (in feet) be $x \times y \times z$. We want to maximize $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = xy + 2xz + 2yz - 48 = 0$. The Lagrangian function is $F(x, y, z, \lambda) = xyz + \lambda(xy + 2xz + 2yz - 48)$. To find the critical points of F, we solve the system $\begin{cases} F_x = yz + \lambda y + 2\lambda z = 0 & (1) \\ F_y = xz + \lambda x + 2\lambda z = 0 & (2) \\ F_z = xy + 2\lambda x + 2\lambda y = 0 & (3) \\ F_\lambda = xy + 2xz + 2yz - 48 = 0 & (4) \end{cases}$ Multiplying (1) by x and (2) by y, we obtain $\begin{cases} xyz + \lambda xy + 2\lambda xz = 0 & (5) \\ xyz + \lambda xy + 2\lambda yz = 0 & (6) \end{cases}$ Subtracting (5) from (6), we have $2\lambda z(y - x) = 0$. Since λ and z cannot be zero, we see that $y = x$. Next, multiplying (3) by z, we obtain $xyz + 2\lambda xz + 2\lambda yz = 0$ (7). Subtracting (6) from (7) gives $\lambda x(2z - y) = 0$, so $z = \frac{1}{2}y$. Substituting $x = y$ and $z = \frac{1}{2}y$ into (4) gives $y^2 + 2y\left(\frac{1}{2}y\right) + 2y\left(\frac{1}{2}y\right) - 48 = 0$, so $3y^2 = 48$ and $y = 4$. Thus, $x = 4$ and $z = 2$. The dimensions of the required box are therefore $4' \times 4' \times 2'$.

21. Let the dimensions of the box (in feet) be $x \times y \times z$. We want to maximize $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = xyz + 2xz + 2yz - 12 = 0$. The Lagrangian function is $F(x, y, z, \lambda) = xyz + \lambda(xy + 2xz + 2yz - 48)$. To find the critical points of F, we solve the system $\begin{cases} F_x = yz + \lambda y + 2\lambda z = 0 & (1) \\ F_y = xz + \lambda x + 2\lambda z = 0 & (2) \\ F_z = xy + 2\lambda x + 2\lambda y = 0 & (3) \\ F_\lambda = xy + 2xz + 2yz - 12 = 0 & (4) \end{cases}$ Multiplying (1) by x and (2) by y, we obtain $\begin{cases} xyz + \lambda xy + 2\lambda xz = 0 & (5) \\ xyz + \lambda xy + 2\lambda yz = 0 & (6) \end{cases}$ Subtracting (5) from (6), we have $2\lambda z(y - x) = 0$. Since λ and z cannot be zero, we see that $y = x$. Next, multiplying (3) by z, we obtain $xyz + 2\lambda xz + 2\lambda yz = 0$ (7). Subtracting (6) from (7) gives $\lambda x(2z - y) = 0$, so $z = \frac{1}{2}y$. Substituting $x = y$ and $z = \frac{1}{2}y$ into (4) gives $y^2 + 2y\left(\frac{1}{2}y\right) + 2y\left(\frac{1}{2}y\right) - 12 = 0$, so $3y^2 = 12$ and $y = 2$. Thus $x = 2$ and $z = 1$. The dimensions of the required box are therefore $2' \times 2' \times 1'$.

22. We want to maximize the profit function $P(x,y) = 4x + 2y$ subject to the constraint $g(x,y) = 2x^2 + y - 3 = 0$. The Lagrangian function is $F(x,y,\lambda) = P(x,y) + \lambda g(xy) = 4x + 2y + \lambda\left(2x^2 + y - 3\right)$. To find the critical point of F, we solve the system $\begin{cases} F_x = 4 + 4\lambda x = 0 \\ F_y = 2 + \lambda = 0 \\ F_\lambda = 2x^2 + y - 3 = 0 \end{cases}$ Solving the second equation yields $\lambda = -2$. Substituting this value into the first equation, we obtain $x = -\frac{4}{4(-2)} = \frac{1}{2}$. Substituting this value of x into the third equation in the system, we have $y = 3 - 2\left(\frac{1}{2}\right)^2 = \frac{5}{2}$. Thus, the company should produce 500 type A and 2500 type B souvenirs per week.

23. Let the dimensions of the box (in feet) be $x \times y \times z$. We want to maximize $f(x,y,z) = xyz$ subject to the constraint $g(x,y,z) = 2x + y + 2z - 108 = 0$. The Lagrangian function is $F(x,y,\lambda) = xyz + \lambda(2x + y + 2z - 108)$. To find the critical points of F, we solve the system $\begin{cases} F_x = yz + 2\lambda = 0 & (1) \\ F_y = xz + \lambda = 0 & (2) \\ F_z = xy + 2\lambda = 0 & (3) \\ F_\lambda = 2x + y + 2z - 108 = 0 & (4) \end{cases}$ Multiplying (1) by x and (2) by y, we obtain $\begin{cases} xyz + 2\lambda x = 0 & (5) \\ xyz + \lambda y = 0 & (6) \end{cases}$ Subtracting (5) from (6), we have $\lambda(y - 2x) = 0$. Since $\lambda \neq 0$, we see that $y = 2x$. Next, multiplying (3) by z, we obtain $xyz + 2\lambda z = 0$ (7). Subtracting (5) from (7) gives $2\lambda(z - x) = 0$, so $z = x$, and substituting $y = 2x$ and $z = x$ into (4) gives $2x + 2x + 2x = 108$, so $6x = 108$, giving $x = 18$. Thus, $y = 2(18) = 36$ and $z = 18$, and so the required dimensions are $18'' \times 18'' \times 36''$.

24. $V = \pi r^2 \ell$. The constraint is $2\pi r + \ell = 130$, so $g(r,\ell) = 2\pi r + \ell - 130$. The Lagrangian function is $F(r,\ell,\lambda) = \pi r^2 \ell + \lambda(2\pi + \ell - 130)$, so we solve the system $\begin{cases} F_r = 2\pi r\ell + 2\pi\lambda = 0 \\ F_\ell = \pi r^2 + \lambda = 0 \\ F_\lambda = 2\pi r + \ell - 130 = 0 \end{cases}$

The second equation gives $\lambda = -\pi r^2$. Substituting into the first equation gives $2\pi r\ell + 2\pi\left(-\pi r^2\right) = 0$, so $2\pi r(\ell - \pi r) = 0$. Because $r \neq 0$, we have $\ell = \pi r$, which we substitute into the third equation to obtain $2\pi r + \pi r - 130 = 0$, $3\pi r = 130$, and so $r = \frac{130}{3\pi}$. Therefore, $\ell = \pi\left(\frac{130}{3\pi}\right) = \frac{130}{3}$, or $43\frac{1}{3}''$. The volume is $\pi r^2 \ell = \pi\left(\frac{130}{3\pi}\right)^2\left(\frac{130}{3}\right) = \frac{2{,}197{,}000}{27\pi}$, or $\frac{2{,}197{,}000}{27\pi}$ in^3.

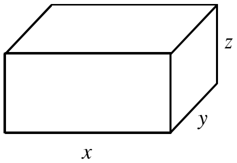
25. We want to minimize the function $C(r,h) = 8\pi rh + 6\pi r^2$ subject to the constraint $\pi r^2 h - 64 = 0$. We form the Lagrangian function $F(r,h,\lambda) = 8\pi rh + 6\pi r^2 - \lambda\left(\pi r^2 h - 64\right)$ and solve the system

$$\begin{cases} F_r = 8\pi h + 12\pi r - 2\lambda\pi rh = 0 \\ F_h = 8\pi r - \lambda\pi r^2 = 0 \\ F_\lambda = \pi r^2 h - 64 = 0 \end{cases}$$

Solving the second equation for λ yields $\lambda = 8/r$, which when substituted into the first equation yields $8\pi h + 12\pi r - 2\pi rh\left(\dfrac{8}{r}\right) = 0$, $12\pi r = 8\pi h$, and $h = \frac{3}{2}r$. Substituting this value of h into the third equation of the system, we find $3r^2\left(\frac{3}{2}r\right) = 64$, $r^3 = \frac{128}{3\pi}$, so $r = \frac{4}{3}\sqrt[3]{\frac{18}{\pi}}$ and $h = 2\sqrt[3]{\frac{18}{\pi}}$.

26. We form the Lagrangian function $F(x,y,\lambda) = xyz + \lambda(3xy + 2xz + 2yz - 36)$ and solve the system

$$\begin{cases} F_x = yz + 3\lambda y + 2\lambda z = 0 \\ F_y = xz + 3\lambda x + 2\lambda z = 0 \\ F_z = xy + 2\lambda x + 2\lambda y = 0 \\ F_\lambda = 3xy + 2xz + 2yz - 36 = 0 \end{cases}$$



Multiplying the first, second, and third equations by x, y, and z respectively, we obtain

$$\begin{cases} xyz + 3\lambda xy + 2\lambda xz = 0 \\ xyz + 3\lambda xy + 2\lambda yz = 0 \\ xyz + 2\lambda xz + 2\lambda yz = 0 \end{cases}$$

Subtracting the second equation from the first and the third equation from the second yields $\begin{cases} 2\lambda(x-y)z = 0 \\ \lambda x(3y - 2z) = 0 \end{cases}$

Solving this system, we find that $x = y$ and $x = \frac{3}{2}y$. Substituting these values into the third equation, we find that $3y^2 + 2y\left(\frac{3}{2}\right)y + 2y\left(\frac{3}{2}\right) - 36 = 0$, and so $y = \pm 2$. We reject the negative root find that $x = 2$, $y = 2$, and $z = 3$ provides the desired relative maximum. Thus, the dimensions are $2' \times 2' \times 3'$.

27. Let the box have dimensions x by y by z feet. Then $xyz = 4$. We want to minimize $C = 2xz + 2yz + \frac{3}{2}(2xy) = 2xz + 2yz + 3xy$. We form the Lagrangian function $F(x,y,z,\lambda) = 2xz + 2yz + 3xy + \lambda(xyz - 4)$ and solve the system

$$\begin{cases} F_x = 2z + 3y + \lambda yz = 0 \\ F_y = 2z + 3x + \lambda xz = 0 \\ F_z = 2x + 2y + \lambda xy = 0 \\ F_\lambda = xyz - 4 = 0 \end{cases}$$

Multiplying the first, second, and third equations by x, y, and z respectively, we have

$$\begin{cases} 2xz + 3xy + \lambda xyz = 0 \\ 2yz + 3xy + \lambda xyz = 0 \\ 2xz + 2yz + \lambda xyz = 0 \end{cases}$$

The first two equations imply that $2z(x-y) = 0$. Because $z \neq 0$, we see that $x = y$. The second and third equations imply that $x(3y - 2z) = 0$ or $x = \frac{3}{2}y$. Substituting these values into the fourth equation in the system, we find $y^2\left(\frac{3}{2}y\right) = 4$, so $y^3 = \frac{8}{3}$. Therefore, $y = \dfrac{2}{3^{1/3}} = \frac{2}{3}\sqrt[3]{9}$, $x = \frac{2}{3}\sqrt[3]{9}$, and $z = \sqrt[3]{9}$, and the dimensions (in feet) are $\frac{2}{3}\sqrt[3]{9} \times \frac{2}{3}\sqrt[3]{9} \times \sqrt[3]{9}$.

28. Let x, y, and z denote the length, width, and height of the box. We can assume without loss of generality that the cost of the material for constructing the sides is $\$1/\text{ft}^2$. Then the total cost is $C = f(x, y, z) = 3xy + 2xz + 2yz$. We want to minimize f subject to the constraint $g(x, y, z) = xyz - 12 = 0$. We form the Lagrangian

$F(x, y, z, \lambda) = 3xy + 2xz + 2yz - \lambda(xyz - 12)$ and solve the system $\begin{cases} F_x = 3y + 2z - \lambda yz = 0 \\ F_y = 3x + 2z - \lambda xz = 0 \\ F_z = 2x + 2y - \lambda xy = 0 \\ F_\lambda = xyz - 12 = 0 \end{cases}$

From the first and second equations, we find $\lambda = \dfrac{3y + 2z}{yz} = \dfrac{3x + 2z}{xz} \Rightarrow 3xyz + 2xz^2 = 3xyz + 2yz^2 \Rightarrow x = y$. From the second and third equations, we have $\lambda = \dfrac{3x + 2z}{xz} = \dfrac{2x + 2y}{xy} \Rightarrow 3x^2y + 2xyz = 2x^2z + 2xyz \Rightarrow$ $z = \frac{3}{2}y$. Substituting into the fourth equation, we have $y(y)\left(\frac{3}{2}y\right) = 12 \Rightarrow y^3 = 8 \Rightarrow y = 2$. Thus, $x = 2$ and $z = 3$. The dimensions of the box are $2' \times 2' \times 3'$.

29. Let x, y, and z denote the length, width, and height of the box. We can assume without loss of generality that the cost of the material for constructing the sides and top is $\$1/\text{ft}^2$. Then the total cost is $C = f(x, y, z) = 3xy + 2xz + 2yz$. We want to minimize f subject to the constraint $g(x, y, z) = xyz - 16 = 0$. We form the Lagrangian

$F(x, y, z, \lambda) = 3xy + 2xz + 2yz - \lambda(xyz - 16)$ and solve the system $\begin{cases} F_x = 3y + 2z - \lambda yz = 0 \\ F_y = 3x + 2z - \lambda xz = 0 \\ F_z = 2x + 3y - \lambda xy = 0 \\ F_\lambda = xyz - 16 = 0 \end{cases}$

From the first and second equations, we find $\lambda = \dfrac{3y + 2z}{yz} = \dfrac{3x + 2z}{xz} \Rightarrow 3xyz + 2xz^2 = 3xyz + 2yz^2 \Rightarrow x = y$. From the second and third equations, we have $\lambda = \dfrac{3x + 2z}{xz} = \dfrac{2x + 2y}{xy} \Rightarrow 3x^2y + 2xyz = 2x^2z + 2xyz \Rightarrow$ $z = \frac{3}{2}y$. Substituting into the fourth equation, we have $y(y)\left(\frac{3}{2}y\right) = 16 \Rightarrow y^3 = \frac{32}{3} \Rightarrow y = \frac{2}{3}\sqrt[3]{36}$. Thus, $x = \frac{2}{3}\sqrt[3]{36}$ and $z = \sqrt[3]{36}$. The dimensions of the box are $\frac{2}{3}\sqrt[3]{36}' \times \frac{2}{3}\sqrt[3]{36}' \times \sqrt[3]{36}'$.

30. We want to maximize $f(x, y) = 90x^{1/4}y^{3/4}$ subject to $x + y = 60{,}000$. We form the Lagrangian function $F(x, y, \lambda) = 90x^{1/4}y^{3/4} + \lambda(x + y - 60{,}000)$ and solve the system

$\begin{cases} F_x = \frac{45}{2}x^{-3/4}y^{3/4} + \lambda = 0 \\ F_y = \frac{135}{2}x^{1/4}y^{-1/4} + \lambda = 0 \\ F_\lambda = x + y - 60{,}000 = 0 \end{cases}$ Eliminating λ in the first two equations gives $\dfrac{45}{2}\left(\dfrac{y}{x}\right)^{3/4} - \dfrac{135}{2}\left(\dfrac{x}{y}\right)^{1/4} = 0$,

so $\dfrac{y}{x} - 3 = 0$, and $y = 3x$. Substituting this value into the third equation in the system, we find $x + 3x = 60{,}000$, so $x = 15{,}000$, and $y = 45{,}000$. Thus, the company should spend \$15,000 on newspaper advertisements and \$45,000 on television advertisements.

31. We want to maximize $P(x, y)$ subject to $g(x, y) = px + qy - C = 0$. First, we form the Lagrangian $F(x, y) = P(x, y) + \lambda(px + qy - C)$. Next, we solve the system $F_x = P_x + \lambda p = 0$, $F_y = Py + \lambda q = 0$, $F_\lambda = px + qy - C = 0$. From the first two equations, we see that $\lambda = -\dfrac{P_x}{p} = -\dfrac{P_y}{q}$, so if (x^*, y^*) gives rise to a relative maximum value of P subject to the constraint $g(x, y) = 0$, then $\dfrac{P_x(x^*, y^*)}{p} = \dfrac{P_y(x^*, y^*)}{q}$ or $\dfrac{P_x(x^*, y^*)}{P_y(x^*, y^*)} = \dfrac{p}{q}$.

32. Using the result of Exercise 31, we have $\dfrac{f_x(x^*, y^*)}{f_y(x^*, y^*)} = \dfrac{p}{q}$. But $f_x(x, y) = abx^{b-1}y^{1-b}$ and $f_y(x, y) = a(1-b)x^b y^{-b}$, so $\dfrac{f_x(x, y)}{f_y(x, y)} = \dfrac{abx^{b-1}y^{1-b}}{a(1-b)x^b y^{-b}} = \dfrac{bx^{-1}y}{1-b} = \dfrac{by}{(1-b)x}$. At the maximum level of production, $\dfrac{by^*}{(1-b)x^*} = \dfrac{p}{q}$ or $y^* = \dfrac{(1-b)px^*}{bq}$. Substituting this into the equation $px^* + qy^* = C$ gives $px^* + \dfrac{(1-b)pq}{bq}x^* = C$, whence $\left[p + \dfrac{(1-b)p}{b}\right]x^* = C$, so $x^* = \dfrac{bC}{p}$ and thus $y^* = \dfrac{(1-b)p}{bq} \cdot \dfrac{bC}{p} = \dfrac{(1-b)C}{q}$. Thus, the amounts to be spent on labor and capital are $\dfrac{bC}{p} \cdot p = bC$ and $\dfrac{(1-b)C}{q} \cdot q = (1-b)C$, respectively.

33. We want to maximize $f(x, y) = 100x^{3/4}y^{1/4}$ subject to $100x + 200y = 200{,}000$. We form the Lagrangian function $F(x, y, \lambda) = 100x^{3/4}y^{1/4} + \lambda(100x + 200y - 200{,}000)$ and solve the system

$$\begin{cases} F_x = 75x^{-1/4}y^{1/4} + 100\lambda = 0 \\ F_y = 25x^{3/4}y^{-3/4} + 200\lambda = 0 \\ F_\lambda = 100x + 200y - 200{,}000 = 0 \end{cases}$$

The first two equations imply that $150x^{-1/4}y^{1/4} - 25x^{3/4}y^{-3/4} = 0$ or, upon multiplying by $x^{1/4}y^{3/4}$, $150y - 25x = 0$, which implies that $x = 6y$. Substituting this value of x into the third equation of the system, we have $600y + 200y - 200{,}000 = 0$, giving $y = 250$, and therefore $x = 1500$. So to maximize production, he should buy 1500 units of labor and 250 units of capital.

34. We want to minimize $C = 2xy + 8xz + 6yz$ subject to $xyz = 12{,}000$. We form the Lagrangian function $F(x, y, z, \lambda) = 2xy + 8xz + 6yz + \lambda(xyz - 12{,}000)$ and solve the system

$$\begin{cases} F_x = 2y + 8z + \lambda yz = 0 \\ F_y = 2x + 6z + \lambda xz = 0 \\ F_z = 8x + 6y + \lambda xy = 0 \\ F_\lambda = xyz - 12{,}000 = 0 \end{cases}$$

Multiplying the first, second, and third equations by x, y, and z, we obtain

$$\begin{cases} 2xy + 8yz + \lambda xyz = 0 \\ 2xy + 6yz + \lambda xyz = 0 \\ 8xz + 6yz + \lambda xyz = 0 \end{cases}$$

The first two equations imply that $z(8x - 6y) = 0$ so, because $z \neq 0$, we have $x = \frac{3}{4}y$. The second and third equations imply that $x(8z - 2y) = 0$, so $x = \frac{1}{4}y$. Substituting these values into the third equation of the system, we have $\left(\frac{3}{4}y\right)(y)\left(\frac{1}{4}y\right) = 12{,}000$, so $y^3 = 64{,}000$ and $y = 40$. Therefore, $x = 30$ and $z = 10$. The heating cost is thus $C = 2(30)(40) + 8(30)(10) + 6(40)(10) = 7200$, or \$7200, as obtained earlier.

35. We use the result of Exercise 33 with $P(x,y) = f(x,y) = 100x^{3/4}y^{3/4}$, $P = 100$, $q = 200$, and $C = 200{,}000$. Here $P_x(x,y) = f_x(x,y) = 100\left(\frac{3}{4}x^{-1/4}y^{1/4}\right) = 75\left(\frac{y}{x}\right)^{1/4}$ and $P_y(x,y) = f_y(x,y) = 100\left(\frac{1}{4}x^{3/4}y^{-3/4}\right) = 25\left(\frac{x}{y}\right)^{3/4}$. Thus, $\dfrac{P_x(x,y)}{P_y(x,y)} = \dfrac{p}{q}$ gives $\dfrac{75(y/x)^{1/4}}{25(x/y)^{3/4}} = \dfrac{100}{200}$, $\dfrac{3y^{1/4}y^{3/4}}{x^{1/4}x^{3/4}} = \dfrac{1}{2}$, $\dfrac{y}{x} = \dfrac{1}{6}$, and $y = \frac{1}{6}x$. Substituting this into the constraint equation $100x + 200y = 200{,}000$ yields $100x + \frac{200}{6}x = 200{,}000$, $600x + 200x = 12{,}000{,}000$, and $x = 1500$, and so $y = \frac{1500}{6} = 250$. Therefore, 1500 units should be expended on labor and 250 units on capital, as obtained earlier.

36. We want to minimize $C(x,y) = px + qy$ subject to $P(x,y) = k$. The Lagrangian is $F(x,y) = px + qy + \lambda(P(x,y) - k)$. We solve the system $\begin{cases} F_x(x,y) = p + \lambda P_x(x,y) = 0 \\ F_y(x,y) = q + \lambda P_y(x,y) = 0 \\ F_\lambda(x,y) = P(x,y) - k = 0 \end{cases}$

From the first two equations, we find $\lambda = -\dfrac{p}{P_x(x,y)} = -\dfrac{q}{P_y(x,y)}$, so if (x^*, y^*) gives a relative minimum value of C subject to $F_\lambda(x,y) = P(x,y) - k = 0$, then we must have $-\dfrac{p}{P_x(x^*,y^*)} = -\dfrac{q}{P_y(x^*,y^*)}$, and so $\dfrac{P_x(x^*,y^*)}{P_y(x^*,y^*)} = \dfrac{p}{q}$.

37. We use the result of Exercise 36 with $P(x,y) = f(x,y) = ax^by^{1-b}$. Here $P_x(x,y) = abx^{b-1}y^{1-b}$ and $P_y(x,y) = a(1-b)x^by^{-b}$, so $\dfrac{P_x(x,y)}{P_y(x,y)} = \dfrac{abx^{b-1}y^{1-b}}{a(1-b)x^by^{-b}} = \dfrac{bx^{-1}y}{1-b} = \dfrac{by}{(1-b)x}$. At the production level with minimum cost, we have $\dfrac{by}{(1-b)x} = \dfrac{p}{q}$, so $y = \dfrac{(1-b)px}{bq}$. Substituting this into the equation $ax^by^{1-b} = k$, we obtain $ax^b\left[\dfrac{(1-b)px}{bq}\right]^{1-b} = k$, whence $ax^b\left[\dfrac{(1-b)p}{bq}\right]^{1-b}x^{1-b} = k$, $ax\left[\dfrac{(1-b)p}{bq}\right]^{1-b} = k$, and $x = \dfrac{k}{a}\left[\dfrac{bq}{(1-b)p}\right]^{1-b}$, so $y = \dfrac{(1-b)p}{bq}\cdot\dfrac{k}{a}\left[\dfrac{bq}{(1-b)p}\right]^{1-b} = \dfrac{k}{a}\left[\dfrac{bq}{(1-b)p}\right]^{-b} = \dfrac{k}{a}\left[\dfrac{(1-b)p}{bq}\right]^{b}$. Thus, $x^*p = \dfrac{kp}{a}\left[\dfrac{bq}{(1-b)p}\right]^{1-b}$ should be spent on labor and $y^*q = \dfrac{kq}{a}\left[\dfrac{1-b}{bq}p\right]^{b}$ should be spent on capital.

38. We use the result of Exercise 37 with $a = 100$, $b = \frac{3}{4}$, $p = 100$, $q = 200$, and $k = 2000$, and find that $x^* = \dfrac{2000}{100}\left[\dfrac{\frac{3}{4}(200)}{\left(1-\frac{3}{4}\right)(100)}\right]^{1-(3/4)} = 20\left(\dfrac{150}{25}\right)^{1/4} = 20(6)^{1/4}$ and $y^* = \dfrac{2000}{100}\left(\dfrac{25}{150}\right)^{3/4} = 20\left(\frac{1}{6}\right)^{3/4}$. Thus, he should spend $100(20)(6)^{1/4} \approx 3130$ (dollars) on labor and $200(20)\left(\frac{1}{6}\right)^{3/4} \approx 1043$ (dollars) on capital.

39. False. See Example 1.

40. False. See Example 1.

41. True. We form the Lagrangian function $F(x,y,\lambda) = f(x,y) + \lambda g(x,y)$. Then $F_x = 0$, $F_y = 0$, and $F_\lambda = 0$ at (a,b) and $f_x(a,b) + \lambda g_x(a,b) = 0$, so $f_x(a,b) = -\lambda g_x(a,b)$, and $f_y(a,b) + \lambda(a,b) = 0$, so $f_y(a,b) = -\lambda g_y(a,b)$ and $g(a,b) = 0$.

42. True.

8.6 Double Integrals

Concept Questions page 630

1. An iterated integral is a single integral such as $\int_a^b f(x, y)\,dx$, where we think of y as a constant. It is evaluated as follows: $\int_R \int f(x, y)\,dA = \int_c^d \left[\int_a^b f(x, y)\,dx\right] dy$.

2. $\int_R \int f(x, y)\;dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\right] dx$.

3. $\int_R \int f(x, y)\,dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\right] dy$.

4. It gives the volume of the solid region bounded above by the graph of f and below by the region R.

5. The average value is $\dfrac{\int_R \int f(x, y)\,dA}{\int_R \int dA}$.

Exercises page 631

1. $\int_1^2 \int_0^1 (y + 2x)\,dy\,dx = \int_1^2 \left(\frac{1}{2}y^2 + 2xy\right)\Big|_{y=0}^{y=1} dx = \int_1^2 \left(\frac{1}{2} + 2x\right) dx = \left(\frac{1}{2}x + x^2\right)\Big|_1^2 = 5 - \frac{3}{2} = \frac{7}{2}$.

2. $\int_0^2 \int_{-1}^2 (x + 2y)\,dx\,dy = \int_0^2 \left(\frac{1}{2}x^2 + 2xy\right)\Big|_{x=-1}^{x=2} dy = \int_0^2 \left[(2 + 4y) - \left(\frac{1}{2} - 2y\right)\right] dy = \int_0^2 \left(\frac{3}{2} + 6y\right) dy$

$= \left(\frac{3}{2}y + 3y^2\right)\Big|_0^2 = 3 + 12 = 15.$

3. $\int_{-1}^1 \int_0^1 xy^2\,dy\,dx = \int_{-1}^1 \frac{1}{3}xy^3\Big|_{y=0}^{y=1} dx = \int_{-1}^1 \frac{1}{3}x\,dx = \frac{1}{6}x^2\Big|_{-1}^1 = \frac{1}{6} - \left(\frac{1}{6}\right) = 0.$

4. $\int_0^1 \int_0^2 \left(12xy^2 + 8y^3\right) dy\,dx = \int_0^1 \left(4xy^3 + 2y^4\right)\Big|_{y=0}^{y=2} dx = \int_0^1 (32x + 32)\,dx = \left(16x^2 + 32x\right)\Big|_0^1 = 48.$

5. $\displaystyle\int_{-1}^2 \int_1^{e^3} \frac{x}{y}\,dy\,dx = \int_{-1}^2 x \ln y\Big|_{y=1}^{y=e^3} dx = \int_{-1}^2 x \ln e^3\,dx = \int_{-1}^2 3x\,dx = \frac{3}{2}x^2\Big|_{-1}^2 = \frac{3}{2}(4) - \frac{3}{2}(1) = \frac{9}{2}.$

6. $\displaystyle\int_0^1 \int_{-2}^2 \frac{xy}{1 + y^2}\,dx\,dy = \int_0^1 \left[\frac{1}{2}\left(\frac{y}{1 + y^2}\right) x^2\right]_{x=-2}^{x=2} dy = \int_0^1 0\,dy = 0.$

7. $\int_{-2}^0 \int_0^1 4xe^{2x^2+y}\,dx\,dy = \int_{-2}^0 e^{2x^2+y}\Big|_{x=0}^{x=1} dy = \int_{-2}^0 \left(e^{2+y} - e^y\right) dy = \left(e^{2+y} - e^y\right)\Big|_{-2}^0 = \left(e^2 - 1\right) - \left(e^0 - e^{-2}\right)$

$= e^2 - 2 + e^{-2} = \left(e^2 - 1\right)\left(1 - e^{-2}\right).$

8. $\displaystyle\int_0^1 \int_1^2 \frac{y}{x^2} e^{y/x}\,dx\,dy = \int_0^1 \left(-e^{y/x}\right)\Big|_{x=1}^{x=2} dy = \int_0^1 \left(-e^{y/2} + e^y\right) dy = \left(-2e^{y/2} + e^y\right)\Big|_0^1$

$= \left(-2e^{1/2} + e\right) - (-2 + 1) = -2e^{1/2} + e + 1.$

9. $\int_0^1 \int_1^e \ln y\,dy\,dx = \int_0^1 (y \ln y - y)\Big|_{y=1}^{y=e} dx = \int_0^1 dx = 1.$

10. $\displaystyle\int_1^e \int_1^{e^2} \frac{\ln y}{x}\,dx\,dy = \int_1^e (\ln y)(\ln x)\big|_{x=1}^{x=e^2}\,dy = \int_1^e 2\ln y\,dy = 2\,(y\ln y - y)\big|_1^e = 2\,[(e-e)-(-1)] = 2.$

11. $\int_0^1 \int_0^x (x+2y)\,dy\,dx = \int_0^1 \left(xy+y^2\right)\big|_{y=0}^{y=x}\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}.$

12. $\int_0^1 \int_0^x xy\,dy\,dx = \int_0^1 \frac{1}{2}xy^2\Big|_{y=0}^{y=x}\,dx = \int_0^1 \frac{1}{2}x^3\,dx = \frac{1}{8}x^4\Big|_0^1 = \frac{1}{8}.$

13. $\int_1^3 \int_0^{x+1} (2x+4y)\,dy\,dx = \int_1^3 \left(2xy+2y^2\right)\big|_{y=0}^{y=x+1}\,dx = \int_1^3 \left[2x\,(x+1)+2\,(x+1)^2\right]dx = \int_1^3 \left(4x^2+6x+2\right)dx$

$= \left(\frac{4}{3}x^3+3x^2+2x\right)\Big|_1^3 = (36+27+6) - \left(\frac{4}{3}+3+2\right) = \frac{188}{3}.$

14. $\int_0^2 \int_{-1}^{1-y} (2-y)\,dx\,dy = \int_0^2 (2x-yx)\big|_{x=-1}^{x=1-y}\,dy = \int_0^2 \left[2\,(1-y) - y\,(1-y) - (-2+y)\right]dy$

$= \int_0^2 \left(4-4y+y^2\right)dy = \left(4y-2y^2+\frac{1}{3}y^3\right)\Big|_0^2 = 8-2\,(4)+\frac{1}{3}\,(8) = \frac{8}{3}.$

15. $\int_0^4 \int_0^{\sqrt{y}} (x+y)\,dx\,dy = \int_0^4 \left(\frac{1}{2}x^2+xy\right)\Big|_{x=0}^{x=\sqrt{y}}\,dy = \int_0^4 \left(\frac{1}{2}y+y^{3/2}\right)dy = \left(\frac{1}{4}y^2+\frac{2}{5}y^{5/2}\right)\Big|_0^4 = 4+\frac{64}{5} = \frac{84}{5}.$

16. $\int_0^1 \int_{x^3}^{x^2} x^2y^2\,dy\,dx = \int_0^1 \frac{1}{3}x^2y^3\Big|_{y=x^3}^{y=x^2}\,dx = \int_0^1 \frac{1}{3}\left(x^8-x^{11}\right)dx = \frac{1}{3}\left(\frac{1}{9}x^9-\frac{1}{12}x^{12}\right)\Big|_0^1 = \frac{1}{3}\left(\frac{1}{9}-\frac{1}{12}\right) = \frac{1}{108}.$

17. $\int_0^2 \int_0^{\sqrt{4-y^2}} y\,dx\,dy = \int_0^2 xy\big|_{x=0}^{x=\sqrt{4-y^2}}\,dy = \int_0^2 y\sqrt{4-y^2}\,dy = -\frac{1}{2}\left(\frac{2}{3}\right)\left(4-y^2\right)^{3/2}\Big|_0^2 = \frac{1}{3}\left(4^{3/2}\right) = \frac{8}{3}.$

18. $\displaystyle\int_0^1 \int_0^x \frac{y}{x^3+2}\,dy\,dx = \int_0^1 \frac{1}{2}\left(\frac{y^2}{x^3+2}\right)\Bigg|_0^x\,dx = \frac{1}{2}\int_0^1 \frac{x^2}{x^3+2}\,dx = \frac{1}{6}\ln\left(x^3+2\right)\Big|_0^1 = \frac{1}{6}\,(\ln 3 - \ln 2) = \frac{1}{6}\ln\frac{3}{2}.$

19. $\int_0^1 \int_0^x 2xe^y\,dy\,dx = \int_0^1 2xe^y\big|_{y=0}^{y=x}\,dx = \int_0^1 (2xe^x-2x)\,dx = 2\,(x-1)\,e^x - x^2\big|_0^1 = (-1)+2 = 1.$

20. $\int_0^1 \int_y^{e^{2y}} 2x\,dx\,dy = \int_0^1 x^2\big|_{x=y}^{x=e^{2y}}\,dy = \int_0^1 \left(e^{4y}-y^2\right)dy = \left(\frac{1}{4}e^{4y}-\frac{1}{3}y^3\right)\Big|_0^1 = \frac{1}{4}e^4-\frac{1}{3}-\frac{1}{4} = \frac{1}{12}\left(3e^4-7\right).$

21. $\int_0^1 \int_x^{\sqrt{x}} ye^x\,dy\,dx = -\int_0^1 \int_{\sqrt{x}}^x ye^x\,dy\,dx = \int_0^1 \left(-\frac{1}{2}y^2e^x\right)\Big|_{y=\sqrt{x}}^{y=x}\,dx = -\frac{1}{2}\int_0^1 \left(x^2e^x-xe^x\right)dx$

$= -\frac{1}{2}\left(x^2e^x\big|_0^1 - 2\int_0^1 xe^x\,dx - \int_0^1 xe^x\,dx\right) = -\frac{1}{2}\left(x^2e^x\big|_0^1 - 3\int_0^1 xe^x\,dx\right)$

$= -\frac{1}{2}\left(x^2e^x-3xe^x+3e^x\right)\Big|_0^1 = -\frac{1}{2}\,(e-3e+3e-3) = \frac{1}{2}\,(3-e).$

22. $\int_0^4 \int_0^{\sqrt{y}} xe^{-y^2}\,dx\,dy = \int_0^4 \frac{1}{2}x^2e^{-y^2}\Big|_{x=0}^{x=\sqrt{y}}\,dy = \frac{1}{2}\int_0^4 ye^{-y^2}\,dy = -\frac{1}{4}e^{-y^2}\Big|_0^4 = -\frac{1}{4}\left(e^{-16}-1\right) = \frac{1}{4}\left(1-e^{-16}\right).$

23. $\int_0^1 \int_{2x}^2 e^{y^2}\,dy\,dx = \int_0^2 \int_0^{y/2} e^{y^2}\,dx\,dy = \int_0^2 xe^{y^2}\Big|_{x=0}^{x=y/2}\,dy = \int_0^2 \frac{1}{2}ye^{y^2}\,dy = \frac{1}{4}e^{y^2}\Big|_0^2 = \frac{1}{4}\left(e^4-1\right).$

24. $\int_0^{\ln x} \int_1^e y\,dx\,dy = \int_0^{\ln x} yx\big|_{x=1}^{x=e}dy = \int_0^{\ln x} (e-1)\,y\,dy = (e-1)\,\frac{1}{2}y^2\Big|_0^{\ln x} = \frac{1}{2}\,(e-1)\,(\ln x)^2.$

25. $\int_0^2 \int_{y/2}^1 ye^{x^3}\,dx\,dy = \int_0^1 \int_0^{2x} ye^{x^3}\,dy\,dx = \int_0^1 \frac{1}{2}y^2e^{x^3}\Big|_{y=0}^{y=2x}\,dx = \int_0^1 2x^2e^{x^3}\,dx = \frac{2}{3}e^{x^3}\Big|_0^1 = \frac{2}{3}\,(e-1).$

26. $V = \iint_R f(x,y)\,dA = \int_0^3 \int_0^4 (6-y)\,dy\,dx = \int_0^3 \left[6y - \frac{1}{2}y^2\right]_{y=0}^{y=4} dx = \int_0^3 16\,dx = 48$

27. $V = \int_0^4 \int_0^3 \left(4 - x + \frac{1}{2}y\right) dx\,dy = \int_0^4 \left(4x - \frac{1}{2}x^2 + \frac{1}{2}xy\right)\Big|_{x=0}^{x=3} dy = \int_0^4 \left(\frac{15}{2} + \frac{3}{2}y\right) dy$

$= \left(\frac{15}{2}y + \frac{3}{4}y^2\right)\Big|_0^4 = 42.$

28. $V = \int_0^2 \int_x^{4-x} 5\,dy\,dx = \int_0^2 5y|_{y=x}^{y=4-x}\,dx = 5\int_0^2 (4-2x)\,dx = 5\left(4x - x^2\right)\big|_0^2 = 20.$

29. $V = \int_0^2 \int_0^{3-(3/2)z} (6 - 2y - 3z)\,dy\,dz = \int_0^2 \left(6y - y^2 - 3yz\right)\big|_{y=0}^{y=3-(3/2)z}\,dz$

$= \int_0^2 \left[6\left(3 - \frac{3}{2}z\right) - \left(3 - \frac{3}{2}z\right)^2 - 3\left(3 - \frac{3}{2}z\right)z\right] dz = \left[-2\left(3 - \frac{3}{2}z\right)^2 - \frac{2}{9}\left(3 - \frac{3}{2}z\right)^3 - \frac{9}{2}z^2 + \frac{3}{2}z^3\right]_0^2$

$= (-18 + 12) - (-18 + 6) = 6.$

30. $V = \iint_R f(x,y)\,dA = 4\int_0^2 \int_0^{\sqrt{4-x^2}} \left(4 - x^2 - y^2\right) dy\,dx = 4\int_0^2 \left[4y - x^2y - \frac{1}{3}y^3\right]_{y=0}^{y=\sqrt{4-x^2}} dx$

$= 4\int_0^2 \left[\left(4 - x^2\right)\sqrt{4 - x^2} - \frac{1}{3}\left(4 - x^2\right)^{3/2}\right] dx = \frac{8}{3}\int_0^2 \left(2^2 - x^2\right)^{3/2}\,dx.$

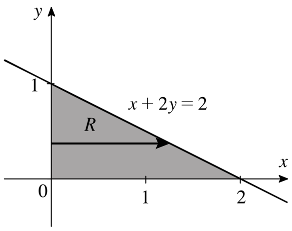
Using a calculator to evaluate the last integral, we find that $V = 8\pi \approx 25.1327$.

31. $V = \iint_R f(x,y)\,dA = \int_0^1 \int_0^{2-2y} \left(4 - x^2 - y^2\right) dx\,dy$

$= \int_0^1 \left[\left(4 - y^2\right)x - \frac{1}{3}x^3\right]_{x=0}^{x=2-2y} dy$

$= \int_0^1 \left[\left(4 - y^2\right)(2 - 2y) - \frac{1}{3}(2 - 2y)^3\right] dy$

$= \int_0^1 \left(\frac{14}{3}y^3 - 10y^2 + \frac{16}{3}\right) dy = \left(\frac{7}{6}y^4 - \frac{10}{3}y^3 + \frac{16}{3}y\right)\Big|_0^1 = \frac{19}{6}$



32. $V = \iint_R f(x,y)\,dA = \int_0^4 \int_0^2 \left(4 - x^2\right) dx\,dy = \int_0^4 \left[4x - \frac{1}{3}x^3\right]_0^2 dy = \int_0^4 \frac{16}{3}\,dy = \frac{64}{3}$

33. $V = \iint_R f(x,y)\,dA = \int_0^2 \int_0^2 2e^{-x}e^{-y}\,dx\,dy = \int_0^2 \left[-2e^{-x}e^{-y}\right]_{x=0}^{x=2} dy = \int_0^2 \left(-2e^{-2}e^{-y} + 2e^{-y}\right) dy$

$= \left(2e^{-2}e^{-y} - 2e^{-y}\right)\big|_0^2 = \frac{2\left(e^2 - 1\right)^2}{e^4}$

34. $V = \int_0^1 \int_0^2 (4 - 2x - y)\,dy\,dx = \int_0^1 \left(4y - 2xy - \frac{1}{2}y^2\right)\Big|_{y=0}^{y=2} dx = \int_0^1 (8 - 4x - 2)\,dx = \int_0^1 (6 - 4x)\,dx$

$= \left(6x - 2x^2\right)\big|_0^1 = 6 - 2 = 4.$

35. $V = \int_0^2 \int_0^{2x} (2x + y)\,dy\,dx = \int_0^2 \left(2xy + \frac{1}{2}y^2\right)\Big|_0^{2x} dx = \int_0^2 \left(4x^2 + 2x^2\right) dx = \int_0^2 6x^2\,dx = 2x^3\big|_0^2 = 16.$

36. $V = \int_0^1 \int_0^2 \left(x^2 + y^2\right) dy\,dx = \int_0^1 \left(x^2y + \frac{1}{3}y^3\right)\Big|_{y=0}^{y=2} dx = \int_0^1 \left(2x^2 + \frac{8}{3}\right) dx = \left(\frac{2}{3}x^3 + \frac{8}{3}x\right)\Big|_0^1$

$= \frac{2}{3} + \frac{8}{3} = \frac{10}{3}.$

37. $V = \int_0^1 \int_0^{-x+1} e^{x+2y}\,dy\,dx = \int_0^1 \frac{1}{2}e^{x+2y}\Big|_{y=0}^{y=-x+1} dx = \frac{1}{2}\int_0^1 \left(e^{-x+2} - e^x\right) dx = \frac{1}{2}\left(-e^{-x+2} - e^x\right)\Big|_0^1$

$= \frac{1}{2}\left(-e - e + e^2 + 1\right) = \frac{1}{2}\left(e^2 - 2e + 1\right) = \frac{1}{2}(e-1)^2.$

38. $V = \int_0^2 \int_x^2 2xe^y\,dy\,dx = \int_0^2 2xe^y\big|_{y=x}^{y=2} dx = \int_0^2 \left(2xe^2 - 2xe^x\right) dx = \left[e^2x^2 - 2(x-1)e^x\right]_0^2$ (by parts)

$= 4e^2 - 2e^2 - 2 = 2\left(e^2 - 1\right).$

39. $V = \displaystyle\int_0^4 \int_0^{\sqrt{x}} \frac{2y}{1+x^2}\,dy\,dx = \int_0^4 \frac{y^2}{1+x^2}\Big|_0^{\sqrt{x}} dx = \int_0^4 \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln\left(1+x^2\right)\Big|_0^4 = \frac{1}{2}(\ln 17 - \ln 1) = \frac{1}{2}\ln 17.$

40. $V = \int_0^1 \int_{x^2}^x 2x^2y\,dy\,dx = \int_0^1 x^2y^2\big|_{y=x^2}^{y=x} dx = \int_0^1 \left(x^4 - x^6\right) dx = \frac{1}{5} - \frac{1}{7} = \frac{2}{35}.$

41. $V = \int_0^4 \int_0^{\sqrt{16-x^2}} x\,dy\,dx = \int_0^4 xy\big|_{y=0}^{y=\sqrt{16-x^2}} dx = \int_0^4 x\left(16-x^2\right)^{1/2} dx = \left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(16-x^2\right)^{3/2}\Big|_0^4$

$= \frac{1}{3}(16)^{3/2} = \frac{64}{3}.$

42. $A = \frac{1}{6}\int_0^3 \int_0^2 6x^2y^3\,dx\,dy = \int_0^3 \frac{1}{3}x^3y^3\Big|_0^2 dy = \frac{8}{3}\int_0^3 y^3\,dy = \frac{2}{3}y^4\Big|_0^3 = 54.$

43. $A = \frac{1}{1/2}\int_0^1 \int_0^x (x+2y)\,dy\,dx = 2\int_0^1 \left(xy + y^2\right)\big|_0^x dx = 2\int_0^1 \left(x^2 + x^2\right) dx = 4\int_0^1 x^2\,dx = \frac{4}{3}x^3\Big|_0^1 = \frac{4}{3}.$

44. The area of R is $\frac{1}{2}(2)(1) = 1$, so the average value of f is

$\int_0^1 \int_y^{2-y} xy\,dx\,dy = \int_0^1 \frac{1}{2}x^2y\Big|_{x=y}^{x=2-y} dy = \int_0^1 \left[\frac{1}{2}(2-y)^2y - \frac{1}{2}y^3\right] dy = \int_0^1 \left(2y - 2y^2\right) dy = \left(y^2 - \frac{2}{3}y^3\right)\Big|_0^1 = \frac{1}{3}.$

45. The area of R is $\frac{1}{2}$, so the average value of f is

$\frac{1}{1/2}\int_0^1 \int_0^x e^{-x^2}\,dy\,dx = 2\int_0^1 e^{-x^2}y\Big|_{y=0}^{y=x} dx = 2\int_0^1 xe^{-x^2}\,dx = -e^{-x^2}\Big|_0^1 = -e^{-1} + 1 = 1 - \frac{1}{e}.$

46. The area of R is $\frac{1}{2}$, so the average value of f is

$2\int_0^1 \int_0^x xe^y\,dy\,dx = 2\int_0^1 xe^y\big|_{y=0}^{y=x} dx = 2\int_0^1 \left(xe^x - x\right) dx = 2\left(xe^x - e^x - \frac{1}{2}x^2\right)\Big|_0^1 = 2\left(e - e - \frac{1}{2} + 1\right) = 1.$

47. By elementary geometry, the area of the region is $4 + \frac{1}{2}(2)(4) = 8$. Therefore, the required average value is

$A = \frac{1}{8}\int_1^3 \int_0^{2x} \ln x\,dy\,dx = \frac{1}{8}\int_1^3 (\ln x)\,y\Big|_{y=0}^{y=2x} dx = \frac{1}{4}\int_1^3 x\ln x\,dx = \frac{1}{4}\left(\frac{1}{4}x^2\right)(2\ln x - 1)\Big|_1^3$ (by parts)

$= \frac{9}{16}(2\ln 3 - 1) - \frac{1}{16}(-1) = \frac{1}{8}(9\ln 3 - 4).$

48. The population is

$$2\int_0^5 \int_{-2}^0 \frac{10{,}000e^y}{1+0.5x}\,dy\,dx = 20{,}000\int_0^5 \frac{e^y}{1+0.5x}\Big|_{y=-2}^{y=0} dx = 20{,}000\left(1-e^{-2}\right)\int_0^5 \frac{1}{1+0.5x}\,dx$$

$$= 20{,}000\left(1-e^{-2}\right)\cdot 2\ln(1+0.5x)\Big|_0^5 = 40{,}000\left(1-e^{-2}\right)\ln 3.5 \approx 43{,}329.$$

49. The average population density inside R is $\frac{43{,}329}{20} \approx 2166$ people per square mile.

50. By symmetry, it suffices to compute the population in the first quadrant. In the first quadrant, $f(x,y) = \dfrac{50{,}000xy}{(x^2+20)(y^2+36)}$. Therefore, the population in R is given by

$$\iint_R f(x,y)\,dA = 4\int_0^{15}\left[\int_0^{20} \frac{50{,}000xy}{(x^2+20)(y^2+36)}\,dy\right]dx = 4\int_0^{15}\left[\frac{50{,}000x\left(\frac{1}{2}\right)\ln(y^2+36)}{x^2+20}\right]_0^{20} dx$$

$$= 100{,}000\,(\ln 436 - \ln 36)\int_0^{15}\frac{x}{x^2+20}\,dx = 100{,}000\,(\ln 436 - \ln 36)\left(\tfrac{1}{2}\right)\ln\left(x^2+20\right)\Big|_0^{15}$$

$= 50{,}000\,(\ln 436 - \ln 36)(\ln 245 - \ln 20) \approx 312{,}455$, or approximately 312,455 people.

51. The average weekly profit is

$$P = \tfrac{1}{(20)(20)}\int_{100}^{120}\int_{180}^{200}\left(-0.2x^2 - 0.25y^2 - 0.2xy + 100x + 90y - 4000\right)dx\,dy$$

$$= \tfrac{1}{400}\int_{100}^{120}\left(-\tfrac{1}{15}x^3 - 0.25y^2x - 0.1x^2y + 50x^2 + 90xy - 4000x\right)\Big|_{x=180}^{x=200} dy$$

$$= \tfrac{1}{400}\int_{100}^{120}\left(-144{,}533.33 - 5y^2 - 760y + 380{,}000 + 1800y - 80{,}000\right)dy$$

$$= \tfrac{1}{400}\int_{100}^{120}\left(155{,}466.67 - 5y^2 + 1040y\right)dy = \tfrac{1}{400}\left(155{,}466.67y - \tfrac{5}{3}y^3 + 520y^2\right)\Big|_{100}^{120}$$

$= \tfrac{1}{400}\,(3{,}109{,}333.40 - 1{,}213{,}333.30 + 2{,}288{,}000) \approx 10{,}460$, or \$10,460.

52. The average price is

$$P = \tfrac{1}{2}\int_0^1\int_0^2\left[200 - 10\left(x-\tfrac{1}{2}\right)^2 - 15\,(y-1)^2\right]dy\,dx = \tfrac{1}{2}\int_0^1\left[200y - 10\left(x-\tfrac{1}{2}\right)^2 y - 5\,(y-1)^3\right]_0^2 dx$$

$$= \tfrac{1}{2}\int_0^1\left[400 - 20\left(x-\tfrac{1}{2}\right)^2 - 5 - 5\right]dx = \tfrac{1}{2}\int_0^1\left[390 - 20\left(x-\tfrac{1}{2}\right)^2\right]dx = \tfrac{1}{2}\left[390x - \tfrac{20}{3}\left(x-\tfrac{1}{2}\right)^3\right]_0^1$$

$= \tfrac{1}{2}\left[390 - \tfrac{20}{3}\left(\tfrac{1}{8}\right) - \tfrac{20}{3}\left(\tfrac{1}{8}\right)\right] \approx 194.17$, or approximately \$194 per square foot.

53. True. This result follows from the definition.

54. False. Let $f(x,y) = \dfrac{x}{y-2}$, $a = 0$, $b = 3$, $c = 0$, and $d = 1$. Then $\int_{R_1}\int f(x,y)\,dA$ is defined on $R_1 = \{(x,y)\mid 0 \le x \le 3, 0 \le y \le 1\}$, but $\int_{R_2}\int f(x,y)\,dA$ is not defined on $R_2 = \{(x,y)\mid 0 \le x \le 1, 0 \le y \le 3\}$, because f is discontinuous on R_2 where $y = 2$.

55. True. $\int_R\int g(x,y)\,dA$ gives the volume of the solid bounded above by the surface $z = g(x,y)$ and $\int_R\int f(x,y)\,dA$ gives the volume of the solid bounded above by the surface $z = f(x,y)$. Therefore, $\int_R\int g(x,y)\,dA - \int_R\int f(x,y)\,dA = \int_R\int\left[g(x,y) - f(x,y)\right]dA$ gives the volume of the solid bounded above by $z = g(x,y)$ and below by $z = f(x,y)$.

56. True. The average value is $V_A = \dfrac{\int_R\int f(x,y)\,dA}{\int_R\int dA}$, and so $V_A\int_R\int dA = \int_R\int f(x,y)\,dA$. The quantity on the left-hand side is the volume of such a cylinder.

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1. xy, ordered pair, real number, $f(x,y)$

2. independent, dependent, value

3. $z = f(x, y)$, f, surface

4. $f(x, y) = k$, level curve, level curves, k

5. constant, x

6. slope, $(a, b, f(a, b))$, x, b

7. $\leq$, (a, b), $\leq$, domain

8. domain, $f_x(a, b) = 0$ and $f_y(a, b) = 0$, exist, candidate

9. scatter, minimizing, least-squares, normal

10. $g(x, y) = 0$, $f(x, y) + \lambda g(x, y)$, $F_x = 0$, $F_y = 0$, $F_\lambda = 0$, extrema

11. volume, solid

12. iterated, $\int_3^5 \int_0^1 \left(2x + y^2\right) dx\, dy$

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1. $f(x, y) = \dfrac{xy}{x^2 + y^2}$, so $f(0, 1) = 0$, $f(1, 0) = 0$, $f(1, 1) = \dfrac{1}{1+1} = \dfrac{1}{2}$, and $f(0, 0)$ does not exist because the point $(0, 0)$ does not lie in the domain of f.

2. $f(x, y) = \dfrac{xe^y}{1 + \ln xy}$, so $f(1, 1) = \dfrac{e}{1 + \ln 1} = e$, $f(1, 2) = \dfrac{e^2}{1 + \ln 2}$, $f(2, 1) = \dfrac{2e}{1 + \ln 2}$, and $f(1, 0)$ does not exist because the point $(0, 0)$ does not lie in the domain of f.

3. $h(x, y, z) = xye^z + \dfrac{x}{y}$, so $h(1, 1, 0) = 1 + 1 = 2$, $h(-1, 1, 1) = -e - 1 = -(e + 1)$, and $h(1, -1, 1) = -e - 1 = -(e + 1)$.

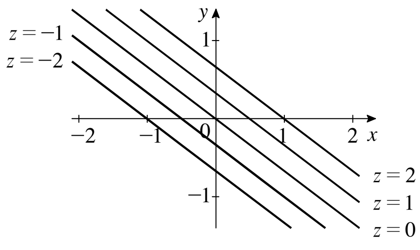
4. $f(u, v) = \dfrac{\sqrt{u}}{u - v}$. The domain of f is the set of all ordered pairs (u, v) of real numbers such that $u \geq 0$ and $u \neq v$.

5. $f(x, y) = \dfrac{x - y}{x + y}$, so $D = \{(x, y) \mid y \neq -x\}$.

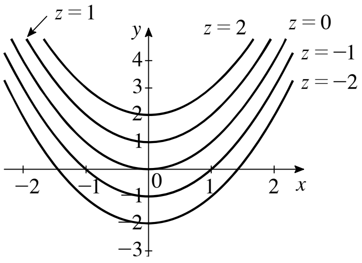
6. $f(x, y) = x\sqrt{y} + y\sqrt{1 - x}$, so $D = \{(x, y) \mid x \leq 1, y \geq 0\}$.

7. $f(x, y, z) = \dfrac{xy\sqrt{z}}{(1 - x)(1 - y)(1 - z)}$. The domain of f is the set of all ordered triples (x, y, z) of real numbers such that $z \geq 0$, $x \neq 1$, $y \neq 1$, and $z \neq 1$.

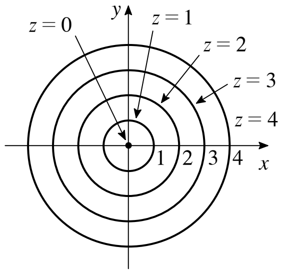
8. $2x + 3y = z$



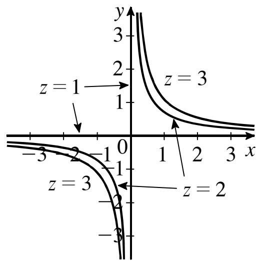
9. $z = y - x^2$



10. $z = \sqrt{x^2 + y^2}$



11. $z = e^{xy}$



12. $f(x, y) = x^2y^3 + 3xy^2 + \dfrac{x}{y}$, so $f_x = 2xy^3 + 3y^2 + \dfrac{1}{y}$ and $f_y = 3x^2y^2 + 6xy - \dfrac{x}{y^2}$.

13. $f(x, y) = x\sqrt{y} + y\sqrt{x}$, so $f_x = \sqrt{y} + \dfrac{y}{2\sqrt{x}}$ and $f_y = \dfrac{x}{2\sqrt{y}} + \sqrt{x}$.

14. $f(u, v) = \sqrt{uv^2 - 2u}$, so $f_u = \frac{1}{2}\left(uv^2 - 2u\right)^{-1/2}\left(v^2 - 2\right) = \dfrac{v^2 - 2}{2\sqrt{uv^2 - 2u}}$ and

$f_v = \frac{1}{2}\left(uv^2 - 2u\right)^{-1/2}(2uv) = \dfrac{uv}{\sqrt{uv^2 - 2u}}$.

15. $f(x, y) = \dfrac{x - y}{y + 2x}$, so $f_x = \dfrac{(y + 2x) - (x - y)(2)}{(y + 2x)^2} = \dfrac{3y}{(y + 2x)^2}$ and

$f_y = \dfrac{(y + 2x)(-1) - (x - y)}{(y + 2x)^2} = \dfrac{-3x}{(y + 2x)^2}$.

16. $g(x, y) = \dfrac{xy}{x^2 + y^2}$, so $g_x = \dfrac{\left(x^2 + y^2\right)y - xy(2x)}{\left(x^2 + y^2\right)^2} = \dfrac{y^3 - x^2y}{\left(x^2 + y^2\right)^2} = \dfrac{y\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2} = \dfrac{y(y - x)(y + x)}{\left(x^2 + y^2\right)^2}$ and

$g_y = \dfrac{\left(x^2 + y^2\right)x - xy(2y)}{\left(x^2 + y^2\right)^2} = \dfrac{x\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2} = \dfrac{x(x - y)(x + y)}{\left(x^2 + y^2\right)^2}$.

17. $h(x, y) = \left(2xy + 3y^2\right)^5$, so $h_x = 10y\left(2xy + 3y^2\right)^4$ and $h_y = 10(x + 3y)\left(2xy + 3y^2\right)^4$.

18. $f(x, y) = (xe^y + 1)^{1/2}$, so $f_x = \frac{1}{2}(xe^y + 1)^{-1/2}e^y = \dfrac{e^y}{2(xe^y + 1)^{1/2}}$ and

$f_y = \frac{1}{2}(xe^y + 1)^{-1/2}xe^y = \dfrac{xe^y}{2(xe^y + 1)^{1/2}}$.

19. $f(x, y) = \left(x^2 + y^2\right)e^{x^2+y^2}$, so $f_x = 2xe^{x^2+y^2} + \left(x^2 + y^2\right)(2x)e^{x^2+y^2} = 2x\left(x^2 + y^2 + 1\right)e^{x^2+y^2}$ and

$f_y = 2ye^{x^2+y^2} + \left(x^2 + y^2\right)(2y)e^{x^2+y^2} = 2y\left(x^2 + y^2 + 1\right)e^{x^2+y^2}$.

20. $f(x, y) = \ln\left(1 + 2x^2 + 4y^4\right)$, so $f_x = \dfrac{4x}{1 + 2x^2 + 4y^4}$ and $f_y = \dfrac{16y^3}{1 + 2x^2 + 4y^4}$.

21. $f(x, y) = \ln\left(1 + \dfrac{x^2}{y^2}\right)$, so $f_x = \dfrac{2x/y^2}{1 + \left(x^2/y^2\right)} = \dfrac{2x}{x^2 + y^2}$ and $f_y = \dfrac{-2x^2/y^3}{1 + \left(x^2/y^2\right)} = -\dfrac{2x^2}{y\left(x^2 + y^2\right)}$.

22. $f(x,y) = x^3 - 2x^2y + y^2 + x - 2y$, so $f_x = 3x^2 - 4xy + 1$ and $f_y = -2x^2 + 2y - 2$. Therefore, $f_{xx} = 6x - 4y$, $f_{xy} = f_{yx} = -4x$, and $f_{yy} = 2$.

23. $f(x,y) = x^4 + 2x^2y^2 - y^4$, so $f_x = 4x^3 + 4xy^2$ and $f_y = 4x^2y - 4y^3$. Therefore, $f_{xx} = 12x^2 + 4y^2$, $f_{xy} = 8xy = f_{yx}$, and $f_{yy} = 4x^2 - 12y^2$.

24. $f(x,y) = \left(2x^2 + 3y^2\right)^3$, so $f_x = 3\left(2x^2 + 3y^2\right)^2(4x) = 12x\left(2x^2 + 3y^2\right)^2$ and $f_y = 3\left(2x^2 + 3y^2\right)^2(6y) = 18y\left(2x^2 + 3y^2\right)^2$. Therefore,

$$\begin{aligned} f_{xx} &= 12\left(2x^2 + 3y^2\right)^2 + 12x(2)\left(2x^2 + 3y^2\right)(4x) = 12\left(2x^2 + 3y^2\right)^2\left[\left(2x^2 + 3y^2\right) + 8x^2\right] \\ &= 12\left(2x^2 + 3y^2\right)\left(10x^2 + 3y^2\right), \end{aligned}$$

$f_{xy} = 12x(2)\left(2x^2 + 3y^2\right)(6y) = 144xy\left(2x^2 + 3y^2\right) = f_{yx}$, and

$$\begin{aligned} f_{yy} &= 18\left(2x^2 + 3y^2\right)^2 + 18y(2)\left(2x^2 + 3y^2\right)(6y) = 18\left(2x^2 + 3y^2\right)\left[\left(2x^2 + 3y^2\right) + 12y^2\right] \\ &= 18\left(2x^2 + 3y^2\right)\left(2x^2 + 15y^2\right). \end{aligned}$$

25. $g(x,y) = \dfrac{x}{x + y^2}$, so $g_x = \dfrac{\left(x + y^2\right) - x}{\left(x + y^2\right)^2} = \dfrac{y^2}{\left(x + y^2\right)^2}$ and

$g_y = \dfrac{-2xy}{\left(x + y^2\right)^2}$. Therefore, $g_{xx} = -2y^2\left(x + y^2\right)^{-3} = -\dfrac{2y^2}{\left(x + y^2\right)^3}$,

$$g_{xy} = \frac{\left(x + y^2\right)2y - y^2(2)\left(x + y^2\right)2y}{\left(x + y^2\right)^4} = \frac{2\left(x + y^2\right)\left(xy + y^3 - 2y^3\right)}{\left(x + y^2\right)^4} = \frac{2y\left(x - y^2\right)}{\left(x + y^2\right)^3} = g_{yx}, \text{ and}$$

$$g_{yy} = \frac{\left(x + y^2\right)^2(-2x) + 2xy(2)\left(x + y^2\right)2y}{\left(x + y^2\right)^4} = \frac{2x\left(x^2 + y^2\right)\left(-x - y^2 + 4y^2\right)}{\left(x + y^2\right)^4} = \frac{2x\left(3y^2 - x\right)}{\left(x + y^2\right)^3}$$

26. $g(x,y) = e^{x^2+y^2}$, so $g_x = 2xe^{x^2+y^2}$ and $g_y = 2ye^{x^2+y^2}$. Therefore, $g_{xx} = 2e^{x^2+y^2} + (2x)^2e^{x^2+y^2} = 2\left(1 + 2x^2\right)e^{x^2+y^2}$, $g_{xy} = 4xye^{x^2+y^2} = g_{yx}$, and $g_{yy} = 2e^{x^2+y^2} + (2y)^2e^{x^2+y^2} = 2\left(1 + 2y^2\right)e^{x^2+y^2}$.

27. $h(s,t) = \ln\left(\dfrac{s}{t}\right)$. Write $h(s,t) = \ln s - \ln t$. Then $h_s = \dfrac{1}{s}$ and $h_t = -\dfrac{1}{t}$, so $h_{ss} = -\dfrac{1}{s^2}$, $h_{st} = h_{ts} = 0$, and $h_{tt} = \dfrac{1}{t^2}$.

28. $f(x,y,z) = x^3y^2z + xy^2z + 3xy - 4z$, so $f_x(1,1,0) = \left.\left(3x^2yz + y^2z + 3y\right)\right|_{(1,1,0)} = 3$; $f_y(1,1,0) = \left.\left(2x^3yz + 2xyz + 3x\right)\right|_{(1,1,0)} = 3$, and $f_z(1,1,0) = \left.\left(x^3y^2 + xy^2 - 4\right)\right|_{(1,1,0)} = -2$.

29. $f(x,y) = 2x^2 + y^2 - 8x - 6y + 4$. To find the critical points of f, we solve the system $\begin{cases} f_x = 4x - 8 = 0 \\ f_y = 2y - 6 = 0 \end{cases}$ obtaining $x = 2$ and $y = 3$. Therefore, the sole critical point of f is $(2,3)$. Next, $f_{xx} = 4$, $f_{xy} = 0$, and $f_{yy} = 2$, so $D(2,3) = f_{xx}(2,3)\,f_{yy}(2,3) - f_{xy}(2,3)^2 = 8 > 0$. Because $f_{xx}(2,3) > 0$, we see that $f(2,3) = -13$ is a relative minimum value.

30. $f(x,y)=x^2+3xy+y^2-10x-20y+12$. We solve the system $\begin{cases} f_x=2x+3y-10=0 \\ f_y=3x+2y-20=0 \end{cases}$ obtaining $x=8$ and $y=-2$, so $(8,-2)$ is the only critical point of f. Next, we compute $f_{xx}=2$, $f_{xy}=3$, and $f_{yy}=2$, so $D(8,-2)=f_{xx}(8,-2)f_{yy}(8,-2)-f_{xy}^2(8,-2)=(2)(2)-3^2=-5<0$. Because $D<0$, we see that $(8,-2)$ gives rise to a saddle point of f. Because $f(8,-2)=-8$, the saddle point is $(8,-2,-8)$.

31. $f(x,y)=x^3-3xy+y^2$. We solve the system of equations $\begin{cases} f_x=3x^2-3y=0 \\ f_y=-3x+2y=0 \end{cases}$ obtaining $x^2-y=0$, and so $y=x^2$. Then $-3x+2x^2=0$, $x(2x-3)=0$, and so $x=0$ or $x=\frac{3}{2}$. The corresponding values of y are $y=0$ and $y=\frac{9}{4}$, so the critical points are $(0,0)$ and $\left(\frac{3}{2},\frac{9}{4}\right)$. Next, $f_{xx}=6x$, $f_{xy}=-3$, and $f_{yy}=2$, and so $D(x,y)=12x-9=3(4x-3)$. Therefore, $D(0,0)=-9$, and so $(0,0)$ is a saddle point and $f(0,0)=0$. $D\left(\frac{3}{2},\frac{9}{4}\right)=3(6-3)=9>0$ and $f_{xx}\left(\frac{3}{2},\frac{9}{4}\right)>0$, and so $f\left(\frac{3}{2},\frac{9}{4}\right)=\frac{27}{8}-\frac{81}{8}+\frac{81}{16}=-\frac{27}{16}$ is a relative minimum value.

32. $f(x,y)=x^3+y^2-4xy+17x-10y+8$. To find the critical points of f, we solve the system $\begin{cases} f_x=3x^2-4y+17=0 \\ f_y=2y-4x-10=0 \end{cases}$ From the second equation, we have $y=2x+5$ which, when substituted into the first equation, gives $3x^2-8x-20+17=0$, so $3x^2-8x-3=(3x+1)(x-3)=0$. The solutions are $x=-\frac{1}{3}$ and $x=3$, so the critical points of f are $\left(-\frac{1}{3},\frac{13}{3}\right)$ and $(3,11)$. Next, we compute $f_{xx}=6x$, $f_{xy}=-4$, and $f_{yy}=2$, and so $D(x,y)=f_{xx}f_{yy}-f_{xy}^2=12x-16$. Because $D\left(-\frac{1}{3},\frac{13}{3}\right)=-20<0$, we see that $\left(-\frac{1}{3},\frac{13}{3}\right)$ gives a saddle point and $f\left(-\frac{1}{3},\frac{13}{3}\right)=-\frac{445}{27}$; and because $D(3,11)=20>0$ and $f_{xx}(3,11)=18>0$, we see that $(3,11)$ gives a relative minimum value of f, namely $f(3,11)=-35$.

33. $f(x,y)=f(x,y)=e^{2x^2+y^2}$. To find the critical points of f, we solve the system $\begin{cases} f_x=4xe^{2x^2+y^2}=0 \\ f_y=2ye^{2x^2+y^2}=0 \end{cases}$ giving $(0,0)$ as the only critical point of f. Next, $f_{xx}=4\left(e^{2x^2+y^2}+4x^2e^{2x^2+y^2}\right)=4\left(1+4x^2\right)e^{2x^2+y^2}$, $f_{xy}=8xye^{2x^2+y^2}=f_{yx}$, and $f_{yy}=2\left(1+2y^2\right)e^{2x^2+y^2}$, so $D=f_{xx}(0,0)f_{yy}(0,0)-f_{xy}^2(0,0)=(4)(2)-0=8>0$. Because $f_{xx}(0,0)>0$, we see that $(0,0)$ gives a relative minimum of f. The minimum value of f is $f(0,0)=e^0=1$.

34. $f(x,y)=\ln\left(x^2+y^2-2x-2y+4\right)$. We solve the system $\begin{cases} f_x=\dfrac{2x-2}{x^2+y^2-2x-2y+4}=0 \\ f_y=\dfrac{2y-2}{x^2+y^2-2x-2y+4}=0 \end{cases}$ obtaining $x=1$ and $y=1$ and giving $(1,1)$ as the only critical point of f. Next, we compute $f_{xx}=\dfrac{\left(x^2+y^2-2x-2y+4\right)(2)-(2x-2)^2}{\left(x^2+y^2-2x-2y+4\right)^2}$,

$$f_{xy}=\frac{\left(x^2+y^2-2x-2y+4\right)(0)-(2x-2)(2y-2)}{\left(x^2+y^2-2x-2y+4\right)^2}=\frac{-4(x-1)(y-1)}{\left(x^2+y^2-2x-2y+4\right)^2}=f_{yx}, \text{ and}$$

$f_{yy}=\dfrac{\left(x^2+y^2-2x-2y+4\right)(2)-(2y-2)^2}{\left(x^2+y^2-2x-2y+4\right)^2}$. In particular, $f_{xx}(1,1)=\dfrac{2}{2^2}=\dfrac{1}{2}$, $f_{xy}(1,1)=0$, and $f_{yy}(1,1)=1$, so $D=f_{xx}(1,1)f_{yy}(1,1)-f_{xy}(1,1)=\frac{1}{2}>0$. Because $f_{xx}(1,1)=1>0$, we conclude that $(1,1)$ gives rise to a relative minimum of f. The relative minimum value is $f(1,1)=\ln 2$.

35. We form the Lagrangian function $F(x,y,\lambda)=-3x^2-y^2+2xy+\lambda(2x+y-4)$. Next, we solve the system $\begin{cases} F_x=6x+2y+2\lambda=0 \\ F_y=-2y+2x+\lambda=0 \\ F_\lambda=2x+y-4=0 \end{cases}$ Multiplying the second equation by 2 and subtracting the resulting equation from the first equation yields $6y-10x=0$ so $y=\frac{5}{3}x$. Substituting this value of y into the third equation of the system gives $2x+\frac{5}{3}x-4=0$, so $x=\frac{12}{11}$ and consequently $y=\frac{20}{11}$. Therefore, $\left(\frac{12}{11},\frac{20}{11}\right)$ gives the maximum value $f\left(\frac{12}{11},\frac{20}{11}\right)=-\frac{32}{11}$ for f subject to the given constraint.

36. We form the Lagrangian function $F(x,y,\lambda)=2x^2+3y^2-6xy+4x-9y+10+\lambda(x+y-1)$. Next, we solve the system $\begin{cases} F_x=4x-6y+4+\lambda=0 \\ F_y=6y-6x-9+\lambda=0 \\ F_\lambda=x+y-1=0 \end{cases}$ Subtracting the second equation from the first, we obtain $10x-12y+13=0$. Adding this equation to the equation obtained by multiplying the third equation in the system by 12, we obtain $22x+1=0$, so $x=-\frac{1}{22}$ and therefore $y=\frac{23}{22}$. Thus, the point $\left(-\frac{1}{22},\frac{23}{22}\right)$ gives the minimum value $f\left(-\frac{1}{22},\frac{23}{22}\right)=\frac{175}{44}$ of f subject to the given constraint.

37. The Lagrangian function is $F(x,y,\lambda)=2x-3y+1+\lambda\left(2x^2+3y^2-125\right)$. Next, we solve the system of equations $\begin{cases} F_x=2+4\lambda x=0 \\ F_y=-3+6\lambda y=0 \\ F_\lambda=2x^2+3y^2-125=0 \end{cases}$ Solving the first equation for x gives $x=-\frac{1}{2}\lambda$, and the second equation gives $y=\frac{1}{2}\lambda$. Substituting these values of x and y into the third equation gives $2\left(-\dfrac{1}{2\lambda}\right)^2+3\left(\dfrac{1}{2\lambda}\right)^2-125=0$, so $\dfrac{1}{2\lambda^2}+\dfrac{3}{4\lambda^2}-125=0$, $2+3-500\lambda^2=0$, and so $\lambda=\pm\frac{1}{10}$. Therefore, $x=\pm5$ and $y=\pm5$, and so the critical points of f are $(-5,5)$ and $(5,-5)$. Next, we compute $f(-5,5)=2(-5)-3(5)+1=-24$ and $f(5,-5)=2(5)-3(-5)+1=26$. We conclude that f has a maximum value of 26 at $(5,-5)$ and a minimum value of -24 at $(-5,5)$.

38. We form the Lagrangian function $F(x, y, \lambda) = e^{x-y} + \lambda\left(x^2 + y^2 - 1\right)$. Next, we solve the system

$$\begin{cases} F_x = e^{x-y} + 2\lambda x = 0 \\ F_y = -e^{x-y} + 2\lambda y = 0 \\ F_\lambda = x^2 + y^2 - 1 = 0 \end{cases}$$

Adding the first two equations, we obtain $2\lambda(x + y) = 0$. Because $\lambda \neq 0$, (otherwise we have $e^{x-y} = 0$ which is impossible), we find $y = -x$. Substituting this value of y into the third equation of the system gives $2x^2 - 1 = 0$, or $x = \pm\sqrt{2}/2$. The corresponding values of y are $\pm\sqrt{2}/2$. We see that $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ gives a minimum of f with value $e^{-\sqrt{2}}$, while $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ gives a maximum of f with value $e^{\sqrt{2}}$.

39. $\int_{-1}^{2} \int_{2}^{4} (3x - 2y)\,dx\,dy = \int_{-1}^{2} \left(\frac{3}{2}x^2 - 2xy\right)\Big|_{x=2}^{x=4} dy = \int_{-1}^{2} \left[(24 - 8y) - (6 - 4y)\right] dy = \int_{-1}^{2} (18 - 4y)\,dy$

$= \left(18y - 2y^2\right)\Big|_{-1}^{2} = (36 - 8) - (-18 - 2) = 48.$

40. $\int_0^1 \int_0^2 e^{-x-2y}\,dx\,dy = \int_0^1 \left(-e^{-x-2y}\right)\Big|_{x=0}^{x=2} dy = \int_0^1 \left(-e^{-2-2y} + e^{-2y}\right) dy = \left(\frac{1}{2}e^{-2-2y} - \frac{1}{2}e^{-2y}\right)\Big|_0^1$

$= \left(\frac{1}{2}e^{-4} - \frac{1}{2}e^{-2}\right) - \left(\frac{1}{2}e^{-2} - \frac{1}{2}\right) = \frac{1}{2}\left(e^{-4} - 2e^{-2} + 1\right) = \frac{1}{2}\left(e^{-2} - 1\right)^2.$

41. $\int_0^1 \int_{x^3}^{x^2} 2x^2 y\,dy\,dx = \int_0^1 \left[x^2y^2\right]_{y=x^3}^{y=x^2} dx = \int_0^1 x^2\left(x^4 - x^6\right) dx = \int_0^1 \left(x^6 - x^8\right) dx = \left[\frac{1}{7}x^7 - \frac{1}{9}x^9\right]_0^1 = \frac{1}{7} - \frac{1}{9} = \frac{2}{63}.$

42. $\displaystyle\int_1^2 \int_1^x \frac{y}{x}\,dy\,dx = \int_1^2 \frac{1}{x}\left(\frac{y^2}{2}\right)\Big|_{y=1}^{y=x} dx = \int_1^2 \left(\frac{x}{2} - \frac{1}{2x}\right) dx = \left(\frac{1}{4}x^2 - \frac{1}{2}\ln x\right)\Big|_1^2$

$= \left(1 - \frac{1}{2}\ln 2\right) - \frac{1}{4} = \frac{3}{4} - \frac{1}{2}\ln 2 = \frac{1}{4}(3 - 2\ln 2).$

43. $\int_0^2 \int_0^1 \left(4x^2 + y^2\right) dy\,dx = \int_0^2 \left(4x^2y + \frac{1}{3}y^3\right)\Big|_{y=0}^{y=1} dx = \int_0^2 \left(4x^2 + \frac{1}{3}\right) dx = \left(\frac{4}{3}x^3 + \frac{1}{3}x\right)\Big|_0^2 = \frac{32}{3} + \frac{2}{3} = \frac{34}{3}.$

44. $V = \int_0^4 \int_{y/4}^{\sqrt{y}} (x + y)\,dx\,dy = \int_0^4 \left(\frac{1}{2}x^2 + xy\right)\Big|_{x=y/4}^{x=\sqrt{y}} dy = \int_0^4 \left(\frac{1}{2}y - y^{3/2} - \frac{1}{32}y^2 - \frac{1}{4}y^2\right) dy$

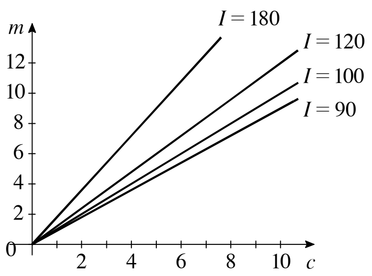
$= \left(\frac{1}{4}y^2 + \frac{2}{5}y^{5/2} - \frac{3}{32}y^3\right)\Big|_0^4 = 4 + \frac{64}{5} - 6 = \frac{54}{5}.$

45. The area of R is $\int_0^2 \int_{x^2}^{2x} dy\,dx = \int_0^2 y\big|_{y=x^2}^{y=2x} dx = \int_0^2 \left(2x - x^2\right) dx = \left(x^2 - \frac{1}{3}x^3\right)\Big|_0^2 = \frac{4}{3}$. Thus,

$AV = \frac{1}{4/3} \int_0^2 \int_{x^2}^{2x} (xy + 1)\,dy\,dx = \frac{3}{4} \int_0^2 \left(\frac{1}{2}xy^2 + y\right)\Big|_{x^2}^{2x} dx = \frac{3}{4} \int_0^2 \left(-\frac{1}{2}x^5 + 2x^3 - x^2 + 2x\right) dx$

$= \frac{3}{4}\left(-\frac{1}{12}x^6 + \frac{1}{2}x^4 - \frac{1}{3}x^3 + x^2\right)\Big|_0^2 = \frac{3}{4}\left(-\frac{16}{3} + 8 - \frac{8}{3} + 4\right) = 3.$

46. $k = \dfrac{100m}{c}$. The level curves are straight lines.



47. **a.** $R(x, y) = px + qy = -0.02x^2 - 0.2xy - 0.05y^2 + 80x + 60y$.

b. The domain of R is the set of all points satisfying $0.02x + 0.1y \leq 80$, $0.1x + 0.05y \leq 60$, $x \geq 0$, and $y \geq 0$.

c.
$$R(100, 300) = -0.02(100)^2 - 0.2(100)(300) - 0.05(300)^2 + 80(100)^2 + 60(300) = 15{,}300,$$
giving revenue of \$15,300 realized from the sale of 100 sixteen-speed and 300 ten-speed electric blenders.

48. $f(p, q) = 900 - 9p - e^{0.4q}$ and $g(p, q) = 20{,}000 - 3000q - 4p$. We compute $\frac{\partial f}{\partial q} = -0.4e^{0.4q}$ and $\frac{\partial g}{\partial p} = -4$. Because $\frac{\partial f}{\partial q} < 0$ and $\frac{\partial g}{\partial p} < 0$ for all $p > 0$ and $q > 0$, we conclude that compact disc players and audio discs are complementary commodities.

49. **a.** We summarize the data at right. The normal equations are $5b + 25m = 2011$ and $25b + 165m = 10{,}383$, and the solutions are $b = 361.2$ and $m = 8.2$. Therefore, the least-squares line has equation $y = 8.2x + 361.2$.

b. The average daily viewing time in 2014 (when $x = 11$) is $y = 8.2(11) + 361.2 = 451.4 = 7.52$, or 7 hr 31 min.

	x	y	x^2	xy
	1	369	1	369
	3	390	9	1170
	5	396	25	1980
	7	420	49	2940
	9	436	81	3924
Sum	25	2011	165	10,383

50. **a.** We summarize the data at right. The normal equations are $6b + 150m = 125.5$ and $150b + 5500m = 3240$, and the solutions are $b = 19.45$ and $m = 0.0586$. Therefore, $y = 0.059x + 19.5$.

b. In 2040, the life expectancy beyond 65 of a 65-year-old female is $y = 0.059(40) + 19.5 = 21.86$, or 21.9 years. This is close to the given datum of 21.8 years.

c. In 2030, the life expectancy beyond 65 of a 65-year-old female is $y = 0.059(30) + 19.5 = 21.27$, or 21.3 years. This is close to the given datum of 21.2 years.

	x	y	x^2	xy
	0	19.5	0	0
	10	20	100	200
	20	20.6	400	412
	30	21.2	900	636
	40	21.8	1600	872
	50	22.4	2500	1120
Sum	150	125.5	5500	3240

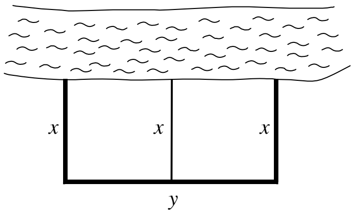
51. **a.** We summarize the calculations as follows:

	x	y	x^2	xy
	0	547.2	0	0
	1	638.9	1	638.9
	2	750.1	4	1500.2
	3	861.2	9	2583.6
	4	929.8	16	3719.2
Sum	10	3727.2	30	8441.9

The normal equations are $5b + 10m = 3727.2$ and $10b + 30m = 8441.9$. The solutions are $m = 547.94$ and $b = 98.75$, so the required equation is $y = 98.75x + 547.94$.

b. The estimated number of users in 2014 is $f(7) = 98.75(7) + 547.95 = 1239.19$, or approximately 1,239.2 million.

52. We want to maximize the function $R(x,y) = -x^2 - 0.5y^2 + xy + 8x + 3y + 20$. To find the critical point of R, we solve the system $\begin{cases} R_x = -2x + y + 8 = 0 \\ R_y = -y + x + 3 = 0 \end{cases}$ Adding the two equations, we obtain $-x + 11 = 0$, or $x = 11$, and so $y = 14$. Therefore, $(11, 14)$ is a critical point of R. Next, we compute $R_{xx} = -2$, $R_{xy} = 1$, and $R_{yy} = -1$, so $D(x,y) = R_{xx}R_{yy} - R_{xy}^2 = 2 - 1 = 1$. In particular, $D(11,14) = 1 > 0$. Because $R_{xx}(11,14) = -2 < 0$, we see that $(11, 14)$ gives a relative maximum of R. The nature of the problem suggests that this is in fact an absolute maximum. So the company should spend \$11,000 on advertising and employ 14 agents in order to maximize its revenue.



53. We want to minimize $C(x,y) = 3(2x) + 2(x) + 3y = 8x + 3y$ subject to $xy = 303{,}750$. The Lagrangian function is $F(x,y,\lambda) = 8x + 3y + \lambda(xy - 303{,}750)$, so we solve the system

$$\begin{cases} F_x = 8 + \lambda y = 0 \\ F_y = 3 + \lambda x = 0 \\ F_\lambda = xy - 303,750 = 0 \end{cases}$$

Solving the first equation for y gives $y = -\frac{8}{\lambda}$. The second equation gives $x = -\frac{3}{\lambda}$. Substituting these values into the third equation gives $\left(-\frac{3}{\lambda}\right)\left(-\frac{8}{\lambda}\right) = 303{,}750$, so or $\lambda = \pm\frac{2}{225}$. Therefore, $x = 337.5$ and $y = 900$, and so the required dimensions of the pasture are 337.5 yd by 900 yd.

54. The total weight of the fish is $W = \left(8 - x - \frac{1}{2}y\right)x + \left(11 - \frac{1}{2}x - 2y\right)y = -x^2 - 2y^2 - xy + 8x + 11y$. To find the critical point of W, we solve the system $\begin{cases} W_x = -2x - y + 8 = 0 \\ W_y = -x - 4y + 11 = 0 \end{cases}$ or $\begin{cases} 2x + y = 8 \\ x + 4y = 11 \end{cases}$ obtaining $x = 3$ and $y = 2$. So the sole critical point is $(3, 2)$. Next, we find $W_{xx} = -2$, $W_{xy} = -1$, and $W_{yy} = -4$. Since $D(3,2) = (-2)(-4) - (-1)^2 = 7 > 0$ and $W_{xx}(3,2) < 0$, we conclude that $(3, 2)$ gives a relative (and therefore absolute) maximum of W. Hence, the desired numbers of bass and trout are 300 and 200, respectively.

55. We want to maximize the function Q subject to the constraint $x + y = 100$. We form the Lagrangian function $f(x,y,\lambda) = x^{3/4}y^{1/4} + \lambda(x + y - 100)$. To find the critical points of F, we solve $\begin{cases} F_x = \frac{3}{4}\left(\frac{y}{x}\right)^{1/4} + \lambda = 0 \\ F_y = \frac{1}{4}\left(\frac{x}{y}\right)^{3/4} + \lambda = 0 \\ F_\lambda = x + y - 100 = 0 \end{cases}$ Solving the first equation for λ and substituting this value into the second equation yields $\frac{1}{4}\left(\frac{x}{y}\right)^{3/4} - \frac{3}{4}\left(\frac{y}{x}\right)^{1/4} = 0$, $\left(\frac{x}{y}\right)^{3/4} = 3\left(\frac{y}{x}\right)^{1/4}$, and so $x = 3y$. Substituting this value of x into the third equation, we have $4y = 100$, so $y = 25$ and $x = 75$. Therefore, 75 units should be spent on labor and 25 units on capital.

56. $\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}kx^a y^{1-a} = kax^{a-1}y^{1-a} = ka\left(\frac{y}{x}\right)^{1-a}$ and $\frac{\partial P}{\partial y} = k(1-x)x^a y^{-a} = k(1-a)\left(\frac{x}{y}\right)^a$. Therefore, $x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} = \frac{kaxy^{1-a}}{x^{1-a}} + \frac{k(1-a)yx^a}{y^a} = kax^a y^{1-a} + k(1-a)x^a y^{1-a} = kx^a y^{1-a} = P$, as was to be shown.

CHAPTER 8 Before Moving On... page 638

1. In order for $f(x, y) = \dfrac{\sqrt{x} + \sqrt{y}}{(1 - x)(2 - y)}$ to be defined, we must have $x \geq 0$, $y \geq 0$, $x \neq 1$ and $y \neq 2$. Therefore, the domain of f is $D = \{(x, y) \mid x \geq 0, y \geq 0, x \neq 1, y \neq 2\}$.

2. $f(x, y) = x^2 y + e^{xy}$, so $f_x = 2xy + ye^{xy}$, $f_y = x^2 + xe^{xy}$, $f_{xx} = 2y + y^2 e^{xy}$, $f_{xy} = 2x + (1 + xy)\,e^{xy} = f_{yx}$, and $f_{yy} = x \cdot xe^{xy} = x^2 e^{xy}$.

3. $f(x, y) = 2x^3 + 2y^3 - 6xy - 5$. Solving $f_x = 6x^2 + 6y = 6\left(x^2 - y^2\right) = 0$ and $f_y = 6y^2 - 6x = 6\left(y^2 - x\right) = 0$ simultaneously gives $y = x^2$ and $x = y^2$. Therefore, $x = x^4$, $x^4 - x = x\left(x^3 - 1\right) = 0$, and so $x = 0$ or 1. The critical points of f are $(0, 0)$ and $(1, 1)$. $f_{xx} = 12x$, $f_{xy} = -6$, and $f_{yy} = 12y$, so $D(x, y) = 144x^2 + 144y^2 - 36$. In particular, $D(0, 0) = -36 < 0$, and so $(0, 0)$ does not give a relative extremum; and $D(1, 1) = 252 > 0$ and $f_{xx}(1, 1) = 12 > 0$, and so $f(1, 1)$ gives a relative minimum value of $f(1, 1) = 2(1)^3 + 2(1)^3 - 6(1)(1) - 5 = -7$.

4. We summarize the data at right. The normal equations are $5b + 11m = 36.8$ and $11b + 39m = 111.1$. Solving, we find $m \approx 2.04$ and $b \approx 2.88$. Thus, the least-squares line has equation $y = 2.04x + 2.88$.

	x	y	x^2	xy
	0	2.9	0	0
	1	5.1	1	5.1
	2	6.8	4	13.6
	3	8.8	9	26.4
	5	13.2	25	66
Sum	11	36.8	39	113.1

5. $F(x, y, \lambda) = 3x^2 + 3y^2 + 1 + \lambda(x + y - 1)$, so we solve the system $\begin{cases} F_x = 6x + \lambda = 0 \\ F_y = 6y + \lambda = 0 \\ F_\lambda = x + y - 1 = 0 \end{cases}$ We find $\lambda = -6x - 6y$, so $y = x$. Substituting into the third equation gives $2x = 1$, so $x = \frac{1}{2}$ and $y = \frac{1}{2}$. Therefore, $\left(\frac{1}{2}, \frac{1}{2}, \frac{5}{2}\right)$ is the required minimum.

6. $$\iint_R (1 - xy)\,dA = \int_0^1 \int_{x^2}^{x} (1 - xy)\,dy\,dx = \int_0^1 \left(y - \tfrac{1}{2}xy^2\right)\Big|_{y=x^2}^{y=x} dx = \int_0^1 \left(x - \tfrac{1}{2}x^3 - x^2 + \tfrac{1}{2}x^5\right) dx$$
$$= \left(\tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{3}x^3 + \tfrac{1}{12}x^6\right)\Big|_0^1 = \tfrac{1}{2} - \tfrac{1}{8} - \tfrac{1}{3} + \tfrac{1}{12} = \tfrac{1}{8}.$$

CHAPTER 8 Explore & Discuss

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$f(y) = P(a, y)$ gives the total profit realized by the company through the sales of y units of the second product with the sales of the first product held fixed at a units. Next, $g(x) = P(x, b)$ gives the total profit realized by the company through the sales of x units of the first product when the sales of the second product are held fixed at b units.

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Because the sales of the first product are fixed at a units, the profit is a function of only y, the number of units of the second product produced and sold. This function is $f(y) = P(a, y)$. In order to determine the number of units of the second product to be produced and sold so as to maximize the profit, we should determine the value of y that maximizes the function f. In the second case, we maximize the function $g(x) = P(x, b)$ with respect to x.

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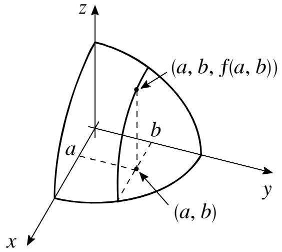
1. $f_x(a, b) = \lim_{h\to 0} \dfrac{f(a+h, b) - f(a, b)}{h} = \lim_{h\to 0} \dfrac{g(a+h) - g(a)}{h} = g'(a)$, justifying the procedure for calculating $f_x(a, b)$. Similarly, we let $h(y) = f(a, y)$ and compute $f_y(a, b) = \lim_{k\to 0} \dfrac{f(a, b+k) - f(a, b)}{k} = \lim_{k\to 0} \dfrac{h(b+k) - h(b)}{k} = h'(b)$, and this suggests that we can compute $f_y(a, b)$ by finding the derivative $h'(b)$.

   A geometric interpretation of the first process: For the value of y fixed at b, the graph of $g(x) = f(x, b)$ is a curve C passing through the point $(a, b, f(a, b))$. The derivative $g'(a)$ gives the slope of the tangent line to C at this point and is precisely the value of $f_x(a, b)$.

2. First, we let $b = 2$ and put $g(x) = f(x, 2) = x^2\left(2^3\right) - 3x^2(2) + 2 = 2x^2 + 2$. Therefore, $g'(x) = 4x$ and $g'(1) = 4$. This gives $f_x(1, 2) = 4$. Next, letting $a = 1$ and $h(y) = f(1, y) = (1)y^3 - 3(1)y + 2 = y^3 - 3y + 2$, we find $h'(y) = 3y^2 - 3$ and $h'(2) = 3(4) - 3 = 9$.

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If f has a relative extremum at (a, b) and f is differentiable, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$. But $f_x(a, b) = g'(x)\big|_{x=a} = g'(a)$ and $f_y(a, b) = h'(y)\big|_{y=b} = h'(b)$. Therefore, $g'(a) = 0$ and $h'(b) = 0$.

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1. Yes. The condition $f_{xx}(a, b) < 0$ in part 2a can be replaced by the condition $f_{yy}(a, b) < 0$. This follows because the surface is "locally" concave downward at a relative maximum. So this is true in the x-direction ($f_{xx}(a, b) < 0$) and it must be true in the y-direction as well; that is, $f_{yy}(a, b) < 0$.

2. **a.** $f(x, y) = x^4 + y^4$, so $f_x = 4x^3$ and $f_y = 4y^3$, giving $(0, 0)$ as a critical point of f. Next, $f_{xx} = 12x^2$, $f_{xy} = 0$, and $f_{yy} = 12y^2$. Therefore, $D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 - 0$ and $D(0, 0) = 0$.

   b. f has an absolute minimum at $(0, 0)$ because $f(x, y) = x^4 + y^4 > 0$ for all $(x, y) \neq (0, 0)$. This does not contradict the second derivative test, which says that if $D(a, b) > 0$ at a critical point (a, b), then f has a relative extremum. It does not say that if f has a relative extremum at a critical point, then $D(a, b) > 0$.

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If we write $z = \sqrt{4 - x^2 - y^2} = f(x, y)$, we see that this equation is equivalent to $\begin{cases} x^2 + y^2 + z^2 = 4 \\ z \geq 0 \end{cases}$ and so the integrand is the upper hemisphere with radius 2 centered at (0, 0, 0). Therefore, the integral gives the volume of the hemisphere, and $\int_R \int \sqrt{4 - x^2 - y^2}\, dA = \frac{1}{2}\left[\frac{4}{3}\pi (2)^3\right] = \frac{16\pi}{3}$. Note that the volume of a sphere is $\frac{4}{3}\pi r^3$.

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1. $h_1(y) = y$ and $h_2(y) = \sqrt{y}$. Also, $c = 0$ and $d = 1$.

2. $\int_R \int f(x, y)\, dA = \int_0^1 \left(\int_y^{\sqrt{y}} xe^y\, dx\right) dy$.

3. $\int_0^1 \left(\int_y^{\sqrt{y}} xe^y\, dx\right) dy = \int_0^1 \left(\frac{1}{2}x^2e^y\right)\Big|_{x=y}^{x=\sqrt{y}} dy = \int_0^1 \left(\frac{1}{2}ye^y - \frac{1}{2}y^2e^y\right) dy$. Then

$\iint\limits_R f(x, y)\, dA = \frac{1}{2}\int_0^1 ye^y\, dy - \frac{1}{2}y^2e^y\Big|_0^1 + \frac{1}{2}(2)\int_0^1 ye^y\, dy = \frac{3}{2}\int_0^1 ye^y\, dy - \frac{1}{2}e$. Next,

$\iint\limits_R f(x, y)\, dA = \frac{3}{2}(y - 1)\, e^y\Big|_0^1 - \frac{1}{2}e = \frac{3}{2} - \frac{1}{2}e = \frac{1}{2}(3 - e)$, as obtained in Example 3.

4. Clearly, viewing the region R as in Example 3 leads to an integral that is much easier to evaluate.

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1. It certainly makes sense to define $\int_R \int f(x, y)\, dA = \int_0^\infty \left[\int_0^\infty f(x, y)\, dx\right] dy = \int_0^\infty \left[\int_0^\infty f(x, y)\, dy\right] dx$. Then, using the definition of improper integrals of functions of one variable, we might define $\int_0^\infty \int_0^\infty f(x, y)\, dA = \lim_{N\to\infty} \int_0^N \left[\lim_{M\to\infty} \int_0^M f(x, y)\, dx\right] dy = \lim_{M\to\infty} \int_0^M \left[\lim_{N\to\infty} \int_0^N f(x, y)\, dy\right] dx$, provided that the limits exist.

2, 3. Let D^c denote the plane region in the first quadrant outside the region R. Then the population outside the rectangular region is

$$
\begin{aligned}
4\int_D \int f(x, y)\, dA - 4\int_R \int f(x, y)\, dA &= 4\int_0^\infty \left(\int_0^\infty 10{,}000e^{-0.2x-0.1y} dx\right) dy - 680{,}438 \\
&= 4\int_0^\infty \left(\lim_{M\to\infty} 10{,}000e^{-0.2x-0.1y} dx\right) dy - 680{,}438 \\
&= 4\int_0^\infty \lim_{M\to\infty} \left[-\frac{10{,}000}{0.2}\left(e^{-0.2M-0.1y}\right)\right]_{x=0}^{x=M} dy - 680{,}438 \\
&= 4\int_0^\infty \lim_{M\to\infty} \left[-50{,}000\left(e^{-0.2M-0.1y} - e^{-0.1y}\right)\right] dy - 680{,}438 \\
&= 200{,}000\int_0^\infty e^{-0.1y} dy - 680{,}438 = \lim_{N\to\infty} \frac{200{,}000}{-0.1}e^{-0.1y}\Big|_0^N - 680{,}438 \\
&= \lim_{N\to\infty} \left(-2{,}000{,}000e^{-0.1N} + 2{,}000{,}000\right) - 680{,}438 \\
&= 2{,}000{,}000 - 680{,}438 = 1{,}319{,}562.
\end{aligned}
$$

CHAPTER 8 Exploring with Technology

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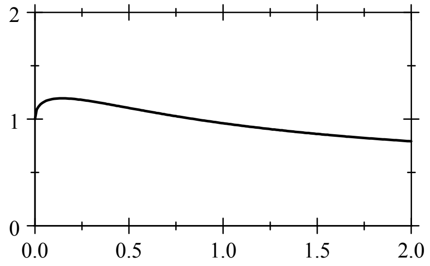
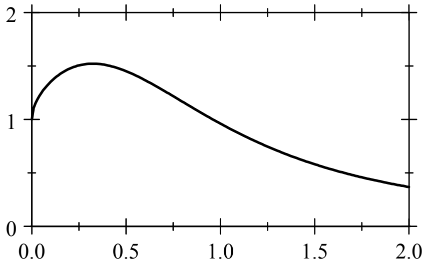
1. $g(x) = f(x,1) = \dfrac{e^{\sqrt{x}}}{(1+x)^{3/2}}.$



2. $g'(1) = f_x(1,1) = -\frac{\sqrt{2}}{16}e \approx -0.240264.$

3. $h(y) = f(1,y) = \dfrac{e^{\sqrt{y}}}{\left(1+y^2\right)^{3/2}}.$



4. $h'(1) = f_y(1,1) = -\frac{\sqrt{2}}{4}e \approx -0.961058.$